Comparison of Macdonald’s and Koornwinder’s relations for the double affine Hecke algebra of type \((C_1^\vee, C_1)\)

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Macdonald [2] gives relations (6.4.2), (6.4.6), (6.4.8) for the double affine Hecke algebra of type \((C_1^\vee, C_1)\) with generators \(T_1, T_0, T'_1, T'_0\) and parameters \(\tau_1, \tau_0, \tau'_1, \tau'_0\):

\[
(T_1 - \tau_1)(T_1 + \tau_1^{-1}) = 0, \quad (T_0 - \tau_0)(T_0 + \tau_0^{-1}) = 0,
\]

\[
(T'_1 - \tau'_1)(T'_1 + \tau'_1^{-1}) = 0, \quad (T'_0 - \tau'_0)(T'_0 + \tau'_0^{-1}) = 0,
\]

\[
T'_0 T_0 T'_1 T_1 = q^{-\frac{1}{2}}.
\]

Koornwinder [1] equivalently gives relations (3.1)–(3.4) for the double affine Hecke algebra of type \((C_1^\vee, C_1)\) with generators \(\tilde{T}_1, \tilde{T}_0, \tilde{Z}, \tilde{Z}^{-1}\) and parameters \(a, b, c, d\):

\[
(\tilde{T}_1 + ab)(\tilde{T}_1 + 1) = 0, \quad (\tilde{T}_0 + q^{-1}cd)(\tilde{T}_0 + 1) = 0,
\]

\[
(\tilde{T}_1 \tilde{Z} + a)(\tilde{T}_1 \tilde{Z} + b) = 0, \quad (q\tilde{T}_0 \tilde{Z}^{-1} + c)(q\tilde{T}_0 \tilde{Z}^{-1} + d) = 0.
\]

We can go from Macdonald’s relations to Koornwinder’s relations by putting

\[
T_1 = \tau_1^{-1}\tilde{T}_1, \quad T_0 = \tau_0^{-1}\tilde{T}_0, \quad T'_1 = \tau_1 \tilde{Z}^{-1}\tilde{T}_1^{-1}, \quad T'_0 = q^{-\frac{1}{2}}\tau_0 \tilde{Z}\tilde{T}_0^{-1},
\]

\[
\tau_1 = i(ab)^{\frac{1}{2}}, \quad \tau_0 = i(cd/q)^{\frac{1}{2}}, \quad \tau'_1 = -i(b/a)^{\frac{1}{2}}, \quad \tau'_0 = -i(d/c)^{\frac{1}{2}}.
\]

Conversely we can go from Koornwinder’s relations to Macdonald’s relations by putting

\[
a = -\tau_1/\tau'_1, \quad b = \tau_1 \tau'_1, \quad c = -q^{\frac{1}{2}}\tau_0/\tau'_0, \quad d = q^{\frac{1}{2}}\tau_0 \tau'_0,
\]

\[
\tilde{T}_1 = \tau_1 T_1, \quad \tilde{T}_0 = \tau_0 T_0, \quad \tilde{Z} = q^{\frac{1}{2}}T'_0 T_0 = T_1^{-1}T'_1^{-1}.
\]

References
