A hypergeometric evaluation connected with the birthday problem

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By induction with respect to m (in fact by telescoping) we see that

$$1=\sum_{k=0}^{m}\frac{(-a)_k\,(-1)^k}{(a+1)^k}\,\frac{k+1}{a+1}+\frac{(-a)_{m+1}\,(-1)^{m+1}}{(a+1)^{m+1}}\,. \tag{1}$$

Hence, for $n\in\mathbb{Z}_{\ge0}$ we have

$$1=\sum_{k=0}^{n}\frac{n!}{(n-k)!\,(n+1)^k}\,\frac{k+1}{n+1} \tag{2}$$

$$=\frac{1}{n+1}\,{}_2F_0\left(\begin{matrix}-n,2\\ —\end{matrix};-\frac{1}{n+1}\right) \tag{3}$$

$$=\frac{n!}{(n+1)^n}\,{}_1F_1\left(\begin{matrix}-n\\ -n-1\end{matrix};n+1\right). \tag{4}$$

Note that (3) can be rewritten as an evaluation for Charlier polynomials in a special case:

$$C_n(-2;n+1)=n+1. \tag{5}$$

Rewrite (2) as

$$1=\sum_{k=1}^{n}\frac{(n-1)!}{(n-k)!\,n^{k-1}}\,\frac{k}{n}\,. \tag{6}$$

Note that $\frac{(n-1)!}{(n-k)!\,n^{k-1}}$ is the chance that in a row of objects of n possible types, each type with equal probability, the first k objects have different types. Furthermore, the term $\frac{(n-1)!}{(n-k)!\,n^{k-1}}\,\frac{k}{n}$ is the chance that the first k objects have different types but among the first $k+1$ objects there are two of equal type. Therefore, the sum in the right-hand side of (6) must be equal to 1.