About the paper *On the zeros of a certain class of polynomials and related analytic functions* by A. Aziz and Q. G. Mohammad

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This is a comment to the paper

Their Theorem 5 states:

Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0 \) be analytic in \( |z| \leq t \). If \( a_j > 0 \) and \( a_{j-1} - t a_j \geq 0 \), \( j = 1, 2, 3, \ldots \),

then \( f(z) \) does not vanish in \( |z| < t \).

Their proof, using Schwarz’s lemma, is extremely short and simple.

Observe that, without loss of generality, we may assume \( t = 1 \) and \( a_0 = 1 \). This is just a matter of rescaling. Moreover, we do not need to assume that \( f(z) \) is analytic up to the boundary. It suffices to assume that it is analytic on the open disc. This is seen by using the version of Schwarz’s lemma which states:

Let \( g(z) \) be analytic for \( |z| < 1 \), let \( |g(z)| \leq 1 \) for \( |z| < 1 \), and let \( g(0) = 0 \). Then \( |g(z)| \leq |z| \) for \( |z| < 1 \).

Furthermore, Schwarz’s lemma becomes trivial in the following case:
If \( g(z) := \sum_{j=1}^{\infty} c_j z^j \) with \( \sum_{j=1}^{\infty} |c_j| \leq 1 \) then \( |g(z)| \leq \sum_{j=1}^{\infty} |c_j| |z| \leq |z| \) for \( |z| < 1 \).

An example, where \( g(z) \) is continuous on the closed unit disk, but where the convergence of \( \sum_{j=1}^{\infty} c_j z^j \) is non-uniform on every arc of \( |z| = 1 \) (and where thus Schwarz’s lemma cannot be proved in the trivial way), is given in Ch. VIII, Theorem (1.17) of the book

This example goes back to a paper by Fejér (1917).

Now the following version of Theorem 5 can be formulated.

**Theorem**

Let \( 1 = a_0 \geq a_1 \geq a_2 \geq \ldots \geq 0 \). Then

\[
\sum_{j=0}^{\infty} a_j z^j \neq 0 \quad \text{for } |z| < 1.
\]
Proof We have
\[(1 - z)f(z) = 1 - g(z) \quad \text{with} \quad g(z) := \sum_{j=1}^{\infty} (a_{j-1} - a_j)z^j.\]

Then \(a_{j-1} - a_j \geq 0\) and \(\sum_{j=1}^{\infty} (a_{j-1} - a_j) = 1 - \lim_{j \to \infty} a_j \leq 1\). Hence, by the trivial case of Schwarz’s lemma, \(|g(z)| \leq |z|\) for \(|z| < 1\). So
\[|(1 - z)f(z)| \geq 1 - |g(z)| \geq 1 - |z| > 0 \quad \text{for} \quad |z| < 1. \]

Of course, the Theorem remains true if the power series for \(f(z)\) is terminating, by which we have also a short proof of the Eneström-Kakeya theorem.

It is surprising that the elegant and potentially useful Theorem 5, with its simple proof, did not make its way to the textbooks.

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