Comment on the paper “A remarkable identity involving Bessel functions” by D. E. Dominici, P. M. W. Gill and T. Limpanuparb, arXiv:1103.0058v1 [math.CA]

Note by Tom H. Koornwinder, T.H.Koornwinder@uva.nl, March 11, 2011

A more conceptual proof of Corollary 1 is obtained by observing that for \( f, g \in L^2([-\pi, \pi]) \) we have

\[
2\pi \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx = \int_{-\infty}^{\infty} \hat{f}(y) \overline{\hat{g}(y)} \, dy = \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)},
\]

where

\[
\hat{f}(y) := \int_{-\pi}^{\pi} f(x) e^{-ixy} \, dx.
\]

Apply this to

\[
f(x) := (1 - x^2/a^2)^{\mu-k-\frac{1}{2}} C_k^{\mu-k}(x/a) \quad (-a < x < a),
\]

\[
g(x) := (1 - x^2/b^2)^{\nu-\ell-\frac{1}{2}} C_\ell^{\nu-\ell}(x/b) \quad (-b < x < b),
\]

and \( f(x) := 0 \) outside \((-a, a), g(x) := 0 \) outside \((-b, b)\). Assume that \( a, b \in (0, \pi] \) and that the nonnegative integers \( k, \ell \) satisfy \( k < \Re \mu \) and \( \ell < \Re \nu \). Then

\[
\int_{-\infty}^{\infty} t^{k+\ell} \, _0F_1\left(\begin{array}{c} - \\ \mu + 1; -\frac{1}{4}a^2t^2 \end{array}\right) _0F_1\left(\begin{array}{c} - \\ \nu + 1; -\frac{1}{4}b^2t^2 \end{array}\right) \, dt
= \sum_{n=-\infty}^{\infty} n^{k+\ell} \, _0F_1\left(\begin{array}{c} - \\ \mu + 1; -\frac{1}{4}a^2n^2 \end{array}\right) _0F_1\left(\begin{array}{c} - \\ \nu + 1; -\frac{1}{4}b^2n^2 \end{array}\right).
\]

By analytic continuation this remains valid and convergent for \( k + \ell < \Re \mu + \Re \nu \). For \( \mu + \nu = k + \ell + 1 \) we obtain the first equality in Corollary 2. Note that we need for this special case that \( \mu + \nu \) is integer.