On the paper "Nonlinear integral-equation formulation of orthogonal polynomials" by C. M. Bender and E. Ben-Naim"

Informal note by Tom H. Koornwinder, thk@science.uva.nl

November 9, 2007

In [1, (1)] an interesting characterization of orthogonal polynomials is given. Here is an equivalent formulation of this result.

Let $\{P_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials with respect to the measure μ on $[0,\infty)$. Then

$$\int_0^\infty \frac{P_n(x+y)-P_n(x)}{y}\,P_n(y)\,d\mu(y)=0. \tag{1}$$

This is evident because $y^{-1}(P_n(x+y)-P_n(x))$ is a polynomial of degree $<n$ in y.

Conversely, if P_n is a n-th degree polynomial which satisfies (1) then P_n is the n-th degree orthogonal polynomial (determined up to a nonzero constant factor) for the orthogonality measure μ. Indeed, Taylor series expansion in (1) gives

$$\sum_{k=1}^{n}\frac{P_n^{(k)}(x)}{k!}\int_0^\infty P_n(y)y^{k-1}\,d\mu(y)=0.$$

Then we see for successive $k=1,2,\ldots,n$ that the integral in the above formula must be zero because the coefficient of x^{n-k} on the left-hand side must be zero.

If $\int_0^\infty y^{-1}\,d\mu(y)<\infty$ then (1) can be rewritten as

$$\int_0^\infty P_n(x+y)P_n(y)\,y^{-1}\,d\mu(y)=P_n(x)\int_0^\infty P_n(y)\,y^{-1}\,d\mu(y). \tag{2}$$

If $\int_0^\infty P_n(y)\,y^{-1}\,d\mu(y)\neq 0$ then we can renormalize P_n such that $\int_0^\infty P_n(y)\,y^{-1}\,d\mu(y)=1$. Compare this with [1, (1)].

References

[1] C. M. Bender and E. Ben-Naim, *Nonlinear integral-equation formulation of orthogonal polynomials*, J. Phys. A 40 (2007), F9-F15.