In [1, (1)] an interesting characterization of orthogonal polynomials is given. Here is an equivalent formulation of this result.

Let \( \{P_n\}_{n=0}^{\infty} \) be a system of orthogonal polynomials with respect to the measure \( \mu \) on \([0, \infty)\).

Then
\[
\int_0^{\infty} \frac{P_n(x + y) - P_n(x)}{y} P_n(y) \, d\mu(y) = 0.
\] (1)

This is evident because \( y^{-1}(P_n(x + y) - P_n(x)) \) is a polynomial of degree \( < n \) in \( y \).

Conversely, if \( P_n \) is a \( n \)-th degree polynomial which satisfies (1) then \( P_n \) is the \( n \)-th degree orthogonal polynomial (determined up to a nonzero constant factor) for the orthogonality measure \( \mu \). Indeed, Taylor series expansion in (1) gives
\[
\sum_{k=1}^{n} \frac{P_n^{(k)}(x)}{k!} \int_0^{\infty} P_n(y) y^{k-1} \, d\mu(y) = 0.
\]

Then we see for successive \( k = 1, 2, \ldots, n \) that the integral in the above formula must be zero because the coefficient of \( x^{n-k} \) on the left-hand side must be zero.

If \( \int_0^{\infty} y^{-1} \, d\mu(y) < \infty \) then (1) can be rewritten as
\[
\int_0^{\infty} P_n(x + y) P_n(y) y^{-1} \, d\mu(y) = P_n(x) \int_0^{\infty} P_n(y) y^{-1} \, d\mu(y).
\] (2)

If \( \int_0^{\infty} P_n(y) y^{-1} \, d\mu(y) \neq 0 \) then we can renormalize \( P_n \) such that \( \int_0^{\infty} P_n(y) y^{-1} \, d\mu(y) = 1 \). Compare this with [1, (1)].

References