# $q$-Special functions

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**Abstract**

This is an overview article on $q$-special functions, a slightly extended version of an article to appear in the Encyclopedia of Mathematical Physics, Elsevier, 2006.

## 1 Introduction

In this article I give a brief introduction to $q$-special functions, i.e. $q$-analogues of the classical special functions. Here $q$ is a deformation parameter, usually $0 < q < 1$, where $q = 1$ is the classical case. The deformation is such that the calculus simultaneously deforms to a $q$-calculus involving $q$-derivatives and $q$-integrals. The main topics to be treated are $q$-hypergeometric series, with some selected evaluation and transformation formulas, and some $q$-hypergeometric orthogonal polynomials, most notably the Askey-Wilson polynomials. In several variables we discuss Macdonald polynomials associated with root systems, with most emphasis on the $A_n$ case. The rather new theory of elliptic hypergeometric series gets some attention. While much of the theory of $q$-special functions keeps $q$ fixed, some of the deeper aspects with number theoretic and combinatorial flavor emphasize expansion in $q$. Finally we indicate applications and interpretations in quantum groups, Chevalley groups, affine Lie algebras, combinatorics and statistical mechanics.

As general references can be recommended [2], [7] and [14].

**Conventions**
$q \in \mathbb{C}\backslash\{1\}$ in general, but $0 < q < 1$ in all infinite sums and products.
$n, m, N$ will be nonnegative integers unless mentioned otherwise.

## 2 $q$-Hypergeometric series

Standard reference here is Gasper & Rahman [7].

### 2.1 Definitions

For $a, q \in \mathbb{C}$ the *$q$-shifted factorial* $(a;q)_k$ is defined as a product of $k$ factors:

$$(a;q)_k := (1-a)(1-aq)\dots(1-aq^{k-1}) \quad (k \in \mathbb{Z}_{>0}); \qquad (a;q)_0 := 1. \tag{2.1}$$

If $|q| < 1$ this definition remains meaningful for $k = \infty$ as a convergent infinite product:

$$(a;q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j). \tag{2.2}$$

We also write $(a_1, \ldots, a_r; q)_k$ for the product of $r$ $q$-shifted factorials:

$$(a_1, \ldots, a_r; q)_k := (a_1;q)_k \ldots (a_r;q)_k \quad (k \in \mathbb{Z}_{\geq 0} \text{ or } k = \infty). \tag{2.3}$$

A *$q$-hypergeometric series* is a power series (for the moment still formal) in one complex variable $z$ with power series coefficients which depend, apart from $q$, on $r$ complex *upper parameters* $a_1, \ldots, a_r$ and $s$ complex *lower parameters* $b_1, \ldots, b_s$ as follows:

$$\begin{aligned} {}_r\phi_s \left[ \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] &= {}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) \\ &:= \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k \, (q;q)_k} \left( (-1)^k q^{\frac{1}{2}k(k-1)} \right)^{s-r+1} z^k \qquad (r, s \in \mathbb{Z}_{\geq 0}). \end{aligned} \tag{2.4}$$

Clearly the above expression is symmetric in $a_1, \ldots, a_r$ and symmetric in $b_1, \ldots, b_s$. On the right-hand side of (2.4) we have that

$$\frac{(k+1)\text{th term}}{k\text{th term}} = \frac{(1 - a_1 q^k) \ldots (1 - a_r q^k) \, (-q^k)^{s-r+1} \, z}{(1 - b_1 q^k) \ldots (1 - b_s q^k) \, (1 - q^{k+1})} \tag{2.5}$$

is rational in $q^k$. Conversely, any rational function in $q^k$ can be written in the form of the right-hand side of (2.5). Hence, any series $\sum_{k=0}^{\infty} c_k$ with $c_0 = 1$ and $c_{k+1}/c_k$ rational in $q^k$ is of the form of a $q$-hypergeometric series (2.4).

In order to avoid singularities in the terms of (2.4) we assume that $b_1, \ldots, b_s \neq 1, q^{-1}, q^{-2}, \ldots$. If, for some $i$, $a_i = q^{-n}$ then all terms in the series (2.4) with $k > n$ will vanish. If none of the $a_i$ is equal to $q^{-n}$ and if $|q| < 1$ then the radius of convergence of the power series (2.4) equals $\infty$ if $r < s + 1$, $1$ if $r = s + 1$, and $0$ if $r > s + 1$.

We can view the $q$-shifted factorial as a $q$-analogue of the *shifted factorial* (or *Pochhammer symbol*) by the limit formula

$$\lim_{q \to 1} \frac{(q^a;q)_k}{(1-q)^k} = (a)_k := a(a+1) \ldots (a+k-1). \tag{2.6}$$

Hence the *$q$-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \qquad (n, k \in \mathbb{Z}, \ n \geq k \geq 0) \tag{2.7}$$

tends to the binomial coefficient for $q \to 1$:

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}, \tag{2.8}$$

and a suitably renormalized $q$-hypergeometric series tends (at least formally) to a hypergeometric series as $q \uparrow 1$:

$$\lim_{q\uparrow 1} {}_{r+r'}\phi_{s+s'}\left[\begin{matrix} q^{a_1},\ldots,q^{a_r},c_1,\ldots,c_{r'} \\ q^{b_1},\ldots,q^{b_s},d_1,\ldots,d_{s'}\end{matrix};q,(q-1)^{1+s-r}z\right] = {}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};\frac{(c_1-1)\ldots(c_{r'}-1)\,z}{(d_1-1)\ldots(d_{s'}-1)}\right). \tag{2.9}$$

At least formally, there are limit relations between $q$-hypergeometric series with neighbouring $r, s$:

$$\lim_{a_r\to\infty} {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,\frac{z}{a_r}\right] = {}_{r-1}\phi_s\left[\begin{matrix} a_1,\ldots,a_{r-1} \\ b_1,\ldots,b_s\end{matrix};q,z\right], \tag{2.10}$$

$$\lim_{b_s\to\infty} {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,b_s z\right] = {}_r\phi_{s-1}\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_{s-1}\end{matrix};q,z\right]. \tag{2.11}$$

A terminating $q$-hypergeometric series $\sum_{k=0}^n c_k\, z^k$ rewritten as $z^n \sum_{k=0}^n c_{n-k} z^{-k}$ yields another terminating $q$-hypergeometric series, for instance:

$$ {}_{s+1}\phi_s\left[\begin{matrix} q^{-n},a_1,\ldots,a_s \\ b_1,\ldots,b_s\end{matrix};q,z\right] = (-1)^n\, q^{-\frac12 n(n+1)}\,\frac{(a_1,\ldots,a_n;q)_n}{(b_1,\ldots,b_s;q)_n}\,z^n \times {}_{s+1}\phi_s\left[\begin{matrix} q^{-n},q^{-n+1}b_1^{-1},\ldots,q^{-n+1}b_s^{-1} \\ q^{-n+1}a_1^{-1},\ldots,q^{-n+1}a_s^{-1}\end{matrix};q,\frac{q^{n+1}b_1\ldots b_s}{a_1\ldots a_s z}\right]. \tag{2.12}$$

Often, in physics and quantum groups related literature, the following notation is used for $q$-number, $q$-factorial and $q$-Pochhammer symbol:

$$[a]_q := \frac{q^{\frac12 a}-q^{-\frac12 a}}{q^{\frac12}-q^{-\frac12}},\quad [k]_q! := \prod_{j=1}^k [j]_q,\quad ([a]_q)_k := \prod_{j=0}^{k-1}[a+j]_q \qquad (k\in\mathbb{Z}_{\ge 0}). \tag{2.13}$$

For $q \to 1$ these symbols tend to their classical counterparts without the need for renormalization. They are expressed in terms of the standard notation (2.1) as follows:

$$[k]_q! = q^{-\frac14 k(k-1)}\,\frac{(q;q)_k}{(1-q)^k},\qquad ([a]_q)_k = q^{-\frac12 k(a-1)}\,q^{-\frac14 k(k-1)}\,\frac{(q^a;q)_k}{(1-q)^k}. \tag{2.14}$$

## 2.2 Special cases

For $s = r-1$ formula (2.4) simplifies to

$$ {}_r\phi_{r-1}\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_{r-1}\end{matrix};q,z\right] = \sum_{k=0}^\infty \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_{r-1};q)_k\,(q;q)_k}\,z^k, \tag{2.15}$$

which has radius of convergence 1 in the non-terminating case. The case $r = 2$ of (2.15) is the $q$-analogue of the Gauss hypergeometric series.

**$q$-Binomial series**

$$ {}_1\phi_0(a;-;q,z) = \sum_{k=0}^{\infty} \frac{(a;q)_k z^k}{(q;q)_k} = \frac{(az;q)_\infty}{(z;q)_\infty} \quad \text{(if series is not terminating then } |z|<1). \tag{2.16} $$

**$q$-Exponential series**

$$ e_q(z) := {}_1\phi_0(0;-;q,z) = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \qquad (|z|<1), \tag{2.17} $$

$$ E_q(z) := {}_0\phi_0(-;-;q,-z) = \sum_{k=0}^{\infty} \frac{q^{\frac12 k(k-1)} z^k}{(q;q)_k} = (-z;q)_\infty = (e_q(-z))^{-1} \qquad (z\in\mathbb{C}), \tag{2.18} $$

$$ \varepsilon_q(z) := {}_1\phi_1(0;-q^{\frac12};q^{\frac12},-z) = \sum_{k=0}^{\infty} \frac{q^{\frac14 k(k-1)}}{(q;q)_k} z^k \qquad (z\in\mathbb{C}). \tag{2.19} $$

**Jackson's $q$-Bessel functions**

$$ J_\nu^{(1)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\tfrac12 x\right)^\nu \, {}_2\phi_1\left[\begin{matrix} 0,0 \\ q^{\nu+1} \end{matrix}; q, -\tfrac14 x^2\right] \qquad (0<x<2), \tag{2.20} $$

$$ J_\nu^{(2)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\tfrac12 x\right)^\nu \, {}_0\phi_1\left[\begin{matrix} - \\ q^{\nu+1} \end{matrix}; q, -\tfrac14 q^{\nu+1} x^2\right] = (-\tfrac14 x;q)_\infty \, J_\nu^{(1)}(x;q) \quad (x>0), \tag{2.21} $$

$$ J_\nu^{(3)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\tfrac12 x\right)^\nu \, {}_1\phi_1\left[\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}; q, \tfrac14 q x^2\right] \qquad (x>0). \tag{2.22} $$

See (3.26) for the orthogonality relation for $J_\nu^{(3)}(x;q)$.

If $\exp_q(z)$ denotes one of the three $q$-exponentials (2.17)–(2.19) then $\frac12\big(\exp_q(ix)+\exp_q(-ix)\big)$ is a $q$-analogue of the cosine and $-\frac12 i\big(\exp_q(ix)-\exp_q(-ix)\big)$ is a $q$-analogue of the sine. The three $q$-cosines are essentially the case $\nu=-\frac12$ of the corresponding $q$-Bessel functions (2.20)–(2.22), and the three $q$-sines are essentially the case $\nu=\frac12$ of $x$ times the corresponding $q$-Bessel functions. See also Suslov [26].

## 2.3 $q$-Derivative and $q$-integral

The *$q$-derivative* of a function $f$ given on a subset of $\mathbb{R}$ or $\mathbb{C}$ is defined by

$$ (D_q f)(x) := \frac{f(x)-f(qx)}{(1-q)x} \qquad (x\neq 0,\ q\neq 1), \tag{2.23} $$

where $x$ and $qx$ should be in the domain of $f$. By continuity we set $(D_q f)(0) := f'(0)$, provided $f'(0)$ exisits. If $f$ is differentiable on an open interval $I$ then

$$ \lim_{q\uparrow 1} (D_q f)(x) = f'(x) \qquad (x\in I). \tag{2.24} $$

For $a \in \mathbb{R}\backslash\{0\}$ and a function $f$ given on $(0,a]$ or $[a,0)$, we define the *$q$-integral* by

$$\int_0^a f(x)\,d_qx := a(1-q)\sum_{k=0}^\infty f(aq^k)\,q^k = \sum_{k=0}^\infty f(aq^k)\,(aq^k - aq^{k+1}), \tag{2.25}$$

provided the infinite sum converges absolutely (for instance if $f$ is bounded). If $F(a)$ is given by the left-hand side of (2.25) then $D_qF = f$. The right-hand side of (2.25) is an infinite Riemann sum. For $q \uparrow 1$ it converges, at least formally, to $\int_0^a f(x)\,dx$.

For nonzero $a, b \in \mathbb{R}$ we define

$$\int_a^b f(x)\,d_qx := \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx. \tag{2.26}$$

For a $q$-integral over $(0,\infty)$ we have to specify a $q$-lattice $\{aq^k\}_{k\in\mathbb{Z}}$ for some $a > 0$ (up to multiplication by an integer power of $q$):

$$\int_0^{a.\infty} f(x)\,d_qx := a(1-q)\sum_{k=-\infty}^\infty f(aq^k)\,q^k = \lim_{n\to\infty}\int_0^{q^{-n}a} f(x)\,d_qx. \tag{2.27}$$

## 2.4 The $q$-gamma and $q$-beta functions

The *$q$-gamma function* is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty\,(1-q)^{1-z}}{(q^z;q)_\infty} \qquad (z \neq 0,-1,-2,\ldots) \tag{2.28}$$

$$= \int_0^{(1-q)^{-1}} t^{z-1}\,E_q(-(1-q)qt)\,d_qt \qquad (\operatorname{Re} z > 0). \tag{2.29}$$

Then

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\,\Gamma_q(z), \tag{2.30}$$

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n}\,, \tag{2.31}$$

$$\lim_{q\uparrow 1}\Gamma_q(z) = \Gamma(z). \tag{2.32}$$

The *$q$-beta function* is defined by

$$B_q(a,b) := \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)} = \frac{(1-q)\,(q,q^{a+b};q)_\infty}{(q^a,q^b;q)_\infty} \qquad (a,b \neq 0,-1,-2,\ldots), \tag{2.33}$$

$$= \int_0^1 t^{b-1}\,\frac{(qt;q)_\infty}{(q^at;q)_\infty}\,d_qt \qquad (\operatorname{Re} b > 0,\ a \neq 0,-1,-2,\ldots). \tag{2.34}$$

## 2.5 The $q$-Gauss hypergeometric series

**$q$-Analogue of Euler's integral representation**

$$
{}_2\phi_1(q^a,q^b;q^c;q,z)=\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-b)}\int_0^1 t^{b-1}\,\frac{(tq;q)_\infty}{(tq^{c-b};q)_\infty}\,\frac{(tzq^a;q)_\infty}{tz;q)_\infty}\,d_qt\quad(\operatorname{Re}b>0,\ |z|<1).\tag{2.35}
$$

By substitution of (2.25), formula (2.35) becomes a transformation formula:

$$
{}_2\phi_1(a,b;c;q,z)=\frac{(az;q)_\infty}{(z;q)_\infty}\,\frac{(b;q)_\infty}{(c;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b).\tag{2.36}
$$

Note the mixing of argument $z$ and parameters $a,b,c$ on the right-hand side.

**Evaluation formulas in special points**

$$
{}_2\phi_1\big(a,b;c;q,c/(ab)\big)=\frac{(c/a,c/b;q)_\infty}{(c,c/(ab);q)_\infty}\qquad(|c/(ab)|<1),\tag{2.37}
$$

$$
{}_2\phi_1(q^{-n},b;c;q,cq^n/b)=\frac{(c/b;q)_n}{(c;q)_n}\,,\tag{2.38}
$$

$$
{}_2\phi_1(q^{-n},b;c;q,q)=\frac{(c/b;q)_n\,b^n}{(c;q)_n}\,.\tag{2.39}
$$

**Two general transformation formulas**

$$
{}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]=\frac{(az;q)_\infty}{(z;q)_\infty}\;{}_2\phi_2\left[\begin{matrix}a,c/b\\c,az\end{matrix};q,bz\right],\tag{2.40}
$$

$$
=\frac{(abz/c;q)_\infty}{(z;q)_\infty}\;{}_2\phi_1\left[\begin{matrix}c/a,c/b\\c\end{matrix};q,\frac{abz}{c}\right].\tag{2.41}
$$

**Transformation formulas in the terminating case**

$$
{}_2\phi_1\left[\begin{matrix}q^{-n},b\\c\end{matrix};q,z\right]=\frac{(c/b;q)_n}{(c;q)_n}\;{}_3\phi_2\left[\begin{matrix}q^{-n},b,q^{-n}bc^{-1}z\\q^{1-n}bc^{-1},0\end{matrix};q,q\right]\tag{2.42}
$$

$$
=(q^{-n}bc^{-1}z;q)_n\;{}_3\phi_2\left[\begin{matrix}q^{-n},cb^{-1},0\\c,qcb^{-1}z^{-1}\end{matrix};q,q\right]\tag{2.43}
$$

$$
=\frac{(c/b;q)_n}{(c;q)_n}\,b^n\;{}_3\phi_1\left[\begin{matrix}q^{-n},b,qz^{-1}\\q^{1-n}bc^{-1}\end{matrix};q,\frac{z}{c}\right].\tag{2.44}
$$

**Second order $q$-difference equation**

$$
z(q^c-q^{a+b+1}z)(D_q^2u)(z)+\left(\frac{1-q^c}{1-q}-\left(q^b\frac{1-q^a}{1-q}+q^a\frac{1-q^{b+1}}{1-q}\right)z\right)(D_qu)(z)
-\frac{1-q^a}{1-q}\,\frac{1-q^b}{1-q}\,u(z)=0.\tag{2.45}
$$

Some special solutions of (2.45) are:

$$u_1(z) := {}_2\phi_1(q^a, q^b; q^c; q, z), \tag{2.46}$$
$$u_2(z) := z^{1-c}\,{}_2\phi_1(q^{1+a-c}, q^{1+b-c}; q^{2-c}; q, z), \tag{2.47}$$
$$u_3(z) := z^{-a}\,{}_2\phi_1(q^a, q^{a-c+1}; q^{a-b+1}; q, q^{-a-b+c+1}z^{-1}). \tag{2.48}$$

They are related by:

$$\begin{aligned} u_1(z) + \frac{(q^a, q^{1-c}, q^{c-b}; q)_\infty}{(q^{c-1}, q^{a-c+1}, q^{1-b}; q)_\infty}\,\frac{(q^{b-1}z, q^{2-b}z^{-1}; q)_\infty}{(q^{b-c}z, q^{c-b+1}z^{-1}; q)_\infty}\,u_2(z) \\ = \frac{(q^{1-c}, q^{a-b+1}; q)_\infty}{(q^{1-b}, q^{a-c+1}; q)_\infty}\,\frac{(q^{a+b-c}z, q^{c-a-b+1}z^{-1}; q)_\infty\, z^a}{(q^{b-c}z, q^{c-b+1}z^{-1}; q)_\infty}\,u_3(z). \end{aligned} \tag{2.49}$$

## 2.6 Summation and transformation formulas for ${}_r\phi_{r-1}$ series

An ${}_r\phi_{r-1}$ series (2.15) is called *balanced* if $b_1 \dots b_{r-1} = qa_1 \dots a_r$ and $z = q$, and the series is called *very-well-poised* if $qa_1 = a_2b_1 = a_3b_2 = \cdots = a_rb_{r-1}$ and $qa_1^{\frac{1}{2}} = a_2 = -a_3$. The following more compact notation is used for very-well-poised series:

$${}_rW_{r-1}(a_1; a_4, a_5, \dots, a_r; q, z) := {}_r\phi_{r-1}\left[\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \dots, a_r \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \dots, qa_1/a_r \end{matrix}; q, z\right]. \tag{2.50}$$

Below only a few of the most important identities are given. See [7] for many more. An important tool for obtaining complicated identities from more simple ones is *Bailey's Lemma*, which can moreover be iterated (*Bailey chain*), see [1, Ch.3].

**The $q$-Saalschütz sum for a terminating balanced ${}_3\phi_2$**

$${}_3\phi_2\left[\begin{matrix} a, b, q^{-n} \\ c, q^{1-n}abc^{-1} \end{matrix}; q, q\right] = \frac{(c/a, c/b; q)_n}{(c, c/(ab); q)_n}. \tag{2.51}$$

**Jackson's sum for a terminating balanced ${}_8W_7$**

$${}_8W_7(a; b, c, d, q^{n+1}a^2/(bcd), q^{-n}; q, q) = \frac{(qa, qa/(bc), qa/(bd), qa/(cd); q)_n}{(qa/b, qa/c, qa/d, qa/(bcd); q)_n}. \tag{2.52}$$

**Watson's transformation of a terminating ${}_8W_7$ into a terminating balanced ${}_4\phi_3$**

$${}_8W_7\left(a; b, c, d, e, q^{-n}; q, \frac{q^{n+2}a^2}{bcde}\right) = \frac{(qa, qa/(de); q)_n}{(qa/d, qa/e; q)_n}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, d, e, qa/(bc) \\ qa/b, qa/c, q^{-n}de/a \end{matrix}; q, q\right]. \tag{2.53}$$

**Sears' transformation of a terminating balanced ${}_4\phi_3$**

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}; q, q\right] = \frac{(e/a, f/a; q)_n}{(e, f; q)_n}\, a^n\; {}_4\phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, q^{1-n}a/e, q^{1-n}a/f \end{matrix}; q, q\right]. \tag{2.54}
$$

By iteration and by symmetries in the upper and in the lower parameters, many other versions of this identity can be found. An elegant comprehensive formulation of all these versions is as follows.
Let $x_1x_2x_3x_4x_5x_6 = q^{1-n}$. Then the following expression is symmetric in $x_1, x_2, x_3, x_4, x_5, x_6$:

$$
\frac{q^{\frac{1}{2}n(n-1)}(x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_3x_6; q)_n}{(x_1x_2x_3)^n}\; {}_4\phi_3\left[\begin{matrix} q^{-n}, x_2x_3, x_1x_3, x_1x_2 \\ x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_3x_6 \end{matrix}; q, q\right]. \tag{2.55}
$$

Similar formulations involving symmetry groups can be given for other transformations, see [27].

**Bailey's transformation of a terminating balanced ${}_{10}W_9$**

$$
\begin{aligned}
{}_{10}W_9\left(a; b, c, d, e, f, \frac{q^{n+2}a^3}{bcdef}, q^{-n}; q, q\right) &= \frac{(qa, qa/(ef), (qa)^2/(bcde), (qa)^2/(bcdf); q)_n}{(qa/e, qa/f, (qa)^2/(bcdef), (qa)^2/(bcd); q)_n} \\
&\quad \times {}_{10}W_9\left(\frac{qa^2}{bcd}; \frac{qa}{cd}, \frac{qa}{bd}, \frac{qa}{bc}, e, f, \frac{q^{n+2}a^3}{bcdef}, q^{-n}; q, q\right).
\end{aligned} \tag{2.56}
$$

## 2.7 Rogers-Ramanujan identities

$$
{}_0\phi_1(-; 0; q, q) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} = \frac{1}{(q, q^4; q^5)_\infty}, \tag{2.57}
$$

$$
{}_0\phi_1(-; 0; q, q^2) = \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q; q)_k} = \frac{1}{(q^2, q^3; q^5)_\infty}. \tag{2.58}
$$

## 2.8 Bilateral series

Definition (2.1) can be extended by

$$
(a; q)_k := \frac{(a; q)_\infty}{(aq^k; q)_\infty} \qquad (k \in \mathbb{Z}). \tag{2.59}
$$

Define a *bilateral q-hypergeometric series* by the Laurent series

$$
\begin{aligned}
&{}_r\psi_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] = {}_r\psi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) \\
&\quad := \sum_{k=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{s-r} z^k \quad (a_1, \ldots, a_r, b_1, \ldots, b_s \neq 0,\ s \geq r).
\end{aligned} \tag{2.60}
$$

The Laurent series is convergent if $|b_1 \ldots b_s/(a_1 \ldots a_r)| < |z|$ and moreover, for $s = r$, $|z| < 1$.

**Ramanajan's ${}_1\psi_1$ summation formula**

$$ {}_1\psi_1(b;c;q,z) = \frac{(q,c/b,bz,q/(bz);q)_\infty}{(c,q/b,z,c/(bz);q)_\infty} \qquad (|c/b|<|z|<1). \tag{2.61}$$

This has as a limit case

$$ {}_0\psi_1(-;c;q,z) = \frac{(q,z,q/z;q)_\infty}{(c,c/z;q)_\infty} \quad (|z|>|c|), \tag{2.62}$$

and as a further specialization the *Jacobi triple product identity*

$$\sum_{k=-\infty}^{\infty} (-1)^k\, q^{\frac12 k(k-1)}\, z^k = (q,z,q/z;q)_\infty \qquad (z\ne 0), \tag{2.63}$$

which can be rewritten as a product formula for a *theta function*:

$$\theta_4(x;q) := \sum_{k=-\infty}^{\infty} (-1)^k\, q^{k^2}\, e^{2\pi i k x} = \prod_{k=1}^{\infty} (1-q^{2k})\big(1-2q^{k-1}\cos(2\pi x)+q^{4k-2}\big). \tag{2.64}$$

# 3 $q$-Hypergeometric orthogonal polynomials

Here we discuss families of orthogonal polynomials $\{p_n(x)\}$ which are expressible as terminating $q$-hypergeometric series $(0<q<1)$ and for which either (i) $P_n(x) := p_n(x)$ or (ii) $P_n(x) := p_n\big(\frac12(x+x^{-1})\big)$ are eigenfunctions of a second order $q$-difference operator, i.e.:

$$A(x)\,P_n(qx) + B(x)\,P_n(x) + C(x)\,P_n(q^{-1}x) = \lambda_n\,P_n(x), \tag{3.1}$$

where $A(x)$, $B(x)$ and $C(x)$ are independent of $n$, and where the $\lambda_n$ are the eigenvalues. The generic cases are the four-parameter classes of *Askey-Wilson polynomials* (continuous weight function) and *q-Racah polynomials* (discrete weights on finitely many points). They are of type (ii) (quadratic $q$-lattice). All other cases can be obtained from the generic cases by specialization or limit transition. In particular, one thus obtains the generic three-parameter classes of type (i) (linear $q$-lattice). These are the *big q-Jacobi polynomials* (orthogonality by $q$-integral) and the *q-Hahn polynomials* (discrete weights on finitely many points). For all these families the standard formulas are given in Koekoek & Swarttouw [10].

## 3.1 Askey-Wilson polynomials

These were introduced by Askey & Wilson [4],

**Definition as $q$-hypergeometric series**

$$p_n(\cos\theta) = p_n(\cos\theta;a,b,c,d\mid q) := \frac{(ab,ac,ad;q)_n}{a^n}\; {}_4\phi_3\left[\begin{matrix} q^{-n},q^{n-1}abcd,ae^{i\theta},ae^{-i\theta} \\ ab,ac,ad\end{matrix};q,q\right]. \tag{3.2}$$

This is symmetric in $a,b,c,d$.

**Orthogonality relation** Assume that $a, b, c, d$ are four reals, or two reals and one pair of complex conjugates, or two pairs of complex conjugates. Also assume that $|ab|, |ac|, |ad|, |bc|, |bd|, |cd| < 1$. Then

$$\int_{-1}^{1} p_n(x)\, p_m(x)\, w(x)\, dx + \sum_k p_n(x_k)\, p_m(x_k)\, \omega_k = h_n\, \delta_{n,m}, \tag{3.3}$$

where

$$2\pi \sin\theta\, w(\cos\theta) = \left| \frac{(e^{2i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta}; q)_\infty} \right|^2, \tag{3.4}$$

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \quad \frac{h_n}{h_0} = \frac{1 - abcdq^{n-1}}{1 - abcdq^{2n-1}} \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_n}, \tag{3.5}$$

and the $x_k$ are the points $\frac{1}{2}(eq^k + e^{-1}q^{-k})$ with $e$ any of the $a, b, c, d$ of absolute value $> 1$; the sum is over the $k \in \mathbb{Z}_{\geq 0}$ with $|eq^k| > 1$. The $\omega_k$ are certain weights which can be given explicitly. The sum in (3.3) does not occur if moreover $|a|, |b|, |c|, |d| < 1$.

A more uniform way of writing the orthogonality relation (3.3) is by the contour integral

$$\frac{1}{2\pi i} \oint_C p_n\big(\tfrac{1}{2}(z + z^{-1})\big)\, p_m\big(\tfrac{1}{2}(z + z^{-1})\big)\, \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty} \frac{dz}{z} = 2h_n \delta_{n,m}, \tag{3.6}$$

where $C$ is the unit circle traversed in positive direction with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to $\infty$.

The case $n = m = 0$ of (3.6) or (3.3) is known as the *Askey-Wilson integral.*

### $q$-Difference equation

$$A(z)P_n(qz) - \big(A(z) + A(z^{-1})\big)P_n(z) + A(z^{-1})P_n(q^{-1}z) = (q^{-n} - 1)(1 - q^{n-1}abcd)P_n(z), \tag{3.7}$$

where $P_n(z) = p_n\big(\frac{1}{2}(z + z^{-1})\big)$ and $A(z) = (1 - az)(1 - bz)(1 - cz)(1 - dz)/\big((1 - z^2)(1 - qz^2)\big)$.

**Special cases** These include the *continuous $q$-Jacobi polynomials* (2 parameters), the *continuous $q$-ultraspherical polynomials* (symmetric one-parameter case of continuous $q$-Jacobi), the *Al-Salam-Chihara polynomials* (Askey-Wilson with $c = d = 0$), and the *continuous $q$-Hermite polynomials* (Askey-Wilson with $a = b = c = d = 0$).

## 3.2 Continuous $q$-ultraspherical polynomials

### Definitions as finite Fourier series and as special Askey-Wilson polynomial

$$C_n(\cos\theta; \beta \mid q) := \sum_{k=0}^{n} \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}}\, e^{i(n-2k)\theta} \tag{3.8}$$

$$= \frac{(\beta; q)_n}{(q; q)_n}\, p_n(\cos\theta; \beta^{\frac{1}{2}}, q^{\frac{1}{2}}\beta^{\frac{1}{2}}, -\beta^{\frac{1}{2}}, -q^{\frac{1}{2}}\beta^{\frac{1}{2}} \mid q). \tag{3.9}$$

**Orthogonality relation** $(-1<\beta<1)$

$$\frac{1}{2\pi}\int_0^\pi C_n(\cos\theta;\beta,q)\,C_m(\cos\theta;\beta,q)\left|\frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty}\right|^2 d\theta=\frac{(\beta,q\beta;q)_\infty}{(\beta^2,q;q)_\infty}\,\frac{1-\beta}{1-\beta q^n}\,\frac{(\beta^2;q)_n}{(q;q)_n}\,\delta_{n,m}. \tag{3.10}$$

**$q$-Difference equation**

$$A(z)P_n(qz)-\big(A(z)+A(z^{-1})\big)P_n(z)+A(z^{-1})P_n(q^{-1}z)=(q^{-n}-1)(1-q^n\beta^2)P_n(z), \tag{3.11}$$

where $P_n(z)=C_n\big(\frac{1}{2}(z+z^{-1});\beta\mid q\big)$ and $A(z)=(1-\beta z^2)(1-q\beta z^2)/\big((1-z^2)(1-qz^2)\big)$.

**Generating function**

$$\frac{(\beta e^{i\theta}z,\beta e^{-i\theta}z;q)_\infty}{(e^{i\theta}z,e^{-i\theta}z;q)_\infty}=\sum_{n=0}^\infty C_n(\cos\theta;\beta\mid q)\,z^n\qquad(|z|<1,\ 0\le\theta\le\pi,\ -1<\beta<1). \tag{3.12}$$

**Special case: the continuous $q$-Hermite polynomials**

$$H_n(x\mid q)=(q;q)_n\,C_n(x;0\mid q). \tag{3.13}$$

**Special cases: the Chebyshev polynomials**

$$C_n(\cos\theta;q\mid q)=U_n(\cos\theta):=\frac{\sin((n+1)\theta)}{\sin\theta}\,, \tag{3.14}$$

$$\lim_{\beta\uparrow 1}\frac{(q;q)_n}{(\beta;q)_n}C_n(\cos\theta;\beta\mid q)=T_n(\cos\theta):=\cos(n\theta)\qquad(n>0). \tag{3.15}$$

### 3.3 $q$-Racah polynomials

**Definition as $q$-hypergeometric series** $(n=0,1,\ldots,N)$

$$R_n(q^{-y}+\gamma\delta q^{y+1};\alpha,\beta,\gamma,\delta\mid q):={}_4\phi_3\left[\begin{matrix}q^{-n},\alpha\beta q^{n+1},q^{-y},\gamma\delta q^{y+1}\\ q\alpha,q\beta\delta,q\gamma\end{matrix};q,q\right]\qquad(\alpha,\ \beta\delta\text{ or }\gamma=q^{-N-1}). \tag{3.16}$$

**Orthogonality relation**

$$\sum_{y=0}^N R_n(q^{-y}+\gamma\delta q^{y+1})\,R_m(q^{-y}+\gamma\delta q^{y+1})\,\omega_y=h_n\delta_{n,m}, \tag{3.17}$$

where $\omega_y$ and $h_n$ can be explicitly given.

### 3.4 Big $q$-Jacobi polynomials

**Definition as $q$-hypergeometric series**

$$P_n(x) = P_n(x;a,b,c;q) := {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{n+1}ab, x \\ qa, qc \end{matrix}; q, q\right]. \tag{3.18}$$

**Orthogonality relation**

$$\int_{qc}^{qa} P_n(x)\, P_m(x)\, \frac{(a^{-1}x, c^{-1}x;q)_\infty}{(x, bc^{-1}x;q)_\infty}\, d_qx = h_n\, \delta_{n,m}, \quad (0 < a < q^{-1},\ 0 < b < q^{-1},\ c < 0), \tag{3.19}$$

where $h_n$ can be explicitly given.

**$q$-Difference equation**

$$A(x)P_n(qx) - (A(x)+C(x))P_n(x) + C(x)P_n(q^{-1}x) = (q^{-n}-1)(1-abq^{n+1})P_n(x), \tag{3.20}$$

where $A(x) = aq(x-1)(bx-c)/x^2$ and $C(x) = (x-qa)(x-qc)/x^2$.

**Limit case: Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$**

$$\lim_{q\uparrow 1} P_n(x; q^\alpha, q^\beta, -q^{-1}d; q) = \frac{n!}{(\alpha+1)_n}\, P_n^{(\alpha,\beta)}\left(\frac{2x+d-1}{d+1}\right). \tag{3.21}$$

**Special case: the little $q$-Jacobi polynomials**

$$p_n(x;a,b;q) = (-b)^{-n} q^{-\frac{1}{2}n(n+1)}\, \frac{(qb;q)_n}{(qa;q)_n}\, P_n(qbx; b, a, 0; q) \tag{3.22}$$

$$= {}_2\phi_1(q^{-n}, q^{n+1}ab; qa; q, qx), \tag{3.23}$$

which satisfy orthogonality relation (for $0 < a < q^{-1}$ and $b < q^{-1}$)

$$\int_0^1 p_n(x;a,b;q)\, p_m(x;a,b;q)\, \frac{(qx;q)_\infty}{(qbx;q)_\infty}\, x^{\log_q a}\, d_qx = \frac{(q, qab;q)_\infty}{(qa, qb;q)_\infty}\, \frac{(1-q)(qa)^n}{1-abq^{2n+1}}\, \frac{(q, qb;q)_n}{(qa, qab;q)_n}\, \delta_{n,m}. \tag{3.24}$$

**Limit case: Jackson's third $q$-Bessel function** (see (2.22) and [16])

$$\lim_{N\to\infty} p_{N-n}(q^{N+k}; q^\nu, b; q) = \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty}\, q^{-\nu(n+k)}\, J_\nu^{(3)}(2q^{\frac{1}{2}(n+k)}; q) \qquad (\nu > -1), \tag{3.25}$$

by which (3.24) tends to the orthogonality relation for $J_\nu^{(3)}(x;q)$:

$$\sum_{k=-\infty}^{\infty} J_\nu^{(3)}(2q^{\frac{1}{2}(n+k)}; q)\, J_\nu^{(3)}(2q^{\frac{1}{2}(m+k)}; q)\, q^k = \delta_{n,m} q^{-n} \qquad (n, m \in \mathbb{Z}). \tag{3.26}$$

### 3.5 $q$-Hahn polynomials

**Definition as $q$-hypergeometric series**

$$Q_n(x;\alpha,\beta,N;q) := {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{n+1}\alpha\beta, x \\ q\alpha, q^{-N} \end{matrix}; q, q\right] \qquad (n = 0, 1, \ldots, N). \tag{3.27}$$

**Orthogonality relation**

$$\sum_{y=0}^{N} Q_n(q^{-y})\, Q_m(q^{-y})\, \frac{(q\alpha, q^{-N}; q)_y\, (q\alpha\beta)^{-y}}{(q^{-N}\beta^{-1}, q; q)_y} = h_n \delta_{n,m}, \tag{3.28}$$

where $h_n$ can be explicitly given.

### 3.6 Stieltjes-Wigert polynomials

**Definition as $q$-hypergeometric series**

$$S_n(x;q) = \frac{1}{(q;q)_n}\, {}_1\phi_1\left[\begin{matrix} q^{-n} \\ 0 \end{matrix}; q, -q^{n+1}x\right]. \tag{3.29}$$

The orthogonality measure is not uniquely determined:

$$\int_0^\infty S_n(q^{\frac{1}{2}}x;q)\, S_m(q^{\frac{1}{2}}x;q)\, w(x)\, dx = \frac{1}{q^n (q;q)_n}\, \delta_{n,m}, \quad \text{where, for instance,}$$

$$w(x) = \frac{q^{\frac{1}{2}}}{\log(q^{-1})\, (q, -q^{\frac{1}{2}}x, -q^{\frac{1}{2}}x^{-1}; q)_\infty} \quad \text{or} \quad \frac{q^{\frac{1}{2}}}{\sqrt{2\pi \log(q^{-1})}} \exp\left(-\frac{\log^2 x}{2\log(q^{-1})}\right). \tag{3.30}$$

### 3.7 Rahman-Wilson biorthogonal rational functions

The following functions are rational in their first argument:

$$R_n\left(\tfrac{1}{2}(z+z^{-1}); a,b,c,d,e\right) := {}_{10}W_9(a/e; q/(be), q/(ce), q/(de), az, a/z, q^{n-1}abcd, q^{-n}; q, q). \tag{3.31}$$

They satisfy the biorthogonality relation

$$\frac{1}{2\pi i} \oint_C R_n\left(\tfrac{1}{2}(z+z^{-1}); a,b,c,d,e\right)\, R_m\left(\tfrac{1}{2}(z+z^{-1}); a,b,c,d,\frac{q}{abcde}\right)\, w(z)\, \frac{dz}{z} = 2h_n\delta_{n,m}, \tag{3.32}$$

where the contour $C$ is as in (3.6), and where

$$w(z) = \frac{(z^2, z^{-2}, abcdez, abcde/z; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z, ez, e/z; q)_\infty}, \tag{3.33}$$

$$h_0 = \frac{(bcde, acde, abde, abce, abcd; q)_\infty}{(q, ab, ac, ad, ae, bc, bd, be, cd, ce, de; q)_\infty}, \tag{3.34}$$

and $h_n/h_0$ can also be given explicitly. For $ab = q^{-N}$, $n, m \in \{0, 1, \ldots, N\}$ there is a related discrete biorthogonality of the form

$$\sum_{k=0}^{N} R_n\left(\tfrac{1}{2}(aq^k + a^{-1}q^{-k}); a, b, c, d, e\right) R_m\left(\tfrac{1}{2}(aq^k + a^{-1}q^{-k}); a, b, c, d, \frac{q}{abcde}\right) w_k = 0 \qquad (n \neq m). \tag{3.35}$$

See Rahman [23] and Wilson [29].

# 4 Identities and functions associated with root systems

## 4.1 $\eta$-Function identities

Let $R$ be a root system on a Euclidean space of dimension $l$. Then Macdonald [18] generalizes Weyl's denominator formula to the case of an affine root system. The resulting formula can be written as an explicit expansion in powers of $q$ of

$$\prod_{n=1}^{\infty}\left((1-q^n)^l \prod_{\alpha\in R}(1-q^n e^{\alpha})\right),$$

which expansion takes the form of a sum over a lattice related to the root system. For root system $A_1$ this reduces to Jacobi's triple product identity (2.63). Macdonald's formula implies a similar expansion in powers of $q$ of $\eta(q)^{l+|R|}$, where $\eta(q)$ is *Dedekind's $\eta$-function* $\eta(q) := q^{1/24}(q;q)_\infty$.

## 4.2 Constant term identitities

Let $R$ be a reduced root system, $R^+$ the positive roots and $k \in \mathbb{Z}_{>0}$. Macdonald [19] conjectured the second equality in

$$\frac{\int_T \prod_{\alpha\in R^+}(e^{-\alpha};q)_k\,(qe^{\alpha};q)_k\,dx}{\int_T dx} = CT\Big(\prod_{\alpha\in R^+}\prod_{i=1}^{k}(1-q^{i-1}e^{-\alpha})(1-q^i e^{\alpha})\Big) = \prod_{i=1}^{l}\begin{bmatrix} kd_i \\ k\end{bmatrix}_q, \tag{4.1}$$

where $T$ is a torus determined by $R$, $CT$ means the constant term in the Laurent expansion in $e^{\alpha}$, and the $d_i$ are the degrees of the fundamental invariants of the Weyl group of $R$. The conjecture was extended for real $k > 0$, for several parameters $k$ (one for each root length), and for root system $BC_n$, where Gustafson's [8] 5-parameter $n$-variable analogue of the Askey-Wilson integral ((3.6) for $n = 0$) settles:

$$\int_{[0,2\pi]^n} |\Delta(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2\,\frac{d\theta_1\ldots\theta_n}{(2\pi)^n} = 2^n n! \prod_{j=1}^{n}\frac{(t, t^{n+j-2}abcd;q)_\infty}{(t^j, q, abt^{j-1}, act^{j-1}, \ldots, cdt^{j-1};q)_\infty}, \tag{4.2}$$

where

$$\Delta(z) := \prod_{1\le i<j\le n}\frac{(z_iz_j, z_i/z_j;q)_\infty}{(tz_iz_j, tz_i/z_j;q)_\infty}\prod_{j=1}^{n}\frac{(z_j^2;q)_\infty}{(az_j, bz_j, cz_j, dz_j;q)_\infty}. \tag{4.3}$$

Further extensions were in Macdonald's conjectures for the quadratic norms of Macdonald polynomials associated with root systems (see §4.4), and finally proved by Cherednik [6].

### 4.3 Macdonald polynomials for root system $A_{n-1}$

Reference for this section is Macdonald [20, Ch. VI]. Let $n \in \mathbb{Z}_{>0}$. We work with *partitions* $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\le n$, where $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$ are integer. On the set of such partitions we take the *partial order* $\lambda \le \mu \iff \lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$ and $\lambda_1 + \cdots + \lambda_i \le \mu_1 + \cdots + \mu_i$ $(i = 1, \ldots, n-1)$. Write $\lambda < \mu$ iff $\lambda \le \mu$ and $\lambda \ne \mu$. The *monomials* are $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ $(\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\ge 0})$. For $\lambda$ a partition the *symmetrized monomials* $m_\lambda(z)$ and the *Schur functions* $s_\lambda(z)$ are defined by:

$$m_\lambda(z) := \sum_\alpha z^\alpha \quad \text{(sum over all distinct permutations } \alpha \text{ of } (\lambda_1, \ldots, \lambda_n)), \tag{4.4}$$

$$s_\lambda(z) := \frac{\det(z_i^{\lambda_j + n - j})_{i,j=1,\ldots,n}}{\det(z_i^{n-j})_{i,j=1,\ldots,n}}\,. \tag{4.5}$$

We integrate a function over the torus $T := \{z \in \mathbb{C}^n \mid |z_1| = \ldots = |z_n| = 1\}$ as

$$\int_T f(z)\,dz := \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i\theta_1}, \ldots, e^{i\theta_n})\,d\theta_1 \ldots d\theta_n. \tag{4.6}$$

**Definition** For $\lambda$ a partition and for $0 \le t \le 1$ the (analytically defined) *Macdonald polynomial* $P_\lambda(z) = P_\lambda(z; q, t)$ is of the form

$$P_\lambda(z) = P_\lambda(z; q, t) = m_\lambda(z) + \sum_{\mu < \lambda} u_{\lambda,\mu} m_\mu(z) \qquad (u_{\lambda,\mu} \in \mathbb{C})$$

such that for all $\mu < \lambda$

$$\int_T P_\lambda(z)\,\overline{m_\mu(z)}\,\Delta(z)\,dz = 0,$$

where

$$\Delta(z) = \Delta(z; q, t) := \prod_{i \ne j} \frac{(z_i z_j^{-1}; q)_\infty}{(t z_i z_j^{-1}; q)_\infty}\,. \tag{4.7}$$

**Orthogonality relation**

$$\frac{1}{n!} \int_T P_\lambda(z)\,\overline{P_\mu(z)}\,\Delta(z)\,dz = \prod_{i<j} \frac{(q^{\lambda_i - \lambda_j} t^{j-i}, q^{\lambda_i - \lambda_j + 1} t^{j-i}; q)_\infty}{(q^{\lambda_i - \lambda_j} t^{j-i+1}, q^{\lambda_i - \lambda_j + 1} t^{j-i-1}; q)_\infty}\,\delta_{\lambda,\mu}. \tag{4.8}$$

**$q$-Difference equation**

$$\sum_{i=1}^n \prod_{j \ne i} \frac{t z_i - z_j}{z_i - z_j}\,\tau_{q,z_i} P_\lambda(z; q, t) = \left(\sum_{i=1}^n q^{\lambda_i} t^{n-i}\right) P_\lambda(z; q, t), \tag{4.9}$$

where $\tau_{q,z_i}$ is the *$q$-shift operator*: $\tau_{q,z_i} f(z_1, \ldots, z_n) := f(z_1, \ldots, q z_i, \ldots, z_n)$. See [20, Ch. VI, §3] for the full system of $q$-difference equations.

**Special value**

$$P_\lambda(1,t,\ldots,t^{n-1};q,t)=\prod_{i=1}^{n}t^{(i-1)\lambda_i}\prod_{i<j}\frac{(tq^{j-i};q)_{\lambda_i-\lambda_j}}{(q^{j-i};q)_{\lambda_i-\lambda_j}}\,. \tag{4.10}$$

**Restriction of number of variables**

$$P_{\lambda_1,\lambda_2,\ldots,\lambda_{n-1},0}(z_1,\ldots,z_{n-1},0;q,t)=P_{\lambda_1,\lambda_2,\ldots,\lambda_{n-1}}(z_1,\ldots,z_{n-1};q,t). \tag{4.11}$$

**Homogeneity**

$$P_{\lambda_1,\ldots,\lambda_n}(z;q,t)=z_1\ldots z_nP_{\lambda_1-1,\ldots,\lambda_n-1}(z;q,t)\qquad(\lambda_n>0). \tag{4.12}$$

**Self-duality** Let $\lambda,\mu$ be partitions.

$$\frac{P_\lambda(q^{\mu_1}t^{n-1},q^{\mu_2}t^{n-2},\ldots,q^{\mu_n};q,t)}{P_\lambda(t^{n-1},t^{n-2},\ldots,1;q,t)}=\frac{P_\mu(q^{\lambda_1}t^{n-1},q^{\lambda_2}t^{n-2},\ldots,q^{\lambda_n};q,t)}{P_\mu(t^{n-1},t^{n-2},\ldots,1;q,t)}. \tag{4.13}$$

**Special cases and limit relations**
*Continuous q-ultraspherical polynomials* (see (3.8)):

$$P_{m,n}(re^{i\theta},re^{-i\theta};q,t)=\frac{(q;q)_{m-n}}{(t;q)_{m-n}}\,r^{m+n}\,C_{m-n}(\cos\theta;t\mid q). \tag{4.14}$$

*Symmetrized monomials* (see (4.4)):

$$P_\lambda(z;q,1)=m_\lambda(z). \tag{4.15}$$

*Schur functions* (see (4.5)):

$$P_\lambda(z;q,q)=s_\lambda(z). \tag{4.16}$$

*Hall-Littlewood polynomials* (see [20, Ch. III]):

$$P_\lambda(z;0,t)=P_\lambda(z;t). \tag{4.17}$$

*Jack polynomials* (see [20, §VI.10]):

$$\lim_{q\uparrow1}P_\lambda(z;q,q^a)=P_\lambda^{(1/a)}(z). \tag{4.18}$$

**Algebraic definition of Macdonald polynomials** Macdonald polynomials can also be defined algebraically. We work now with partitions $\lambda$ ($\lambda_1\ge\lambda_2\ge\cdots\ge0$) of arbitrary length $l(\lambda)$, and with symmetric polynomials in arbitrarily many variables $x_1,x_2,\ldots$, which can be canonically extended to symmetric functions in infinitely many variables $x_1,x_2,\ldots$. The $r$th *power sum* $p_r$ and the symmetric functions $p_\lambda$ are formally defined by

$$p_r=\sum_{i\ge1}x_i^r,\qquad p_\lambda=p_{\lambda_1}\,p_{\lambda_2}\ldots\,. \tag{4.19}$$

Put

$$z_\lambda := \prod_{i\ge1} i^{m_i}\, m_i!\,,\quad \text{where } m_i = m_i(\lambda) \text{ is the number of parts of } \lambda \text{ equal to } i. \tag{4.20}$$

Define an inner product $\langle\ ,\ \rangle_{q,t}$ on the space of symmetric functions such that

$$\langle p_\lambda, p_\mu\rangle_{q,t} = \delta_{\lambda,\mu}\, z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\,. \tag{4.21}$$

For partitions $\lambda,\mu$ the partial ordering $\lambda \ge \mu$ means now that $\sum_{j\ge1}\lambda_j = \sum_{j\ge1}\mu_j$ and $\lambda_1 + \cdots + \lambda_i \ge \mu_1 + \cdots + \mu_i$ for all $i$. The Macdonald polynomial $P_\lambda(x;q,t)$ can now be algebraically defined as the unique symmetric function $P_\lambda$ of the form $P_\lambda = \sum_{\mu\le\lambda} u_{\lambda,\mu} m_\mu$ ($u_{\lambda,\mu}\in\mathbb{C}$, $u_{\lambda,\lambda}=1$) such that

$$\langle P_\lambda, P_\mu\rangle_{q.t} = 0 \qquad \text{if } \lambda\ne\mu. \tag{4.22}$$

If $l(\lambda)\le n$ then the newly defined $P_\lambda(x)$ with $x_{n+1} = x_{n+2} = \ldots = 0$ coincides with $P_\lambda(x;q,t)$ defined analytically, and the new inner product is a constant multiple (depending on $n$) of the old inner product.

**Bilinear sum**

$$\sum_\lambda \frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}}\, P_\lambda(x;q,t) P_\lambda(y;q,t) = \prod_{i,j\ge1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}. \tag{4.23}$$

**Generalized Kostka numbers** The *Kostka numbers* $K_{\lambda,\mu}$ occurring as expansion coefficients in $s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu$ were generalized by Macdonald to coefficients $K_{\lambda,\mu}(q,t)$ occurring in connection with Macdonald polynomials, see [20, §VI.8]. Macdonald's conjecture that $K_{\lambda,\mu}(q,t)$ is a polynomial in $q$ and $t$ with coefficients in $\mathbb{Z}_{\ge0}$ was fully proved in [9].

## 4.4 Macdonald-Koornwinder polynomials

Macdonald [21] also introduced Macdonald polynomials associated with an arbitrary root system. For root system $BC_n$ this yields a three-parameter family which can be extended to the five-parameter *Macdonald-Koornwinder polynomials* (M-K polynomials) [13]. They are orthogonal with respect to the measure occurring in (4.2) with $\Delta(z)$ given by (4.3). The M-K polynomials are $n$-variable analogues of the Askey-Wilson polynomials. All polynomials just discussed tend for $q \uparrow 1$ to *Jacobi polynomials associated with root systems.*

Macdonald conjectured explicit expressions for the quadratic norms of the Macdonald polynomials associated with root systems and of the M-K polynomials. These were proved by Cherednik by considering these polynomials as Weyl group symmetrizations of non-invariant polynomials which are related to double affine Hecke algebras. See Macdonald [22].

## 5 Elliptic hypergeometric series

See Gasper & Rahman [7, Ch. 11] and references given there.
Let $p, q \in \mathbb{C}$, $|p|, |q| < 1$. Define a *modified Jacobi theta function* by

$$\theta(x;p) := (x, p/x; p)_\infty \qquad (x \neq 0), \tag{5.1}$$

and the *elliptic shifted factorial* by

$$(a;q,p)_k := \theta(a;p)\theta(aq;p)\ldots\theta(aq^{k-1};p) \quad (k \in \mathbb{Z}_{>0}); \qquad (a;q,p)_0 := 1, \tag{5.2}$$

$$(a_1,\ldots,a_r;q,p)_k := (a_1;q,p)_k \ldots (a_r;q,p)_k, \tag{5.3}$$

where $a, a_1, \ldots, a_r \neq 0$. For $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$ ($\operatorname{Im}\tau > 0$) and $a \in \mathbb{C}$ we have

$$\frac{\theta(ae^{2\pi i\sigma(x+\sigma^{-1})}; e^{2\pi i\tau})}{\theta(ae^{2\pi i\sigma x}; e^{2\pi i\tau})} = 1, \qquad \frac{\theta(ae^{2\pi i\sigma(x+\tau\sigma^{-1})}; e^{2\pi i\tau})}{\theta(ae^{2\pi i\sigma x}; e^{2\pi i\tau})} = -a^{-1}q^{-x}. \tag{5.4}$$

A series $\sum_{k=0}^\infty c_k$ with $c_{k+1}/c_k$ being an elliptic (i.e. doubly periodic meromorphic) function of $k$ considered as a complex variable, is called an *elliptic hypergeometric series.* In particular, define the ${}_rE_{r-1}$ *theta hypergeometric series* as the formal series

$${}_rE_{r-1}(a_1,\ldots,a_r;b_1,\ldots,b_{r-1};q,p;z) := \sum_{k=0}^\infty \frac{(a_1,\ldots,a_r;q,p)_k}{(b_1,\ldots,b_{r-1};q,p)_k}\,\frac{z^k}{(q;q,p)_k}\,. \tag{5.5}$$

It has $g(k) := c_{k+1}/c_k$ with

$$g(x) = \frac{z\,\theta(a_1q^x;p)\ldots\theta(a_rq^x;p)}{\theta(q^{x+1};p)\,\theta(b_1q^x;p)\ldots\theta(b_{r-1}q^x;p)}\,.$$

By (5.4) $g(x)$ is an elliptic function with periods $\sigma^{-1}$ and $\tau\sigma^{-1}$ ($q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$) if the balancing condition $a_1\ldots a_r = qb_1\ldots b_{r-1}$ is satisfied.

The ${}_rV_{r-1}$ *very-well-poised theta hypergeometric series* (a special ${}_rE_{r-1}$) is defined, in case of argument 1, as:

$${}_rV_{r-1}(a_1;a_6,\ldots,a_r;q,p) := \sum_{k=0}^\infty \frac{\theta(a_1q^{2k};p)}{\theta(a_1;p)}\,\frac{(a_1,a_6,\ldots,a_r;q,p)_k}{(qa_1/a_6,\ldots,qa_1/a_r;q,p)_k}\,\frac{q^k}{(q;q,p)_k}\,. \tag{5.6}$$

The series is called *balanced* if $a_6^2\ldots a_r^2 = a_1^{r-6}q^{r-4}$. The series terminates if, for instance, $a_r = q^{-n}$.

**Elliptic analogue of Jackson's ${}_8W_7$ summation formula** (2.52)

$${}_{10}V_9(a;b,c,d,q^{n+1}a^2/(bcd),q^{-n};q,p) = \frac{(qa,qa/(bc),qa/(bd),qa/(cd);q,p)_n}{(qa/b,qa/c,qa/d,qa/(bcd);q,p)_n}\,. \tag{5.7}$$

**Elliptic analogue of Bailey's ${}_{10}W_9$ transformation formula** (2.56)

$$
{}_{12}V_{11}\left(a;b,c,d,e,f,\frac{q^{n+2}a^3}{bcdef},q^{-n};q,p\right)=\frac{(qa,qa/(ef),(qa)^2/(bcde),(qa)^2/(bcdf);q,p)_n}{(qa/e,qa/f,(qa)^2/(bcdef),(qa)^2/(bcd);q,p)_n}
\times {}_{12}V_{11}\left(\frac{qa^2}{bcd};\frac{qa}{cd},\frac{qa}{bd},\frac{qa}{bc},e,f,\frac{q^{n+2}a^3}{bcdef},q^{-n};q,p\right). \quad (5.8)
$$

Suitable ${}_{12}V_{11}$ functions satisfy a discrete biorthogonality relation which is an elliptic analogue of (3.35).

**Ruijsenaars' elliptic gamma function** (see Ruijsenaars [24])

$$
\Gamma(z;q,p):=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}q^{j+1}p^{k+1}}{1-zq^jp^k}, \quad (5.9)
$$

which is symmetric in $p$ and $q$. Then

$$
\Gamma(qz;q,p)=\theta(z;p)\,\Gamma(z;q,p), \qquad \Gamma(q^nz;q,p)=(z;q,p)_n\,\Gamma(z;q,p). \quad (5.10)
$$

# 6 Applications

## 6.1 Quantum groups

A specific quantum group is usually a Hopf algebra which is a $q$-deformation of the Hopf algebra of functions on a specific Lie group or, dually, of a universal enveloping algebra (viewed as Hopf algebra) of a Lie algebra. The general philosophy is that representations of the Lie group or Lie algebra also deform to representations of the quantum group, and that special functions associated with the representations in the classical case deform to $q$-special functions associated with the representations in the quantum case. Sometimes this is straightforward, but often new subtle phenomena occur.

The representation theoretic objects which may be explicitly written in terms of $q$-special functions include matrix elements of representations with respect to specific bases (in particular spherical elements), Clebsch-Gordan coefficients and Racah coefficients. Many one-variable $q$-hypergeometric functions have found interpretation in some way in connection with a quantum analogue of a three-dimensional Lie group (generically the Lie group $SL(2,\mathbb{C})$ and its real forms). Classical by now are: little $q$-Jacobi polynomials interpreted as matrix elements of irreducible representations of $SU_q(2)$ with respect to the standard basis; Askey-Wilson polynomials similarly interpreted with respect to a certain basis not coming from a quantum subgroup; Jackson's third $q$-Bessel functions as matrix elements of irreducible representations of $E_q(2)$; $q$-Hahn polynomials and $q$-Racah polynomials interpreted as Clebsch-Gordan coefficients and Racah coefficients, respectively, for $SU_q(2)$. See for instance Vilenkin & Klimyk [28] and Koelink [11]

Further developments include: Macdonald polynomials as spherical elements on quantum analogues of compact Riemannian symmetric spaces; $q$-analogues of Jacobi functions as matrix elements of irreducble unitary representations of $SU_q(1,1)$; Askey-Wilson polynomials as matrix elements of representations of the $SU(2)$ dynamical quantum group; an interpretation of discrete ${}_{12}V_{11}$ biorthogonality relations on the elliptic $U(2)$ quantum group (see [12]).

Since the $q$-deformed Hopf algebras are usually presented by generators and relations, identities for $q$-special functions involving non-commuting variables satisfying simple relations are important for further interpretations of $q$-special functions in quantum groups, for instance:

**$q$-Binomial formula with $q$-commuting variables**

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} x^k \qquad (xy = qyx). \tag{6.1}$$

**Functional equations for $q$-exponentials with $xy = qyx$**

$$e_q(x+y) = e_q(y)e_q(x), \qquad E_q(x+y) = E_q(x)E_q(y), \tag{6.2}$$
$$e_q(x+y-yx) = e_q(x)e_q(y), \qquad E_q(x+y+yx) = E_q(y)E_q(x). \tag{6.3}$$

See [15] for further reading.

## 6.2 Various algebraic settings

**Classical groups over finite fields (Chevalley groups)**
$q$-Hahn polynomials and various kinds of $q$-Krawtchouk polynomials have interpretations as spherical and intertwining functions on classical groups ($GL_n$, $SO_n$, $Sp_n$) over a finite field $F_q$ with respect to suitable subgroups, see Stanton [25].

**Affine Kac-Moody algebras** (see [17])
The Rogers-Ramanujan identities (2.57), (2.58) and some of their generalizations were interpreted in the context of characters of representations of the simplest affine Kac-Moody algebra $A_1^{(1)}$. Macdonald's generalization of Weyl's denominator formula to affine root systems (see §4.3) has an interpretation as an identity for the denominator of the character of a representation of an affine Kac-Moody algebra.

## 6.3 Partitions of positve integers

Let $n$ be a positive integer, $p(n)$ the number of partitions of $n$, $p_N(n)$ the number of partitions of $n$ into parts $\le N$, $p_{\text{dist}}(n)$ the number of partitions of $n$ into distinct parts, and $p_{\text{odd}}(n)$ the number of partitions of $n$ into odd parts. Then Euler observed:

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n, \qquad \frac{1}{(q;q)_N} = \sum_{n=0}^{\infty} p_N(n)q^n, \tag{6.4}$$

$$(-q;q)_\infty = \sum_{n=0}^{\infty} p_{\text{dist}}(n)q^n, \qquad \frac{1}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} p_{\text{odd}}(n)q^n, \tag{6.5}$$

and

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}\,, \qquad p_{\text{dist}}(n) = p_{\text{odd}}(n). \tag{6.6}$$

The Rogers-Ramanujan identity (2.57) has the following partition theoretic interpretation: The number of partitions of $n$ with parts differing at least 2 equals the number of partitions of $n$ into parts congruent to 1 or 4 (mod 5). Similarly, (2.58) yields: The number of partitions of $n$ with parts larger than 1 and differing at least 2 equals the number of partitions of $n$ into parts congruent to 2 or 3 (mod 5).

The left-hand sides of the Rogers-Ramanujan identities (2.57) and (2.58) have interpretations in the *Hard Hexagon Model*, see [5]. Much further work has been done on Rogers-Ramanujan type identities in connection with more general models in statistical mechanics. So-called *fermionic expressions* do occur.

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