1. **Left $A$-module algebra and left $A$-module coalgebra (p.109)**

Let $A$ be a bialgebra over over a commutative ring $k$ with identity element. So $\lambda_V: A \to \text{End}_k(V)$ is an algebra homomorphism and we write $a.v := \lambda_V(a)v$ ($a \in A$, $v \in V$). If $W$ is another left $A$-module then $V \otimes W$ becomes a left $A$-module with

$$
\lambda_{V \otimes W}(a) := (\lambda_V \otimes \lambda_W)(\Delta(a)), \quad \text{so} \quad a.(v \otimes w) = \sum_{(a)} (a_{(1)}.v) \otimes (a_{(2)}.w).
$$

Also, $k$ becomes a left $A$-module (the trivial $A$-module) by

$$
\lambda_k(a) := \varepsilon(a), \quad \text{so} \quad a.\alpha := \varepsilon(a)\alpha.
$$

I will give the definitions of left $A$-module algebra and left $A$-module coalgebra.

**1.1. Left $A$-module algebra.** $V$ is moreover an algebra such that the following two diagrams commute for each $a \in A$:

$$
\begin{array}{ccc}
V \otimes V & \xrightarrow{\lambda_{V \otimes V}(a)} & V \otimes V \\
\downarrow \mu_V & & \downarrow \mu_V \\
V & \xrightarrow{\lambda_V(a)} & V \\
\end{array}
\quad
\begin{array}{ccc}
k & \xrightarrow{\varepsilon(a)} & k \\
\downarrow \iota_V & & \downarrow \iota_V \\
V & \xrightarrow{\lambda_V(a)} & V \\
\end{array}
$$

Equivalently, $\mu_V: V \otimes V \to V$ and $\iota_V: k \to V$ are are left $A$-module homomorphisms. Another equivalent way to write this is by:

$$
a.(vw) = \sum_{(a)} (a_{(1)}.v) (a_{(2)}.w), \quad a.1_V = \varepsilon_A(a) 1_V.
$$

**1.2. Left $A$-module coalgebra.** $V$ is moreover a coalgebra such that the following two diagrams commute for each $a \in A$:

$$
\begin{array}{ccc}
V & \xrightarrow{\lambda_V(a)} & V \\
\downarrow \Delta_V & & \downarrow \Delta_V \\
V \otimes V & \xrightarrow{\lambda_{V \otimes V}(a)} & V \otimes V \\
\end{array}
\quad
\begin{array}{ccc}
k & \xrightarrow{\varepsilon(a)} & k \\
\downarrow \varepsilon_V & & \downarrow \varepsilon_V \\
V \otimes V & \xrightarrow{\lambda_V(a)} & V \otimes V \\
\end{array}
$$

Equivalently, $\Delta_V: V \to V \otimes V$ and $\varepsilon_V: V \to k$ are left $A$-module homomorphisms. Another equivalent way to write this is by:

$$
\Delta_V(a.v) = a.\Delta_V(v), \quad \varepsilon_V(a.v) = \varepsilon_A(a) \varepsilon_V(v).
$$
2. Quantum trace and quantum character (p.126)
Assume now that $A$ is a Hopf algebra over a field $k$. Let all left $A$-modules under consideration be finite dimensional. Let $V$ be a left $A$-module and write $V^* := \text{Hom}_k(V, k)$. This gives a pairing between $V^*$ and $V$:

$$\langle \xi, v \rangle := \xi(v) \quad (v \in V, \xi \in V^*).$$

Then $V^*$ becomes a left $A$-module by

$$\lambda_{V^*}(a) := (\lambda_V(S(a)))^*, \quad \text{so} \quad \langle a.\xi, v \rangle = \langle \xi, S(a).v \rangle.$$ 

$V$ and $V^{**}$ can be naturally identified as $k$-modules. However, for the left $A$-module structures we have

$$\langle \xi, \lambda_{V^{**}}(a) v \rangle = \langle \lambda_V(S(a)) \xi, v \rangle = \langle \xi, \lambda_V(S^2(a)) v \rangle.$$ 

So

$$\lambda_{V^{**}}(a) = \lambda_V(S^2(a)).$$

From now on we assume that there is an invertible element $u \in A$ such that $S^2(a) = uau^{-1}$. By Proposition 4.2.3 the element $u := \mu(S \otimes \text{id})(R_{21})$ satisfies this property if $A$ is an almost cocommutative Hopf algebra as in Definition 4.2.1.

Now it follows that the following diagram is commutative for each $a \in A$:

$$\begin{array}{ccc}
V & \xrightarrow{\lambda_V(a)} & V \\
\downarrow{\lambda_V(u)} & & \downarrow{\lambda_V(u)} \\
V^{**} & \xrightarrow{\lambda_{V^{**}}(a)} & V^{**}
\end{array}$$

Identify $W \otimes V^*$ and $\text{Hom}_k(V, W)$ as $k$-modules such that $w \otimes \xi \in W \otimes V^*$ corresponds with $\langle \xi, . \rangle w \in \text{Hom}_k(V, W)$. The $k$-module structure of $W \otimes V^*$ is carried by this identification to $\text{Hom}_k(V, W)$. We obtain

$$(\lambda_{W \otimes V^*}(a) f)(v) = (a.f)(v) = \sum_{(a)} a(1). f(S(a(2)).v) \quad (f \in \text{Hom}_k(V, W), \ a \in A, \ v \in V).$$

The adjoint representation of $A$ on $A$, denoted by $\text{ad}$, is defined by

$$\text{ad}(a) b := \sum_{(a)} a(1) b S(a(2)) \quad (a, b \in A).$$

Now the following diagram commutes for each $a \in A$:

$$\begin{array}{ccc}
A & \xrightarrow{\text{ad}(a)} & A \\
\downarrow{\lambda_V} & & \downarrow{\lambda_V} \\
\text{End}_k(V) & \xrightarrow{\lambda_{V \otimes V^*}(a)} & \text{End}_k(V)
\end{array}$$
So \( \lambda_V : A \to \text{End}_k(V) \) is an intertwining operator for the representations \( \text{ad} \) on \( A \) and \( \lambda_{V \otimes V^*} \) on \( \text{End}_k(V) \). For the proof note that
\[
(a, \lambda_V(b))(v) = \sum_{(a)} a_{(1)} \cdot \lambda_V(b)(S(a_{(2)}).v) = \sum_{(a)} \lambda_V(a_{(1)}bS(a_{(2)})) v = \lambda_V(\text{ad}(a).b) v.
\]

The mapping \( \text{tr} : \xi \otimes v \mapsto \langle \xi, v \rangle : V^* \otimes V \to k \) is a homomorphism of left \( A \)-modules:
\[
\text{tr}(a.(\xi \otimes v)) = \varepsilon(a) \langle \xi, v \rangle = \varepsilon(a) \text{tr}(\xi \otimes v).
\]

However, the mapping \( \text{tr} : v \otimes \xi \mapsto \langle \xi, v \rangle : V \otimes V^* \to k \) is generally not a homomorphism of left \( A \)-modules. Under the identification of \( V \otimes V^* \) and \( \text{End}_k(V) \) the mapping \( \text{tr} : V^* \otimes V \to k \) is carried to the usual trace mapping from \( \text{End}_k(V) \) to \( k \), but this mapping will neither be a homomorphism of left \( A \)-modules in general.

As \( k \)-modules we can identify \( V \otimes V^* \) and \( V^{**} \otimes V^* \), but they are generally different as left \( A \)-modules. We have
\[
\lambda_{V^{**} \otimes V^*}(a)(v \otimes \xi) = \sum_{(a)} (\lambda^{2}(a_{(1)}).v) \otimes (a_{(2)}, \xi).
\]

Hence
\[
\text{tr} \left( \lambda_{V^{**} \otimes V^*}(a)(v \otimes \xi) \right) = \sum_{(a)} (a_{(2)}, \xi, \lambda^{2}(a_{(1)}).v) = \sum_{(a)} \langle \xi, \lambda_V(S(a_{(2)}) \lambda^{2}(a_{(1)})) v \rangle
\]
\[
= \sum_{(a)} \langle \xi, \lambda_V(S(a_{(1)}) a_{(2))} v \rangle = \langle \xi, \lambda_V(\varepsilon(a)1) v \rangle = \varepsilon(a) \langle \xi, v \rangle = \varepsilon(a) \text{tr}(v \otimes \xi).
\]

We conclude that the mapping \( \text{tr} : V^{**} \otimes V^* \to k \) is a homomorphism of left \( A \)-modules.

Now identify \( \text{End}_k(V) \) and \( V^{**} \otimes V^* \) as \( k \)-modules. Carrying the left \( A \)-module structure of \( V^{**} \otimes V^* \) to \( \text{End}(V) \) yields
\[
(\lambda_{V^{**} \otimes V^*}(a)f)(v) = \sum_{(a)} \lambda^{2}(a_{(1)}).f(S(a_{(2)}).v) \quad (f \in \text{End}_k(V), a \in A, v \in V).
\]

Then \( \text{tr} : \text{End}_k(V) \to k \) intertwines the representations \( \lambda_{V^{**} \otimes V^*} \) on \( \text{End}_k(V) \) and \( \varepsilon \) on \( k \).

So the following diagram is commutative for each \( a \in A \):
\[
\begin{array}{ccc}
\text{End}_k(V) & \xrightarrow{\lambda_{V^{**} \otimes V^*}(a)} & \text{End}_k(V) \\
\text{tr} \downarrow & & \text{tr} \downarrow \\
k & \xrightarrow{\varepsilon(a)} & k
\end{array}
\quad (2.2)
\]

Let \( u \in A \) be as before. Then
\[
\lambda_V(u)(\lambda_{V \otimes V^*}(a)f)(v) = \sum_{(a)} u.a_{(1)}.f(S(a_{(2)}).v) = \sum_{(a)} \lambda^{2}(a_{(1)}).u.f(S(a_{(2)}), v)
\]
\[
= \lambda_{V^{**} \otimes V^*}(a)(\lambda_V(u)f(v)).
\]
Hence the following diagram commutes for each \( a \in A \):

\[
\begin{array}{ccc}
\text{End}_k(V) & \xrightarrow{\lambda_T \otimes V^*(a)} & \text{End}_k(V) \\
\downarrow{\lambda_T(u)} & & \downarrow{\lambda_T(u)} \\
\text{End}_k(V) & \xrightarrow{\lambda_T \otimes V^*(a)} & \text{End}_k(V)
\end{array}
\] (2.3)

Here \( \lambda_T(u) \) means left multiplication by \( \lambda_T(u) \) in \( \text{End}_k(V) \).

Combination of the diagrams (2.1), (2.3) and (2.2) yields the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{ad}(a)} & A \\
\downarrow{\lambda_T} & & \downarrow{\lambda_T} \\
\text{End}_k(V) & \xrightarrow{\lambda_T \otimes V^*(a)} & \text{End}_k(V) \\
\downarrow{\lambda_T(u)} & & \downarrow{\lambda_T(u)} \\
\text{End}_k(V) & \xrightarrow{\lambda_T \otimes V^*(a)} & \text{End}_k(V) \\
\downarrow{\text{tr}} & & \downarrow{\text{tr}} \\
k & \xrightarrow{\varepsilon(a)} & k
\end{array}
\]

Define the quantum trace, quantum character and quantum dimension by

\[
\begin{aligned}
qtr_V(f) &:= \text{tr}(\lambda_T(u) f) \quad (f \in \text{End}_k(V)), \\
qch_V(b) &:= qtr_V(\lambda_T(b)) = \text{tr}(\lambda_T(ub)) \quad (b \in A), \\
qdim(V) &:= qch_V(1) = qtr_V(I_V) = \text{tr}(\lambda_T(u)).
\end{aligned}
\]

Then the following two diagrams commute for each \( a \in A \):

\[
\begin{array}{ccc}
\text{End}_k(V) & \xrightarrow{\lambda_T \otimes V^*(a)} & \text{End}_k(V) \\
\downarrow{qtr_V} & & \downarrow{qtr_V} \\
k & \xrightarrow{\varepsilon(a)} & k
\end{array}
\]

In particular, we have

\[
qch_V(\text{ad}(a) b) = \varepsilon(a) qch_V(b) \quad (a, b \in A).
\]

Now assume that the element \( u \) satisfies moreover:

\[
\Delta(u) = u \otimes u.
\]

Then:

\[
\begin{aligned}
\lambda_T \otimes W(u) &= \lambda_T(u) \otimes \lambda_W(u), \\
qtr_V \otimes W(f \otimes g) &= qtr_V(f) qtr_W(g), \\
qch_V \otimes W(b) &= qch_V(b) qch_W(b), \\
qdim(V \otimes W) &= qdim(V) qdim(W).
\end{aligned}
\]
So the quantum trace, quantum character and quantum dimension then have properties quite similar to their classical analogues.

Let $A$ be quasitriangular and take $u := \mu(S \otimes \text{id})(R_{21})$. If $A$ is triangular then $\Delta(u) = u \otimes u$. Otherwise, $A$ can be enlarged with a certain central element $v$ such that $v^2 = u S(u)$ by which $v^{-1}u$ will have the required properties (cf. §4.2C).

If $A$ is the Hopf $*$-algebra for a Woronowicz compact matrix group (or more generally a Dijkhuizen-Koornwinder CQG-algebra) then there is an invertible element $u$ in the dual $A^\circ$ of $A$ such that, for each irreducible unitary corepresentation $\lambda_V$ of $A$, the operator $\lambda_V(u)$ intertwines $\lambda_V$ and $\lambda_V$ and satisfies $\text{tr}(\lambda_V(u)) = \text{tr}(\lambda_V(u^{-1})) > 0$. This element $u$ will have the required properties in $A^\circ$. See §2.4 in the following reference:


3. The inversion map on a Poisson-Lie group is an anti-Poisson map (p.21)

(personal communication by A. Pressley to T. H. Koornwinder)

In the Warning on p.21 it is stated that the inversion map $\iota$ on a Poisson-Lie group $G$ satisfies

$$\{f_1 \circ \iota, f_2 \circ \iota\} = -\{f_1, f_2\} \circ \iota$$

for all $f_1, f_2 \in C^\infty(G)$. Here follows a proof. We have

$$\{f_1 \circ \iota, f_2 \circ \iota\}(g) = \langle w_g, d(f_1 \circ \iota)_g \otimes d(f_2 \circ \iota)_g \rangle$$

$$= \langle (i'_g \otimes i'_g)(w_g), (df_1)_{g^{-1}} \otimes (df_2)_{g^{-1}} \rangle.$$

Differentiating the identity

$$\iota = L_{g^{-1}} \circ \iota \circ R_{g^{-1}}$$

at $g$, and noting that $i'_e = -\text{id}$, gives

$$i'_g = -(L_{g^{-1}})'_e(R_{g^{-1}})'_g$$

$$= -[(L_g)'_{g^{-1}}]^{-1}(R_{g^{-1}})'_g,$$

where the last equation was obtained by differentiating the identity $L_{g^{-1}} \circ L_g = \text{id}$ at $g^{-1}$. So

$$(i'_g \otimes i'_g)(w_g) = \left([(L_g)'_{g^{-1}}]^{-1}(R_{g^{-1}})'_g \otimes [(L_g)'_{g^{-1}}]^{-1}(R_{g^{-1}})'_g \right)(w_g),$$

which, by taking $g' = g^{-1}$ in formula (8) on p.22, we see is exactly $-w_{g^{-1}}$. Thus,

$$\{f_1 \circ \iota, f_2 \circ \iota\}(g) = -\langle (df_1)_{g^{-1}} \otimes (df_2)_{g^{-1}}, w_{g^{-1}} \rangle = -\{f_1, f_2\}(g^{-1}).$$