Errata and comments for the the Lecture Notes
Orthogonal polynomials and special functions by R. Askey

collected by Tom Koornwinder, T.H.Koornwinder@uva.nl

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These are errata and comments for the book

R. Askey, *Special Functions and orthogonal polynomials*, Regional Conference Series in Applied Mathematics 21, SIAM, 1975.

p.1, (1.4): On the right replace the summation sign by a product sign.

p.3, line above (1.14): Replace "equivalent to" by "implied by".

p.7, l.-1: We can rewrite (2.2) as

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(n+\alpha+\beta+1)_k\ (\alpha+k+1)_{n-k}}{k!\,(n-k)!} \left(\frac{x-1}{2}\right)^k,$$

which makes sense for $\alpha,\beta \in \mathbb{C}$.

p.8, (2.16): The argument of the ${}_2F_1$ should be $\frac{1+x}{2}$ instead of $\frac{1-x}{2}$.

p.14, l.11,12: The argument in Askey and Gasper [4, pp. 66–67] that the generalized translation operator for Hahn polynomials $Q_n(x;\alpha,\beta,N)$ is not a positive operator is only given for $\alpha,\beta > -1$. As shown in

C. F. Dunkl, *Spherical functions on compact groups and applications to special functions*, Symposia Mathematica 22, Academic Press, 1977, 145–161 ,

and

M. Rahman, *A positive kernel for Hahn-Eberlein polynomials*, SIAM J. Math. Anal. 9 (1978), 891–905,

there are many instances for $\alpha,\beta < -N$ where this operator is nonnegative.

p.15, (2.42a): On the right replace a^n by a^{-n}.

p.16, (2.47): This formula was first obtained in

J. Meixner, *Erzeugende Funktionen der Charlierschen Polynome*,
Math. Z. 44 (1938), 531–535.

p.20, (3.6): Write $R_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. For positive integers p,q,r with $q>r$ and $\alpha = \frac12 p-1$, $\beta=\frac12 q-1$, $\mu = \frac12 r$ formula (3.6) also follows from the characterization of

$$(x_1^2+\cdots+x_{q+p}^2)^n\, R_n^{(\frac12 p-1,\frac12 q-1)}\left(\frac{(x_1^2+\cdots+x_q^2)-(x_{q+1}^2+\cdots+x_{q+p}^2)}{x_1^2+\cdots+x_{q+p}^2}\right)$$

as an $O(q)\times O(p)$-invariant spherical harmonics of degree $2n$ on $\mathbb{R}^{q+p}$ which is equal to 1 at $(1,0,\ldots,0)$. Then (3.6) means symmetrization of this expression with respect to $O(q-r)\times O(p+r)$:

$$(x_1^2+\cdots+x_{q+p}^2)^n\,R_n^{(\frac12(p+r)-1,\frac12(q-r)-1)}\left(\frac{(x_1^2+\cdots+x_{q-r}^2)-(x_{q-r+1}^2+\cdots+x_{q+p}^2)}{x_1^2+\cdots+x_{q+p}^2}\right)$$
$$=\int_0^{\pi/2}(x_1^2+\cdots+x_{q+p}^2)^n R_n^{(\frac12 p-1,\frac12 q-1)}\left(\frac{(x_1^2+\cdots+x_{q-r}^2)-\cos(2\phi)(x_{q-r+1}^2+\cdots+x_{q+p}^2)}{x_1^2+\cdots+x_{q+p}^2}\right)$$
$$\times(\sin\phi)^{r-1}(\cos\phi)^{p-1}\,d\phi\Big/\int_0^{\pi/2}(\sin\phi)^{r-1}(\cos\phi)^{p-1}\,d\phi.$$

Formula (3.7) can be interpreted in a similar way by symmetrization with respect to $O(q+r)\times O(p-r)$.

p.22, line after (3.23): Replace (3.22) by (3.19). The formal limiting case can be seen by rewriting (3.19) as

$$\frac{1}{m}\sum_{t\in m^{-1}\mathbb{Z}_{\ge0}}\frac{\Gamma(\alpha+\mu+mt)}{\Gamma(\alpha+1+mt)(mt)^{\mu-1}}\,e^{-yt}\,\frac{1}{(mt)^\alpha}\,P_{mt}^{(\alpha,\mu-1)}\left(1-\frac{(xt)^2}{2(mt)^2}\right)\left(\frac{xt}{2}\right)^\alpha t^{\mu-1}$$
$$=\left(\frac{m^{-1}y}{1-e^{-m^{-1}y}}\right)^{\alpha+\mu}\frac{(\frac12x)^\alpha}{y^{\alpha+\mu}}\,{}_2F_1\left(\begin{matrix}\frac12(\alpha+\mu),\frac12(\alpha+\mu+1)\\ \alpha+1\end{matrix};-\frac{x^2}{y^2}\,e^{-y/m}\left(\frac{m^{-1}y}{1-e^{-m^{-1}y}}\right)^2\right).$$

Then let $m\to\infty$ and use (3.24) in the form

$$\lim_{n\to\infty}n^{-\alpha}\,P_n^{(\alpha,\beta)}\left(1-\frac{x^2}{2n^2}\right)\left(\frac{x}{2}\right)^\alpha=J_\alpha(x).$$

The generating function (3.20) is the special case $\beta=-1$ of (3.17) by DLMF, (15.4.17). In a similar way as above we see that this has limiting case (3.20), which is the special case $\mu=0$ of (3.19), again by DLMF, (15.4.17).

A common special case of (3.17) and (3.18) for $\beta=0$ is

$$\sum_{n=0}^\infty P_n^{(\alpha,0)}(x)r^n=2^\alpha R^{-1}(1-r+R)^{-\alpha}.$$

In a similar way as above we see that this has limiting case (3.21), which is the special case $\mu=1$ of (3.19) by DLMF, (15.4.18).

In a similar way as above we see that (3.16), i.e., the special case $\beta=\alpha$ of (3.17), has limiting case (3.23), i.e., the special case $\mu=\alpha+1$ of (3.19).

In a similar way as above we see that the special case $\beta=\alpha+1$ of (3.17) has limiting case (3.24), i.e., the special case $\mu=\alpha+2$ of (3.19).

Since (3.20)–(3.23) all correspond to specializations of μ in (3.19), i.e., specializations of β in (3.17), there is no need to find other generating functions of Jacobi polynomials for general α,β having (3.20)–(3.23) as limiting cases.

p.24, (3.30): As for Vilenkin [1], MR0095986 quotes formula (3.30) and mentions a connection with a group theoretic interpretation. The formula also occurs in Vilenkin [2, Ch. 9, 4.11(5)], where it is derived from the interpretation of Gegenbauer polynomials $C_n^{\frac12 p-1}$ as zonal spherical harmonics on S^{p-1}. By using this interpretation, a proof of (3.30) even simpler than in Vilenkin [2] can be given for $q>p\ge 3$ integers and $\nu=\frac12 q-1$, $\lambda=\frac12 p-1$ (the general case then probably follows by analytic continuation). Indeed,

$$\begin{aligned}&\big(x_1^2+\cdots+x_p^2\big)^{\frac12 n}\,C_n^{\frac12 q-1}\big(x_1/(x_1^2+\cdots+x_p^2)^{\frac12}\big)/C_n^{\frac12 q-1}(1)\\&=\int_0^{\pi/2}\big(x_1^2+\sin^2\phi(x_2^2+\cdots+x_p^2)\big)^{\frac12 n}\,C_n^{\frac12 p-1}\big(x_1/(x_1^2+\sin^2\phi(x_2^2+\cdots+x_p^2))^{\frac12}\big)/C_n^{\frac12 p-1}(1)\\&\qquad\times(\sin\phi)^{p-2}(\cos\phi)^{q-p-1}\,d\phi\bigg/\int_0^{\pi/2}(\sin\phi)^{p-2}(\cos\phi)^{q-p-1}\,d\phi,\end{aligned}$$

because $\big(x_1^2+\cdots+x_p^2\big)^{\frac12 n}\,C_n^{\frac12 p-1}\big(x_1/(x_1^2+\cdots+x_p^2)^{\frac12}\big)$ is a homogeneous harmonic polynomial of degree n on $\mathbb{R}^p$, and therefore also on $\mathbb{R}^q$ if considered as a polynomial in $x_1,\ldots,x_p,x_{p+1},\ldots,x_q$. Then we symmetrize this with respect to the group $SO(q-1)$, i.e., the group of rotations of $\mathbb{R}^q$ which leave $(1,0,\ldots,0)$ fixed.

p.27, l.1: Replace "Theorem 3.3" by "Theorem 3.4".

p.37, 6th line after (4.43): Replace Boursma by Boersma.

p.42, l.-11: This is a limit case of (5.7).

p.45, l. 4–6: On l.4 multiply $p_m(x)p_{n-1}(x)-p_{m-1}(x)p_n(x)$ by a coefficient b_n.
On l.5 multiply $p_1(x)p_{n-m}(x)-p_{n-m+1}(x)$ by a coefficient $b_n\ldots b_{n-m+1}$.
On l.6 multiply $a_{n-m}p_{n-m}(x)+b_{n-m}p_{n-m-1}(x)$ by a coefficient $b_n\ldots b_{n-m+1}$.

p.63, (7.34): In the fraction on the right, before the summation sign, replace the denominator by $(2\gamma+1)_n$.

p.66, Theorem 7.1, l.6: Replace (7.19) by (7.23).

p.76–77: These formulas and positivity results also occur in Askey & Gasper [5], formula (1.16), and in

G. Gasper, *Positivity and special functions*, in: *Theory and application of special functions*, Academic Press, 1975, pp. 375–433, formula (8.12).

Gasper also observes a limit case (see (8.14) in Gasper's paper) for Bessel functions of this positivity result. Indeed, the positivity result implies the positivity (for $\alpha>-1$) of

$$m^{-1}\sum_{t\in m^{-1}\{0,1,\ldots,[mx]\}}P_{mt}^{(\alpha,0)}\left(1-\frac{t^2}{2(mt)^2}\right)(mt)^{-\alpha}(t/2)^{\alpha},$$

which formally tends, as $m \to \infty$, to

$$\int_0^x J_\alpha(t)\,dt.$$

The positivity of the right-hand side of (8.21), which is implied by the positivity of the right-hand side of (8.28) plays a crucial role in the proof of the Bieberbach conjecture, see p.150 in

L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. 154 (1985), 137–152.

p.109, Askey and Gasper [4]: This has appeared in J. Analyse Math. 31 (1977), 48–68.

p.109, Askey and Gasper [5]: This has appeared in Amer. J. Math. 98 (1976), 709–737.