Additions to the formula lists in “Hypergeometric orthogonal polynomials and their q-analogues” by Koekoek, Lesky and Swarttouw

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Abstract

This report gives a rather arbitrary choice of formulas for (q-)hypergeometric orthogonal polynomials which the author missed while consulting Chapters 9 and 14 in the book “Hypergeometric orthogonal polynomials and their q-analogues” by Koekoek, Lesky and Swarttouw. The systematics of these chapters will be followed here, in particular for the numbering of subsections and of references.

Introduction

This report contains some formulas about (q-)hypergeometric orthogonal polynomials which I missed but wanted to use while consulting Chapters 9 and 14 in the book [KLS]:

R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer-Verlag, 2010.

These chapters form together the (slightly extended) successor of the report

R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report 98-17, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1998; http://aw.twi.tudelft.nl/~koekoek/askey/.

Certainly these chapters give complete lists of formulas of special type, for instance orthogonality relations and three-term recurrence relations. But outside these narrow categories there are many other formulas for (q-)orthogonal polynomials which one wants to have available. Often one can find the desired formula in one of the standard references listed at the end of this report. Sometimes it is only available in a journal or a less common monograph. Just for my own comfort, I have brought together some of these formulas. This will possibly also be helpful for some other users.

Usually, any type of formula I give for a special class of polynomials, will suggest a similar formula for many other classes, but I have not aimed at completeness by filling in a formula of such type at all places. The resulting choice of formulas is rather arbitrary, just depending on the formulas which I happened to need or which raised my interest. For each formula I give a suitable reference or I sketch a proof. It is my intention to gradually extend this collection of formulas.

Conventions

The (x.y) and (x.y.z) type subsection numbers, the (x.y.z) type formula numbers, and the [x] type citation numbers refer to [KLS]. The (x) type formula numbers refer to this manuscript and the [Kx] type citation numbers refer to citations which are not in [KLS]. Some standard references like [DLMF] are given by special acronyms.

N is always a positive integer. Always assume n to be a nonnegative integer or, if N is present, to be in $\{0, 1, \ldots, N\}$. Throughout assume $0 < q < 1$.

For each family the coefficient of the term of highest degree of the orthogonal polynomial of degree n can be found in [KLS] as the coefficient of $p_n(x)$ in the formula after the main formula under the heading "Normalized Recurrence Relation". If that main formula is numbered as (x.y.z) then I will refer to the second formula as (x.y.zb).

In the notation of q-hypergeometric orthogonal polynomials we will follow the convention that the parameter list and q are separated by '$\,|\,$' in the case of a q-quadratic lattice (for instance Askey–Wilson) and by ';' in the case of a q-linear lattice (for instance big q-Jacobi). This convention is mostly followed in [KLS], but not everywhere, see for instance little q-Laguerre / Wall.

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Contents

Other references

Generalities

Criteria for uniqueness of orthogonality measure According to Shohat & Tamarkin [K33, p.50] orthonormal polynomials p_n have a unique orthogonality measure (up to positive constant factor) if for some $z \in \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} |p_n(z)|^2 = \infty. \tag{1}$$

Also (see Shohat & Tamarkin [K33, p.59]), monic orthogonal polynomials p_n with three-term recurrence relation $xp_n(x) = p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x)$ (C_n necessarily positive) have a unique orthogonality measure if

$$\sum_{n=1}^{\infty} (C_n)^{-1/2} = \infty. \tag{2}$$

Furthermore, if orthogonal polynomials have an orthogonality measure with bounded support, then this is unique (see Chihara [146]).

Kernel polynomials and the three-term recurrence relation
For given monic orthogonal polynomials $\{p_n\}$ with respect to orthogonality measure μ and with

$$h_n := \int_{\mathbb{R}} p_n(x)^2 \, d\mu(x),$$

there is the *Christoffel–Darboux formula*

$$K_n(x,y) := \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{1}{h_n} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \qquad (x \neq y). \tag{3}$$

Fix $y \in \mathbb{R}$ and suppose that $\operatorname{supp}(\mu) \subseteq (-\infty, y]$. Then $p_n(y) \neq 0$ for all n and the monic polynomials

$$q_n(x) := \frac{h_n}{p_n(y)} K_n(x,y) \tag{4}$$

are orthogonal with respect to $(y-x)\, d\mu(x)$. They are called *kernel polynomials* (see Chihara [146, Ch. 1, §7]). There is a pair of contiguous relations relating the polynomialsd p_n and q_n:

$$(x-y)q_n(x) = p_{n+1}(x) - A_n p_n(x), \tag{5}$$

$$p_n(x) = q_n(x) - C_n q_{n-1}(x), \tag{6}$$

where

$$A_n = \frac{p_{n+1}(y)}{p_n(y)}, \qquad C_n = \frac{h_n}{h_{n-1}} \frac{p_{n-1}(y)}{p_n(y)}. \tag{7}$$

Then the *three-term recurrence relations* for the orthogonal polynomials p_n and q_n can be written in the form (see [K35, §5, Lemma 1])

$$x\, p_n(x) = p_{n+1}(x) + (y - A_n - C_n)p_n(x) + A_{n-1}C_n p_{n-1}(x), \tag{8}$$

$$x\, q_n(x) = q_{n+1}(x) + (y - A_n - C_{n+1})q_n(x) + A_n C_n q_{n-1}(x). \tag{9}$$

In the above formulas put terms containing the factor C_0 equal to 0.

In many cases in [KLS, Chapters 9, 14] the normalized three-term recurrence relation is given in the form (8), already in the Askey–Wilson case (14.1.5), and where it is not written in this way, it can be done so. See for instance (55) for Jacobi.

If we write the normalized recurrence relation for the p_n as

$$x\,p_n(x) = p_{n+1}(x) + b_n\,p_n(x) + c_n\,p_{n-1}(x), \tag{10}$$

and compare it with (8) then

$$b_0 = y - A_0, \qquad b_n = y - A_n - C_n, \qquad c_n = A_{n-1}C_n \qquad (n \ge 1). \tag{11}$$

This can be recursively solved for the A_n, C_n in terms of the b_n, c_n by

$$A_0 = y - b_0, \qquad C_n = \frac{c_n}{A_{n-1}}\,, \quad A_n = y - b_n - C_n \qquad (n \ge 1). \tag{12}$$

Equations (5), (6), (8) correspond to an LU factorization of the Jacobi matrix associated with the OPs p_n, see [K7, Lemma 2.1], where also (12) is given.

Even orthogonality measure If $\{p_n\}$ is a system of orthogonal polynomials with respect to an even orthogonality measure which satisfies the three-term recurrence relation

$$xp_n(x) = a_n p_{n+1}(x) + c_n\,p_{n-1}(x)$$

then

$$\frac{p_{2n}(0)}{p_{2n-2}(0)} = -\,\frac{c_{2n-1}}{a_{2n-1}}\,. \tag{13}$$

Finite systems of OPs of degree up to N with weights on $N+1$ points
Suppose we have OPs $\{p_n\}_{n=0}^N$ which are orthogonal on $\{x_0, x_1, \ldots, x_N\}$ with respect to weights w_i $(i = 0, 1, \ldots, N)$. Then we have recurrence relations

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \quad (n = 0, 1 \ldots, N), \tag{14}$$

where $p_{-1}(x) = 0$, $p_{N+1}(x) = (x - x_0) \ldots (x - x_N)$ and $p_N(x) = A_N x^N +$ terms of lower degree. For a proof of the case $n = N$ note that, for $x \in \{x_0, x_1, \ldots, x_N\}$, we have $xp_n(x) = B_n p_n(x) + C_n p_{n-1}(x)$ by orthogonality and by the fact that $p_0, p_1, \ldots, p_N$ is a basis of the function space on this set. Hence $xp_n(x) - B_n p_n(x) - C_n p_{n-1}(x)$ is a polynomial of degree $N+1$ which vanishes on $\{x_0, x_1, \ldots, x_N\}$ and for which the coefficient of x^{N+1} equals the coefficient of x^N for $p_N(x)$. Hence $xp_n(x) - B_n p_n(x) - C_n p_{n-1}(x) = A_N(x - x_0) \ldots (x - x_N)$.

Appell's bivariate hypergeometric function F_4 This is defined by

$$F_4(a, b; c, c'; x, y) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (c')_n\, m!\, n!}\, x^m y^n \qquad (|x|^{\frac12} + |y|^{\frac12} < 1), \tag{15}$$

see [HTF1, 5.7(9), 5.7(44)] or [DLMF, (16.13.4)]. There is the reduction formula

$$F_4\left(a,b;b,b;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right)=(1-x)^a(1-y)^a\ {}_2F_1\left({a,1+a-b\atop b};xy\right),$$

see [HTF1, 5.10(7)]. When combined with the quadratic transformation [HTF1, 2.11(34)] (here $a-b-1$ should be replaced by $a-b+1$), see also [DLMF, (15.8.15)], this yields

$$\begin{aligned}F_4&\left(a,b;b,b;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right)\\&=\left(\frac{(1-x)(1-y)}{1+xy}\right)^a\ {}_2F_1\left({\frac12a,\frac12(a+1)\atop b};\frac{4xy}{(1+xy)^2}\right).\end{aligned}$$

This can be rewritten as

$$F_4(a,b;b,b;x,y)=(1-x-y)^{-a}\ {}_2F_1\left({\frac12a,\frac12(a+1)\atop b};\frac{4xy}{(1-x-y)^2}\right).\tag{16}$$

Note that, if $x,y\ge0$ and $x^{\frac12}+y^{\frac12}<1$, then $1-x-y>0$ and $0\le\frac{4xy}{(1-x-y)^2}<1$.

q-Hypergeometric series of base q^{-1} By [GR, Exercise 1.4(i)]:

$${}_r\phi_s\left({a_1,\dots,a_r\atop b_1,\dots b_s};q^{-1},z\right)={}_{s+1}\phi_s\left({a_1^{-1},\dots a_r^{-1},0,\dots,0\atop b_1^{-1},\dots,b_s^{-1}};q,\frac{qa_1\dots a_rz}{b_1\dots b_s}\right)\tag{17}$$

for $r\le s+1$, $a_1,\dots,a_r,b_1,\dots,b_s\ne0$. In the non-terminating case, for $0<q<1$, there is convergence if $|z|<b_1\dots b_s/(qa_1\dots a_r)$.

A transformation of a terminating ${}_2\phi_1$ By [GR, Exercise 1.15(i)] we have

$${}_2\phi_1\left({q^{-n},b\atop c};q,z\right)=(bz/(cq);q^{-1})_n\ {}_3\phi_2\left({q^{-n},c/b,0\atop c,cq/(bz)};q,q\right).\tag{18}$$

Very-well-poised q-hypergeometric series The notation of [GR, (2.1.11)] will be followed:

$${}_{r+1}W_r(a_1;a_4,a_5,\dots,a_{r+1};q,z):={}_{r+1}\phi_r\left({a_1,qa_1^{\frac12},-qa_1^{\frac12},a_4,\dots,a_{r+1}\atop a_1^{\frac12},-a_1^{\frac12},qa_1/a_4,\dots,qa_1/a_{r+1}};q,z\right).\tag{19}$$

Theta function The notation of [GR, (11.2.1)] will be followed:

$$\theta(x;q):=(x,q/x;q)_\infty,\qquad\theta(x_1,\dots,x_m;q):=\theta(x_1;q)\dots\theta(x_m;q).\tag{20}$$

9.1 Wilson

Symmetry The Wilson polynomial $W_n(y;a,b,c,d)$ is symmetric in a,b,c,d.
This follows from the orthogonality relation (9.1.2) together with the value of its coefficient of y^n given in (9.1.5b). Alternatively, combine (9.1.1) with [AAR, Theorem 3.1.1].
As a consequence, it is sufficient to give generating function (9.1.12). Then the generating functions (9.1.13), (9.1.14) will follow by symmetry in the parameters.

Hypergeometric representation In addition to (9.1.1) we have (see [513, (2.2)]):

$$W_n(x^2;a,b,c,d)=\frac{(a-\mathrm{i}x)_n(b-\mathrm{i}x)_n(c-\mathrm{i}x)_n(d-\mathrm{i}x)_n}{(-2\mathrm{i}x)_n}$$
$$\times\ {}_7F_6\left(\begin{matrix}2\mathrm{i}x-n,\mathrm{i}x-\frac12 n+1,a+\mathrm{i}x,b+\mathrm{i}x,c+\mathrm{i}x,d+\mathrm{i}x,-n\\ \mathrm{i}x-\frac12 n,1-n-a+\mathrm{i}x,1-n-b+\mathrm{i}x,1-n-c+\mathrm{i}x,1-n-d+\mathrm{i}x,1+2\mathrm{i}x\end{matrix};1\right). \tag{21}$$

The symmetry in a,b,c,d is clear from (21).

Special value

$$W_n(-a^2;a,b,c,d)=(a+b)_n(a+c)_n(a+d)_n\,, \tag{22}$$

and similarly for arguments $-b^2$, $-c^2$ and $-d^2$ by symmetry of W_n in a,b,c,d.

Uniqueness of orthogonality measure Under the assumptions on a,b,c,d for (9.1.2) or (9.1.3) the orthogonality measure is unique up to constant factor.

For the proof assume without loss of generality (by the symmetry in a,b,c,d) that $\operatorname{Re} a\ge 0$. Write the right-hand side of (9.1.2) or (9.1.3) as $h_n\delta_{m,n}$. Observe from (9.1.2) and (22) that

$$\frac{|W_n(-a^2;a,b,c,d)|^2}{h_n}=O(n^{4\operatorname{Re}a-1})\quad\text{as } n\to\infty.$$

Therefore (1) holds, from which the uniqueness of the orthogonality measure follows.

By a similar, but necessarily more complicated argument Ismail et al. [281, Section 3] proved the uniqueness of orthogonality measure for associated Wilson polynomials.

9.2 Racah

Racah in terms of Wilson In the Remark on p.196 Racah polynomials are expressed in terms of Wilson polynomials. This can be equivalently written as

$$R_n\big(x(x-N+\delta);\alpha,\beta,-N-1,\delta\big)$$
$$=\frac{W_n\big(-(x+\frac12(\delta-N))^2;\frac12(\delta-N),\alpha+1-\frac12(\delta-N),\beta+\frac12(\delta+N)+1,-\frac12(\delta+N)\big)}{(\alpha+1)_n(\beta+\delta+1)_n(-N)_n}\,. \tag{23}$$

9.3 Continuous dual Hahn

Symmetry The continuous dual Hahn polynomial $S_n(y; a, b, c)$ is symmetric in a, b, c. This follows from the orthogonality relation (9.3.2) together with the value of its coefficient of y^n given in (9.3.5b). Alternatively, combine (9.3.1) with [AAR, Corollary 3.3.5].
As a consequence, it is sufficient to give generating function (9.3.12). Then the generating functions (9.3.13), (9.3.14) will follow by symmetry in the parameters.

Special value

$$S_n(-a^2; a, b, c) = (a+b)_n (a+c)_n\,, \tag{24}$$

and similarly for arguments $-b^2$ and $-c^2$ by symmetry of S_n in a, b, c.

Uniqueness of orthogonality measure Under the assumptions on a, b, c for (9.3.2) or (9.3.3) the orthogonality measure is unique up to constant factor.

For the proof assume without loss of generality (by the symmetry in a, b, c) that $\mathrm{Re}\, a \ge 0$. Write the right-hand side of (9.3.2) or (9.3.3) as $h_n \delta_{m,n}$. Observe from (9.3.2) and (24) that

$$\frac{|S_n(-a^2; a, b, c)|^2}{h_n} = O(n^{2\mathrm{Re}\, a - 1}) \quad \text{as } n \to \infty.$$

Therefore (1) holds, from which the uniqueness of the orthogonality measure follows.

Special continuous dual Hahn in terms of Wilson

$$S_n\big(x; a, b, \tfrac{1}{2}\big) = \frac{2^{2n}}{(a+b+n)_n}\, W_n\big(\tfrac{1}{4}x; \tfrac{1}{2}a, \tfrac{1}{2}(a+1), \tfrac{1}{2}b, \tfrac{1}{2}(b+1)\big). \tag{25}$$

For the proof compare the weight functions and the values for $x = -a^2$.

Generating functions By (9.3.17) the generating function (9.3.16) has the generating function (9.7.13) for Meixner–Pollaczek polynomials as a limit case.

9.4 Continuous Hahn

Orthogonality relation and parameter symmetry The orthogonality relation (9.4.2) holds under the more general assumption that $\mathrm{Re}\,(a, b, c, d) > 0$ and $(c, d) = (\overline{a}, \overline{b})$ or $(\overline{b}, \overline{a})$.
Thus, under these assumptions, the continuous Hahn polynomial $p_n(x; a, b, c, d)$ is symmetric in a, b and in c, d. This follows from the orthogonality relation (9.4.2) together with the value of its coefficient of x^n given in (9.4.4b).
As a consequence, it is sufficient to give generating function (9.4.11). Then the generating function (9.4.12) will follow by symmetry in the parameters.

Symmetry

$$p_n(-x; a, b, \overline{a}, \overline{b}) = (-1)^n p_n(x; \overline{a}, \overline{b}, a, b). \tag{26}$$

Special value

$$p_n(\mathrm{i}a; a, b, \overline{a}, \overline{b}) = \frac{\mathrm{i}^n (a+\overline{a})_n (a+\overline{b})_n}{n!}\,. \tag{27}$$

Similarly, $p_n(x; a, b, \overline{a}, \overline{b})$ has special values for $x = -\mathrm{i}\overline{a}$, $\mathrm{i}b$ and $-\mathrm{i}\overline{b}$.

Quadratic transformation For $a, b \in \mathbb{R}$ or $b = \overline{a}$ we have [K23, (2.29), (2.30)]

$$\frac{p_{2n}(x; a, b, \overline{a}, \overline{b})}{p_{2n}(\mathrm{i}a; a, b, \overline{a}, \overline{b})} = \frac{W_n(x^2; a, b, \frac12, 0)}{W_n(-a^2; a, b, \frac12, 0)}\,, \quad \frac{p_{2n+1}(x; a, b, \overline{a}, \overline{b})}{p_{2n+1}(\mathrm{i}a; a, b, \overline{a}, \overline{b})} = \frac{x W_n(x^2; a, b, \frac12, 1)}{\mathrm{i}a W_n(-a^2; a, b, \frac12, 1)}\,. \tag{28}$$

Explicit expression For $a, b \in \mathbb{R}$ or $b = \overline{a}$ we have by (28), (9.1.1) and reversion of direction of summation that

$$\begin{aligned} p_n(x; a, b, \overline{a}, \overline{b}) = \frac{(n+a+b+\overline{a}+\overline{b}-1)_n}{n!}\, x^{n-2[\frac12 n]}\, (-\tfrac12 n + \mathrm{i}x + 1)_{[\frac12 n]} (-\tfrac12 n - \mathrm{i}x + 1)_{[\frac12 n]} \\ \times\, {}_4F_3\left(\begin{matrix} -\frac12 n, -\frac12 n + \frac12, -\frac12 n - a + 1, -\frac12 n - b + 1 \\ -n - a - b + \frac32, -\frac12 n + \mathrm{i}x + 1, -\frac12 n - \mathrm{i}x + 1 \end{matrix}; 1\right). \end{aligned} \tag{29}$$

Special cases In the following special case there is a reduction to Meixner–Pollaczek:

$$p_n(x; a, a+\tfrac12, a, a+\tfrac12) = \frac{(2a)_n (2a+\frac12)_n}{(4a)_n}\, P_n^{(2a)}(2x; \tfrac12\pi). \tag{30}$$

See [342, (2.6)] (note that in [342, (2.3)] the Meixner–Pollaczek polyonmials are defined different from (9.7.1), without a constant factor in front).

For $0 < a < 1$ the continuous Hahn polynomials $p_n(x; a, 1-a, a, 1-a)$ are orthogonal on $(-\infty, \infty)$ with respect to the weight function $\big(\cosh(2\pi x) - \cos(2\pi a)\big)^{-1}$ (by straightforward computation from (9.4.2)). For $a = \frac14$ the two special cases coincide: Meixner–Pollaczek with weight function $\big(\cosh(2\pi x)\big)^{-1}$.

Uniqueness of orthogonality measure The coefficient of $p_{n-1}(x)$ in (9.4.4) behaves as $O(n^2)$ as $n \to \infty$. Hence (2) holds, by which the orthogonality measure is unique.

9.5 Hahn

Special values

$$Q_n(0; \alpha, \beta, N) = 1, \quad Q_n(N; \alpha, \beta, N) = \frac{(-1)^n (\beta+1)_n}{(\alpha+1)_n}\,. \tag{31}$$

Use (9.5.1) and compare with (9.8.1) and (54).

From (9.5.3) and (13) it follows that

$$Q_{2n}(N; \alpha, \alpha, 2N) = \frac{(\frac12)_n (N+\alpha+1)_n}{(-N+\frac12)_n (\alpha+1)_n}\,. \tag{32}$$

From (9.5.1) and [DLMF, (15.4.24)] it follows that

$$Q_N(x;\alpha,\beta,N) = \frac{(-N-\beta)_x}{(\alpha+1)_x} \qquad (x=0,1,\ldots,N). \tag{33}$$

Symmetries By the orthogonality relation (9.5.2):

$$\frac{Q_n(N-x;\alpha,\beta,N)}{Q_n(N;\alpha,\beta,N)} = Q_n(x;\beta,\alpha,N), \tag{34}$$

It follows from (41) and (36) that

$$\frac{Q_{N-n}(x;\alpha,\beta,N)}{Q_N(x;\alpha,\beta,N)} = Q_n(x;-N-\beta-1,-N-\alpha-1,N) \qquad (x=0,1,\ldots,N). \tag{35}$$

Duality The Remark on p.208 gives the duality between Hahn and dual Hahn polynomials:

$$Q_n(x;\alpha,\beta,N) = R_x(n(n+\alpha+\beta+1);\alpha,\beta,N) \quad (n,x\in\{0,1,\ldots N\}). \tag{36}$$

9.6 Dual Hahn

Special values By (33) and (36) we have

$$R_n(N(N+\gamma+\delta+1);\gamma,\delta,N) = \frac{(-N-\delta)_n}{(\gamma+1)_n}\,. \tag{37}$$

It follows from (31) and (36) that

$$R_N(x(x+\gamma+\delta+1);\gamma,\delta,N) = \frac{(-1)^x(\delta+1)_x}{(\gamma+1)_x} \qquad (x=0,1,\ldots,N). \tag{38}$$

Symmetries Write the weight in (9.6.2) as

$$w_x(\alpha,\beta,N) := N!\,\frac{2x+\gamma+\delta+1}{(x+\gamma+\delta+1)_{N+1}}\,\frac{(\gamma+1)_x}{(\delta+1)_x}\binom{N}{x}. \tag{39}$$

Then

$$(\delta+1)_N\,w_{N-x}(\gamma,\delta,N) = (-\gamma-N)_N\,w_x(-\delta-N-1,-\gamma-N-1,N). \tag{40}$$

Hence, by (9.6.2),

$$\frac{R_n((N-x)(N-x+\gamma+\delta+1);\gamma,\delta,N)}{R_n(N(N+\gamma+\delta+1);\gamma,\delta,N)} = R_n(x(x-2N-\gamma-\delta-1);-N-\delta-1,-N-\gamma-1,N). \tag{41}$$

Alternatively, (41) follows from (9.6.1) and [DLMF, (16.4.11)].

It follows from (34) and (36) that

$$\frac{R_{N-n}(x(x+\gamma+\delta+1);\gamma,\delta,N)}{R_N(x(x+\gamma+\delta+1);\gamma,\delta,N)} = R_n(x(x+\gamma+\delta+1);\delta,\gamma,N) \qquad (x=0,1,\ldots,N). \tag{42}$$

Re: (9.6.11). The generating function (9.6.11) can be written in a more conceptual way as

$$(1-t)^x\,{}_2F_1\left({x-N,x+\gamma+1 \atop -\delta-N};t\right)=\frac{N!}{(\delta+1)_N}\sum_{n=0}^N \omega_n\,R_n(\lambda(x);\gamma,\delta,N)\,t^n, \tag{43}$$

where

$$\omega_n:=\binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}, \tag{44}$$

i.e., the denominator on the right-hand side of (9.6.2). By the duality between Hahn polynomials and dual Hahn polynomials (see (36)) the above generating function can be rewritten in terms of Hahn polynomials:

$$(1-t)^n\,{}_2F_1\left({n-N,n+\alpha+1 \atop -\beta-N};t\right)=\frac{N!}{(\beta+1)_N}\sum_{x=0}^N w_x\,Q_n(x;\alpha,\beta,N)\,t^x, \tag{45}$$

where

$$w_x:=\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}, \tag{46}$$

i.e., the weight occurring in the orthogonality relation (9.5.2) for Hahn polynomials.

Re: (9.6.15). There should be a closing bracket before the equality sign.

9.7 Meixner–Pollaczek

Re: (9.7.1) In addition to the hypergeometric representation (9.7.1) we have, by the Pfaff transformation [HTF1, 2.9(3)], that

$$P_n^{(\lambda)}(x;\phi)=\frac{(2\lambda)_n}{n!}\,\mathrm{e}^{-\mathrm{i}n\phi}\,{}_2F_1\left({-n,\lambda-\mathrm{i}x \atop 2\lambda};1-\mathrm{e}^{2\mathrm{i}\phi}\right). \tag{47}$$

Special values By (9.7.1) and (47) we have:

$$P_n^{(\lambda)}(\mathrm{i}\lambda;\phi)=\frac{(2\lambda)_n}{n!}\,\mathrm{e}^{\mathrm{i}n\phi},\qquad P_n^{(\lambda)}(-\mathrm{i}\lambda;\phi)=\frac{(2\lambda)_n}{n!}\,\mathrm{e}^{-\mathrm{i}n\phi}. \tag{48}$$

Symmetry

$$P_n^{(\lambda)}(x;\phi)=(-1)^nP_n^{(\lambda)}(-x;\pi-\phi). \tag{49}$$

Quadratic transformations [K23, (2.33), (2.34)]

$$\frac{P_{2n}^{(a)}(x;\frac12\pi)}{P_{2n}^{(a)}(\mathrm{i}a;\frac12\pi)}=\frac{S_n(x^2;a,\frac12,0)}{S_n(-a^2;a,\frac12,0)}\,,\qquad \frac{P_{2n+1}^{(a)}(x;\frac12\pi)}{P_{2n+1}^{(a)}(\mathrm{i}a;\frac12\pi)}=\frac{xS_n(x^2;a,\frac12,1)}{\mathrm{i}aS_n(-a^2;a,\frac12,1)}\,. \tag{50}$$

These are limit cases of (28) by the limits (9.1.16), (9.4.14).

Uniqueness of orthogonality measure The coefficient of $p_{n-1}(x)$ in (9.7.4) behaves as $O(n^2)$ as $n \to \infty$. Hence (2) holds, by which the orthogonality measure is unique.

Generating functions By (9.3.17) the generating function (9.3.16) for continuous dual Hahn polynomials has the generating function (9.7.13) as a limit case. By (9.7.14) formula (9.7.13) has the generating function (9.12.12) for Laguerre polynomials as a limit case.

9.8 Jacobi

Orthogonality relation Write the right-hand side of (9.8.2) as $h_n\,\delta_{m,n}$. Then

$$\begin{aligned} &\frac{h_n}{h_0} = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+\beta+2)_n\,n!}\,, \quad h_0 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,,\\ &\frac{h_n}{h_0\,(P_n^{(\alpha,\beta)}(1))^2} = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{(\beta+1)_n\,n!}{(\alpha+1)_n\,(\alpha+\beta+2)_n}\,. \end{aligned} \tag{51}$$

In (9.8.3) the numerator factor $\Gamma(n+\alpha+\beta+1)$ in the last line should be $\Gamma(\beta+1)$. When thus corrected, (9.8.3) can be rewritten as:

$$\begin{aligned} &\int_1^\infty P_m^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(x)\,(x-1)^\alpha(x+1)^\beta\,\mathrm{d}x = h_n\,\delta_{m,n}\,,\\ &\qquad\qquad -1-\beta>\alpha>-1, \quad m,n<-\tfrac12(\alpha+\beta+1),\\ &\frac{h_n}{h_0} = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+\beta+2)_n\,n!}\,, \quad h_0 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(-\alpha-\beta-1)}{\Gamma(-\beta)}\,. \end{aligned} \tag{52}$$

Following Lesky [382] the Jacobi polynomials in case of orthogonality relation (52) may be called *Romanovski–Jacobi polynomials*.

Symmetry

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n\,P_n^{(\beta,\alpha)}(x). \tag{53}$$

Use (9.8.2) and (9.8.5b) or see [DLMF, Table 18.6.1].

Special values

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}\,, \quad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(\beta+1)_n}{n!}\,, \quad \frac{P_n^{(\alpha,\beta)}(-1)}{P_n^{(\alpha,\beta)}(1)} = \frac{(-1)^n(\beta+1)_n}{(\alpha+1)_n}\,. \tag{54}$$

Use (9.8.1) and (53) or see [DLMF, Table 18.6.1].

Normalized recurrence relation Formula (9.8.5) can be rewritten as

$$x\,p_n(x) = p_{n+1}(x) + (1-A_n-C_n)p_n(x) + A_{n-1}C_n\,p_{n-1}(x), \tag{55}$$

where $p_n(x) = 2^n n!\,P_n^{(\alpha,\beta)}(x)/(n+\alpha+\beta+1)_n$ and

$$A_n = \frac{2(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\,, \qquad C_n = \frac{2n(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\,.$$

Contiguous relations

$$(n+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+1)(1-x)P_n^{(\alpha+1,\beta)}(x) = -(n+1)P_{n+1}^{(\alpha,\beta)}(x) + (n+\alpha+1)P_n^{(\alpha,\beta)}(x), \tag{56}$$

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha+1,\beta)}(x). \tag{57}$$

See [HTF2, 10.8(32) and (35)]. These can be rewritten as

$$(x-1)q_n(x) = p_{n+1}(x) - A_n p_n(x), \tag{58}$$

$$p_n(x) = q_n(x) - C_n q_{n-1}(x), \tag{59}$$

where $q_n(x) = 2^n n!\, P_n^{(\alpha+1,\beta)}(x)/(n+\alpha+\beta+2)_n$ and $p_n(x)$, A_n and C_n are as above.

Formula (55) can be derived from (58), (59) by substituting these last two formulas in the following rewritten form of (55) (compare with (5)–(8)):

$$(x-1)p_n(x) = \big(p_{n+1}(x) - A_n p_n(x)\big) - C_n\big(p_n(x) - A_{n-1}p_{n-1}(x)\big).$$

Generating functions Formula (9.8.15) was first obtained by Brafman [109, (12)]. Alternatively (see [109, (9)] or use [DLMF, (16.16.6)]), the left-hand side of (9.8.15) can be written as Appell's hypergeometric function F_4:

$$F_4\big(\gamma,\alpha+\beta+1-\gamma;\alpha+1,\beta+1;\tfrac{1}{2}t(x-1),\tfrac{1}{2}t(x+1)\big) = \sum_{k=0}^{\infty}\frac{(\gamma)_k(\alpha+\beta+1-\gamma)_k}{(\alpha+1)_k(\beta+1)_k}\,P_k^{(\alpha,\beta)}(x)t^k \tag{60}$$

The generating function (9.12.12) for Laguerre polynomials is a limit case of (60) by (9.8.16).

Formula (9.8.15) with t, x replaced by $\frac{1}{2}(x+y)$, $\frac{1+xy}{x+y}$, respectively, takes the form

$$\begin{aligned}{}_2F_1\left(\begin{matrix}\gamma,\alpha+\beta+1-\gamma\\ \alpha+1\end{matrix};\tfrac{1}{2}(1-x)\right)\,{}_2F_1\left(\begin{matrix}\gamma,\alpha+\beta+1-\gamma\\ \beta+1\end{matrix};\tfrac{1}{2}(1+y)\right)&\\ = \sum_{k=0}^{\infty}\frac{(\gamma)_k(\alpha+\beta+1-\gamma)_k}{(\alpha+1)_k(\beta+1)_k}\,(x+y)^k P_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right).&\end{aligned} \tag{61}$$

In [109, (14)] the case γ nonpositive integer of (9.8.15) is given. When we do this for (61) with $\gamma=-n\in\mathbb{Z}_{\le 0}$ this yields the inverse of Bateman's bilinear sum, as is given in [331, (2.19), (2.20)], [DLMF, (18.18.25), (18.18.26)].

Bilinear generating functions For $0\le r<1$ and $x,y\in[-1,1]$ we have in terms of F_4 (see (15)):

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n\,n!}{(\alpha+1)_n(\beta+1)_n}\,r^n\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y) &= \frac{1}{(1+r)^{\alpha+\beta+1}}\\ &\times F_4\Big(\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha+\beta+2);\alpha+1,\beta+1;\frac{r(1-x)(1-y)}{(1+r)^2},\frac{r(1+x)(1+y)}{(1+r)^2}\Big),\end{aligned} \tag{62}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1}\,\frac{(\alpha+\beta+2)_n\,n!}{(\alpha+1)_n(\beta+1)_n}\,r^n\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y) &= \frac{1-r}{(1+r)^{\alpha+\beta+2}}\\ &\times F_4\Big(\tfrac{1}{2}(\alpha+\beta+2),\tfrac{1}{2}(\alpha+\beta+3);\alpha+1,\beta+1;\frac{r(1-x)(1-y)}{(1+r)^2},\frac{r(1+x)(1+y)}{(1+r)^2}\Big).\end{aligned} \tag{63}$$

Formulas (62) and (63) were first given by Bailey [91, (2.1), (2.3)]. See Stanton [485] for a shorter proof. (However, in the second line of [485, (1)] z and Z should be interchanged.) As observed in Bailey [91, p.10], (63) follows from (62) by applying the operator $r^{-\frac{1}{2}(\alpha+\beta-1)}\,\frac{\mathrm{d}}{\mathrm{d}r}\circ r^{\frac{1}{2}(\alpha+\beta+1)}$ to both sides of (62). In view of (51), formula (63) is the Poisson kernel for Jacobi polynomials. The right-hand side of (63) makes clear that this kernel is positive. See also the discussion in Askey [46, following (2.32)].

Quadratic transformations

$$\frac{C_{2n}^{(\alpha+\frac{1}{2})}(x)}{C_{2n}^{(\alpha+\frac{1}{2})}(1)}=\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)}=\frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)}\,, \tag{64}$$

$$\frac{C_{2n+1}^{(\alpha+\frac{1}{2})}(x)}{C_{2n+1}^{(\alpha+\frac{1}{2})}(1)}=\frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)}=\frac{x\,P_n^{(\alpha,\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,\frac{1}{2})}(1)}\,. \tag{65}$$

See p.221, Remarks, last two formulas together with (54) and (76). Or see [DLMF, (18.7.13), (18.7.14)].

Differentiation formulas Each differentiation formula is given in two equivalent forms.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x)^\alpha P_n^{(\alpha,\beta)}(x)\Big)&=-(n+\alpha)\,(1-x)^{\alpha-1}P_n^{(\alpha-1,\beta+1)}(x),\\ \left((1-x)\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\right)P_n^{(\alpha,\beta)}(x)&=-(n+\alpha)\,P_n^{(\alpha-1,\beta+1)}(x).\end{aligned} \tag{66}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big((1+x)^\beta P_n^{(\alpha,\beta)}(x)\Big)&=(n+\beta)\,(1+x)^{\beta-1}P_n^{(\alpha+1,\beta-1)}(x),\\ \left((1+x)\frac{\mathrm{d}}{\mathrm{d}x}+\beta\right)P_n^{(\alpha,\beta)}(x)&=(n+\beta)\,P_n^{(\alpha+1,\beta-1)}(x).\end{aligned} \tag{67}$$

Formulas (66) and (67) follow from [DLMF, (15.5.4), (15.5.6)] together with (9.8.1). They also follow from each other by (53).

Generalized Gegenbauer polynomials These are defined by

$$S_{2m}^{(\alpha,\beta)}(x):=\text{const.}\,P_m^{(\alpha,\beta)}(2x^2-1),\qquad S_{2m+1}^{(\alpha,\beta)}(x):=\text{const.}\,x\,P_m^{(\alpha,\beta+1)}(2x^2-1) \tag{68}$$

in the notation of [146, p.156] (see also [K5]), while [K12, Section 1.5.2] has $C_n^{(\lambda,\mu)}(x)=\text{const.}\times S_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(x)$. For $\alpha,\beta>-1$ we have the orthogonality relation

$$\int_{-1}^1 S_m^{(\alpha,\beta)}(x)\,S_n^{(\alpha,\beta)}(x)\,|x|^{2\beta+1}(1-x^2)^\alpha\,\mathrm{d}x=0\qquad(m\neq n). \tag{69}$$

For $\beta=\alpha-1$ generalized Gegenbauer polynomials are limit cases of continuous q-ultraspherical polynomials, see (197).

If we define the *Dunkl operator* T_μ by

$$(T_\mu f)(x) := f'(x) + \mu\,\frac{f(x)-f(-x)}{x} \tag{70}$$

and if we choose the constants in (68) as

$$S_{2m}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)_m}{(\beta+1)_m}\,P_m^{(\alpha,\beta)}(2x^2-1),\quad S_{2m+1}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)_{m+1}}{(\beta+1)_{m+1}}\,x\,P_m^{(\alpha,\beta+1)}(2x^2-1) \tag{71}$$

then (see [K6, (1.6)])

$$T_{\beta+\frac12} S_n^{(\alpha,\beta)} = 2(\alpha+\beta+1)\,S_{n-1}^{(\alpha+1,\beta)}. \tag{72}$$

Formula (72) with (71) substituted gives rise to two differentiation formulas involving Jacobi polynomials which are equivalent to (9.8.7) and (67).

Composition of (72) with itself gives

$$T_{\beta+\frac12}^2 S_n^{(\alpha,\beta)} = 4(\alpha+\beta+1)(\alpha+\beta+2)\,S_{n-2}^{(\alpha+2,\beta)},$$

which is equivalent to the composition of (9.8.7) and (67):

$$\left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\,\frac{d}{dx}\right) P_n^{(\alpha,\beta)}(2x^2-1) = 4(n+\alpha+\beta+1)(n+\beta)\,P_{n-1}^{(\alpha+2,\beta)}(2x^2-1). \tag{73}$$

Formula (73) was also given in [332, (2.4)].

9.8.1 Gegenbauer / Ultraspherical

Notation Here the Gegenbauer polynomial is denoted by C_n^λ instead of $C_n^{(\lambda)}$.

Orthogonality relation Write the right-hand side of (9.8.20) as $h_n\,\delta_{m,n}$. Then

$$\frac{h_n}{h_0} = \frac{\lambda}{\lambda+n}\,\frac{(2\lambda)_n}{n!}\,,\quad h_0 = \frac{\pi^{\frac12}\,\Gamma(\lambda+\frac12)}{\Gamma(\lambda+1)},\quad \frac{h_n}{h_0\,(C_n^\lambda(1))^2} = \frac{\lambda}{\lambda+n}\,\frac{n!}{(2\lambda)_n}\,. \tag{74}$$

Hypergeometric representation Beside (9.8.19) we have also

$$C_n^\lambda(x) = \sum_{\ell=0}^{\lfloor n/2\rfloor} \frac{(-1)^\ell(\lambda)_{n-\ell}}{\ell!\,(n-2\ell)!}\,(2x)^{n-2\ell} = (2x)^n\,\frac{(\lambda)_n}{n!}\,{}_2F_1\left(\begin{matrix}-\frac12 n, -\frac12 n+\frac12\\ 1-\lambda-n\end{matrix};\frac{1}{x^2}\right). \tag{75}$$

See [DLMF, (18.5.10)].

Special value

$$C_n^\lambda(1) = \frac{(2\lambda)_n}{n!}\,. \tag{76}$$

Use (9.8.19) or see [DLMF, Table 18.6.1].

Expression in terms of Jacobi

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} = \frac{P_n^{(\lambda-\frac12,\lambda-\frac12)}(x)}{P_n^{(\lambda-\frac12,\lambda-\frac12)}(1)}\,, \qquad C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda+\frac12)_n}\, P_n^{(\lambda-\frac12,\lambda-\frac12)}(x). \tag{77}$$

Re: (9.8.21) By iteration of recurrence relation (9.8.21):

$$x^2 C_n^\lambda(x) = \frac{(n+1)(n+2)}{4(n+\lambda)(n+\lambda+1)}\, C_{n+2}^\lambda(x) + \frac{n^2+2n\lambda+\lambda-1}{2(n+\lambda-1)(n+\lambda+1)}\, C_n^\lambda(x) + \frac{(n+2\lambda-1)(n+2\lambda-2)}{4(n+\lambda)(n+\lambda-1)}\, C_{n-2}^\lambda(x). \tag{78}$$

Bilinear generating functions

$$\sum_{n=0}^\infty \frac{n!}{(2\lambda)_n}\, r^n\, C_n^\lambda(x)\, C_n^\lambda(y) = \frac{1}{(1-2rxy+r^2)^\lambda}\; {}_2F_1\left(\begin{matrix} \frac12\lambda, \frac12(\lambda+1) \\ \lambda+\frac12 \end{matrix}; \frac{4r^2(1-x^2)(1-y^2)}{(1-2rxy+r^2)^2}\right) \quad (r\in(-1,1),\ x,y\in[-1,1]). \tag{79}$$

For the proof put $\beta := \alpha$ in (62), then use (16) and (77). The Poisson kernel for Gegenbauer polynomials can be derived in a similar way from (63), or alternatively by applying the operator $r^{-\lambda+1}\frac{d}{dr}\circ r^\lambda$ to both sides of (79):

$$\sum_{n=0}^\infty \frac{\lambda+n}{\lambda}\,\frac{n!}{(2\lambda)_n}\, r^n\, C_n^\lambda(x)\, C_n^\lambda(y) = \frac{1-r^2}{(1-2rxy+r^2)^{\lambda+1}} \times {}_2F_1\left(\begin{matrix} \frac12(\lambda+1), \frac12(\lambda+2) \\ \lambda+\frac12 \end{matrix}; \frac{4r^2(1-x^2)(1-y^2)}{(1-2rxy+r^2)^2}\right) \quad (r\in(-1,1),\ x,y\in[-1,1]). \tag{80}$$

Formula (80) was obtained by Gasper & Rahman [234, (4.4)] as a limit case of their formula for the Poisson kernel for continuous q-ultraspherical polynomials.

Trigonometric expansions By [DLMF, (18.5.11), (15.8.1)]:

$$C_n^\lambda(\cos\theta) = \sum_{k=0}^n \frac{(\lambda)_k(\lambda)_{n-k}}{k!\,(n-k)!}\, e^{i(n-2k)\theta} = e^{in\theta}\frac{(\lambda)_n}{n!}\; {}_2F_1\left(\begin{matrix} -n, \lambda \\ 1-\lambda-n \end{matrix}; e^{-2i\theta}\right) \tag{81}$$

$$= \frac{(\lambda)_n}{2^\lambda n!}\, e^{-\frac12 i\lambda\pi} e^{i(n+\lambda)\theta}\, (\sin\theta)^{-\lambda}\; {}_2F_1\left(\begin{matrix} \lambda, 1-\lambda \\ 1-\lambda-n \end{matrix}; \frac{ie^{-i\theta}}{2\sin\theta}\right) \tag{82}$$

$$= \frac{(\lambda)_n}{n!} \sum_{k=0}^\infty \frac{(\lambda)_k(1-\lambda)_k}{(1-\lambda-n)_k k!}\, \frac{\cos((n-k+\lambda)\theta + \frac12(k-\lambda)\pi)}{(2\sin\theta)^{k+\lambda}}\,. \tag{83}$$

In (82) and (83) we require that $\frac16\pi < \theta < \frac56\pi$. Then the convergence is absolute for $\lambda > \frac12$ and conditional for $0 < \lambda \le \frac12$.

By [DLMF, (14.13.1), (14.3.21), (15.8.1)]:

$$C_n^\lambda(\cos\theta)=\frac{2\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\,(\sin\theta)^{1-2\lambda}\sum_{k=0}^\infty\frac{(1-\lambda)_k(n+1)_k}{(n+\lambda+1)_k k!}\,\sin\big((2k+n+1)\theta\big) \tag{84}$$

$$=\frac{2\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\,(\sin\theta)^{1-2\lambda}\,\mathrm{Im}\left(\mathrm{e}^{\mathrm{i}(n+1)\theta}\,{}_2F_1\left(\begin{matrix}1-\lambda,n+1\\ n+\lambda+1\end{matrix};\mathrm{e}^{2\mathrm{i}\theta}\right)\right)$$

$$=\frac{2^\lambda\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\,(\sin\theta)^{-\lambda}\,\mathrm{Re}\left(\mathrm{e}^{-\frac12\mathrm{i}\lambda\pi}e^{\mathrm{i}(n+\lambda)\theta}\,{}_2F_1\left(\begin{matrix}\lambda,1-\lambda\\ 1+\lambda+n\end{matrix};\frac{e^{i\theta}}{2i\sin\theta}\right)\right)$$

$$=\frac{2^{2\lambda}\Gamma(\lambda+\frac12)}{\pi^{\frac12}\Gamma(\lambda+1)}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\sum_{k=0}^\infty\frac{(\lambda)_k(1-\lambda)_k}{(1+\lambda+n)_k k!}\,\frac{\cos((n+k+\lambda)\theta-\frac12(k+\lambda)\pi)}{(2\sin\theta)^{k+\lambda}}\,. \tag{85}$$

We require that $0<\theta<\pi$ in (84) and $\frac16\pi<\theta<\frac56\pi$ in (85) The convergence is absolute for $\lambda>\frac12$ and conditional for $0<\lambda\le\frac12$. For $\lambda\in\mathbb{Z}_{>0}$ the above series terminate after the term with $k=\lambda-1$. Formulas (84) and (85) are also given in [Sz, (4.9.22), (4.9.25)].

Fourier transform

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac12)\,\Gamma(\frac12)}\int_{-1}^1\frac{C_n^\lambda(y)}{C_n^\lambda(1)}\,(1-y^2)^{\lambda-\frac12}\,\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y=i^n\,2^\lambda\,\Gamma(\lambda+1)\,x^{-\lambda}\,J_{\lambda+n}(x). \tag{86}$$

See [DLMF, (18.17.17) and (18.17.18)].

Laplace transforms

$$\frac{2}{n!\,\Gamma(\lambda)}\int_0^\infty H_n(tx)\,t^{n+2\lambda-1}\,\mathrm{e}^{-t^2}\,\mathrm{d}t=C_n^\lambda(x). \tag{87}$$

See Nielsen [K29, p.48, (4) with p.47, (1) and p.28, (10)] (1918) or Feldheim [K13, (28)] (1942).

$$\frac{2}{\Gamma(\lambda+\frac12)}\int_0^1\frac{C_n^\lambda(t)}{C_n^\lambda(1)}\,(1-t^2)^{\lambda-\frac12}\,t^{-1}\,(x/t)^{n+2\lambda+1}\,\mathrm{e}^{-x^2/t^2}\,\mathrm{d}t=2^{-n}\,H_n(x)\,\mathrm{e}^{-x^2}\quad(\lambda>-\tfrac12). \tag{88}$$

Use Askey & Fitch [K2, (3.29)] for $\alpha=\pm\frac12$ together with (53), (64), (65), (113) and (114).

Addition formula (see [AAR, (9.8.5′)]])

$$R_n^{(\alpha,\alpha)}\big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\big)=\sum_{k=0}^n\frac{(-1)^k(-n)_k\,(n+2\alpha+1)_k}{2^{2k}((\alpha+1)_k)^2}$$
$$\times(1-x^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-y^2)^{k/2}R_{n-k}^{(\alpha+k,\alpha+k)}(y)\,\omega_k^{(\alpha-\frac12,\alpha-\frac12)}\,R_k^{(\alpha-\frac12,\alpha-\frac12)}(t), \tag{89}$$

where

$$R_n^{(\alpha,\beta)}(x):=P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1),\quad \omega_n^{(\alpha,\beta)}:=\frac{\int_{-1}^1(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x}{\int_{-1}^1(R_n^{(\alpha,\beta)}(x))^2\,(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x}\,.$$

9.8.2 Chebyshev

In addition to the Chebyshev polynomials T_n of the first kind (9.8.35) and U_n of the second kind (9.8.36),

$$T_n(x) := \frac{P_n^{(-\frac12,-\frac12)}(x)}{P_n^{(-\frac12,-\frac12)}(1)} = \cos(n\theta), \quad x = \cos\theta, \tag{90}$$

$$U_n(x) := (n+1)\,\frac{P_n^{(\frac12,\frac12)}(x)}{P_n^{(\frac12,\frac12)}(1)} = \frac{\sin((n+1)\theta)}{\sin\theta}\,, \quad x = \cos\theta, \tag{91}$$

we have Chebyshev polynomials V_n *of the third kind* and W_n *of the fourth kind*,

$$V_n(x) := \frac{P_n^{(-\frac12,\frac12)}(x)}{P_n^{(-\frac12,\frac12)}(1)} = \frac{\cos((n+\frac12)\theta)}{\cos(\frac12\theta)}\,, \quad x = \cos\theta, \tag{92}$$

$$W_n(x) := (2n+1)\,\frac{P_n^{(\frac12,-\frac12)}(x)}{P_n^{(\frac12,-\frac12)}(1)} = \frac{\sin((n+\frac12)\theta)}{\sin(\frac12\theta)}\,, \quad x = \cos\theta, \tag{93}$$

see [K26, Section 1.2.3]. Then there is the symmetry

$$V_n(-x) = (-1)^n W_n(x). \tag{94}$$

The names of Chebyshev polynomials of the third and fourth kind and the notation $V_n(x)$ are due to Gautschi [K14]. The notation $W_n(x)$ was first used by Mason [K25]. Names and notations for Chebyshev polynomials of the third and fourth kind are interchanged in [AAR, Remark 2.5.3] and [DLMF, Table 18.3.1].

9.9 Pseudo Jacobi (or Romanovski-Routh)

In this section in [KLS] the pseudo Jacobi polynomial $P_n(x;\nu,N)$ in (9.9.1) is considered for $N \in \mathbb{Z}_{\ge 0}$ and $n = 0,1,\ldots,n$. However, we can more generally take $-\frac12 < N \in \mathbb{R}$ (so here I overrule my convention formulated in the beginning of this paper), N_0 integer such that $N - \frac12 \le N_0 < N + \frac12$, and $n = 0,1,\ldots,N_0$ (see [382, §5, case A.4]). The orthogonality relation (9.9.2) is valid for $m,n = 0,1,\ldots,N_0$.

History These polynomials were first observed by Routh [K32] in 1885, but not as orthogonal polynomials (see Natanson [K28] about the history). Romanovski [463] (see also Lesky [382]) independently obtained them in 1929 as orthogonal polynomials.

Limit relation: Pseudo big q-Jacobi $\longrightarrow$ Pseudo Jacobi
See also (180).

References See also [Ism, §20.1], [51], [384], [K20], [K24], [K30].

9.10 Meixner

History In 1934 Meixner [406] (see (1.1) and case IV on pp. 10, 11 and 12) gave the orthogonality measure for the polynomials P_n given by the generating function

$$e^{xu(t)}\,f(t)=\sum_{n=0}^{\infty}P_n(x)\,\frac{t^n}{n!}\,,$$

where

$$e^{u(t)}=\left(\frac{1-\beta t}{1-\alpha t}\right)^{\frac{1}{\alpha-\beta}},\quad f(t)=\frac{(1-\beta t)^{\frac{k_2}{\beta(\alpha-\beta)}}}{(1-\alpha t)^{\frac{k_2}{\alpha(\alpha-\beta)}}}\quad (k_2<0;\ \alpha>\beta>0\ \text{ or }\ \alpha<\beta<0).$$

Then P_n can be expressed as a Meixner polynomial:

$$P_n(x)=(-k_2(\alpha\beta)^{-1})_n\,\beta^n\,M_n\left(-\frac{x+k_2\alpha^{-1}}{\alpha-\beta},-k_2(\alpha\beta)^{-1},\beta\alpha^{-1}\right).$$

In 1938 Gottlieb [K18, §2] introduces polynomials l_n "of Laguerre type" which turn out to be special Meixner polynomials: $l_n(x)=\mathrm{e}^{-n\lambda}M_n(x;1,e^{-\lambda})$.

Uniqueness of orthogonality measure The coefficient of $p_{n-1}(x)$ in (9.10.4) behaves as $O(n^2)$ as $n\to\infty$. Hence (2) holds, by which the orthogonality measure is unique.

9.11 Krawtchouk

Special values By (9.11.1) and the binomial formula:

$$K_n(0;p,N)=1,\qquad K_n(N;p,N)=(1-p^{-1})^n. \tag{95}$$

The self-duality (p.240, Remarks, first formula)

$$K_n(x;p,N)=K_x(n;p,N)\qquad (n,x\in\{0,1,\ldots,N\}) \tag{96}$$

combined with (95) yields:

$$K_N(x;p,N)=(1-p^{-1})^x\qquad (x\in\{0,1,\ldots,N\}). \tag{97}$$

Symmetry By the orthogonality relation (9.11.2):

$$\frac{K_n(N-x;p,N)}{K_n(N;p,N)}=K_n(x;1-p,N). \tag{98}$$

By (98) and (96) we have also

$$\frac{K_{N-n}(x;p,N)}{K_N(x;p,N)}=K_n(x;1-p,N)\qquad (n,x\in\{0,1,\ldots,N\}), \tag{99}$$

and, by (99), (98) and (95),

$$K_{N-n}(N-x;p,N) = \left(\frac{p}{p-1}\right)^{n+x-N} K_n(x;p,N) \qquad (n,x \in \{0,1,\ldots,N\}). \tag{100}$$

A particular case of (98) is:

$$K_n(N-x;\tfrac{1}{2},N) = (-1)^n K_n(x;\tfrac{1}{2},N). \tag{101}$$

Hence

$$K_{2m+1}(N;\tfrac{1}{2},2N) = 0. \tag{102}$$

From (9.11.11):

$$K_{2m}(N;\tfrac{1}{2},2N) = \frac{(\frac{1}{2})_m}{(-N+\frac{1}{2})_m}\,. \tag{103}$$

Quadratic transformations

$$K_{2m}(x+N;\tfrac{1}{2},2N) = \frac{(\frac{1}{2})_m}{(-N+\frac{1}{2})_m}\,R_m(x^2;-\tfrac{1}{2},-\tfrac{1}{2},N), \tag{104}$$

$$K_{2m+1}(x+N;\tfrac{1}{2},2N) = -\,\frac{(\frac{3}{2})_m}{N\,(-N+\frac{1}{2})_m}\,x\,R_m(x^2-1;\tfrac{1}{2},\tfrac{1}{2},N-1), \tag{105}$$

$$K_{2m}(x+N+1;\tfrac{1}{2},2N+1) = \frac{(\frac{1}{2})_m}{(-N-\frac{1}{2})_m}\,R_m(x(x+1);-\tfrac{1}{2},\tfrac{1}{2},N), \tag{106}$$

$$K_{2m+1}(x+N+1;\tfrac{1}{2},2N+1) = \frac{(\frac{3}{2})_m}{(-N-\frac{1}{2})_{m+1}}\,(x+\tfrac{1}{2})\,R_m(x(x+1);\tfrac{1}{2},-\tfrac{1}{2},N), \tag{107}$$

where R_m is a dual Hahn polynomial (9.6.1). For the proofs use (9.6.2), (9.11.2), (9.6.4) and (9.11.4).

Recurrence relation Formula (9.11.3) holds for $n=N$ if we replace there the term $p(N-n)K_{n+1}(x;p,N)$ by $(-x)_{N+1}/(p^N N!)$.

Generating functions

$$\sum_{x=0}^{N}\binom{N}{x}K_m(x;p,N)K_n(x;q,N)z^x$$
$$= \left(\frac{p-z+pz}{p}\right)^m \left(\frac{q-z+qz}{q}\right)^n (1+z)^{N-m-n} K_m\left(n;-\frac{(p-z+pz)(q-z+qz)}{z},N\right). \tag{108}$$

This follows immediately from Rosengren [K31, (3.5)], which goes back to Meixner [K27].

9.12 Laguerre

Notation Here the Laguerre polynomial is denoted by L_n^α instead of $L_n^{(\alpha)}$.

Hypergeometric representation

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right) \tag{109}$$

$$= \frac{(-x)^n}{n!}\,{}_2F_0\left(\begin{matrix}-n,-n-\alpha\\ -\end{matrix};-\frac{1}{x}\right) \tag{110}$$

$$= \frac{(-x)^n}{n!}\,C_n(n+\alpha;x), \tag{111}$$

where C_n in (111) is a Charlier polynomial. Formula (109) is (9.12.1). Then (110) follows by reversal of summation. Finally (111) follows by (110) and (123). It is also the remark on top of p.244 in [KLS], and it is essentially [416, (2.7.10)].

Uniqueness of orthogonality measure The coefficient of $p_{n-1}(x)$ in (9.12.4) behaves as $O(n^2)$ as $n \to \infty$. Hence (2) holds, by which the orthogonality measure is unique.

Special value

$$L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}\,. \tag{112}$$

Use (9.12.1) or see [DLMF, 18.6.1)].

Quadratic transformations

$$H_{2n}(x) = (-1)^n\,2^{2n}\,n!\,L_n^{-1/2}(x^2), \tag{113}$$

$$H_{2n+1}(x) = (-1)^n\,2^{2n+1}\,n!\,x\,L_n^{1/2}(x^2). \tag{114}$$

See p.244, Remarks, last two formulas. Or see [DLMF, (18.7.19), (18.7.20)].

Fourier transform

$$\frac{1}{\Gamma(\alpha+1)}\int_0^\infty \frac{L_n^\alpha(y)}{L_n^\alpha(0)}\,\mathrm{e}^{-y}\,y^\alpha\,\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y = \mathrm{i}^n\,\frac{y^n}{(iy+1)^{n+\alpha+1}}\,, \tag{115}$$

see [DLMF, (18.17.34)].

Differentiation formulas Each differentiation formula is given in two equivalent forms.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha L_n^\alpha(x)\right) = (n+\alpha)\,x^{\alpha-1}L_n^{\alpha-1}(x), \qquad \left(x\frac{\mathrm{d}}{\mathrm{d}x}+\alpha\right)L_n^\alpha(x) = (n+\alpha)\,L_n^{\alpha-1}(x). \tag{116}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}L_n^\alpha(x)\right) = -\mathrm{e}^{-x}L_n^{\alpha+1}(x), \qquad \left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)L_n^\alpha(x) = -L_n^{\alpha+1}(x). \tag{117}$$

Formulas (116) and (117) follow from [DLMF, (13.3.18), (13.3.20)] together with (9.12.1).

Generating functions The generating function (9.12.12) is a limit case of the generating function (60) for Jacobi polynomials by (9.8.16). By (9.7.14) the generating function (9.12.12) is also a limit case of the generating function (9.7.13) for Meixner–Pollaczek polynomials.

Generalized Hermite polynomials See [146, p.156], [K12, Section 1.5.1]. These are defined by

$$H^{\mu}_{2m}(x) := \text{const.}\, L_m^{\mu-\frac12}(x^2), \qquad H^{\mu}_{2m+1}(x) := \text{const.}\, x\, L_m^{\mu+\frac12}(x^2). \tag{118}$$

Then for $\mu > -\frac12$ we have orthogonality relation

$$\int_{-\infty}^{\infty} H^{\mu}_m(x)\, H^{\mu}_n(x)\, |x|^{2\mu} \mathrm{e}^{-x^2}\, \mathrm{d}x = 0 \qquad (m \neq n). \tag{119}$$

Let the Dunkl operator T_μ be defined by (70). If we choose the constants in (118) as

$$H^{\mu}_{2m}(x) = \frac{(-1)^m (2m)!}{(\mu+\frac12)_m}\, L_m^{\mu-\frac12}(x^2), \qquad H^{\mu}_{2m+1}(x) = \frac{(-1)^m (2m+1)!}{(\mu+\frac12)_{m+1}}\, x\, L_m^{\mu+\frac12}(x^2) \tag{120}$$

then (see [K6, (1.6)])

$$T_\mu H^{\mu}_n = 2n\, H^{\mu}_{n-1}. \tag{121}$$

Formula (121) with (120) substituted gives rise to two differentiation formulas involving Laguerre polynomials which are equivalent to (9.12.6) and (116).

Composition of (121) with itself gives

$$T_\mu^2 H^{\mu}_n = 4n(n-1)\, H^{\mu}_{n-2},$$

which is equivalent to the composition of (9.12.6) and (116):

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\alpha+1}{x}\,\frac{\mathrm{d}}{\mathrm{d}x}\right) L_n^{\alpha}(x^2) = -4(n+\alpha)\, L_{n-1}^{\alpha}(x^2). \tag{122}$$

9.13 Bessel

Hypergeometric representation The constraint $n = 0, 1, 2, \ldots, N$ can be omitted. All formulas in §9.13 except (9.13.2) remain valid for all integer $n \geq 0$. These more general values of n are even needed in the generating function (9.13.10).

Notation In the notation of Grosswald [255] the left-hand side of (9.13.1) has to be replaced by $y_n(x; a+2)$.

Orthogonality relation
Replace the constraint $a < -2N - 1$ in (9.13.2) by $m, n = 0, 1, \ldots, N = \lceil -(3+a)/2 \rceil$.
Following Lesky [382] the Bessel polynomials in case of orthogonality relation (9.13.2) may be called *Romanovski–Bessel polynomials*.

9.14 Charlier

Hypergeometric representation

$$C_n(x;a) = {}_2F_0\left(\begin{matrix}-n,-x\\ -\end{matrix};-\frac{1}{a}\right) \tag{123}$$

$$= \frac{(-x)_n}{a^n}\,{}_1F_1\left(\begin{matrix}-n\\ x-n+1\end{matrix};a\right) \tag{124}$$

$$= \frac{n!}{(-a)^n}\,L_n^{x-n}(a), \tag{125}$$

where $L_n^\alpha(x)$ is a Laguerre polynomial. Formula (123) is (9.14.1). Then (124) follows by reversal of the summation. Finally (125) follows by (124) and (9.12.1). It is also the Remark on p.249 of [KLS], and it was earlier given in [416, (2.7.10)].

Uniqueness of orthogonality measure The coefficient of $p_{n-1}(x)$ in (9.14.4) behaves as $O(n)$ as $n\to\infty$. Hence (2) holds, by which the orthogonality measure is unique.

9.15 Hermite

Uniqueness of orthogonality measure The coefficient of $p_{n-1}(x)$ in (9.15.4) behaves as $O(n)$ as $n\to\infty$. Hence (2) holds, by which the orthogonality measure is unique.

Fourier transforms

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H_n(y)\,\mathrm{e}^{-\frac12 y^2}\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y = \mathrm{i}^n\,H_n(x)\,\mathrm{e}^{-\frac12 x^2}, \tag{126}$$

see [AAR, (6.1.15) and Exercise 6.11].

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_n(y)\,\mathrm{e}^{-y^2}\,\mathrm{e}^{\mathrm{i}xy}dupy = \mathrm{i}^n\,x^n\,\mathrm{e}^{-\frac14 x^2}, \tag{127}$$

see [DLMF, (18.17.35)].

$$\frac{i^n}{2\sqrt{\pi}}\int_{-\infty}^{\infty} y^n\,\mathrm{e}^{-\frac14 y^2}\,\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}y = H_n(x)\,\mathrm{e}^{-x^2}, \tag{128}$$

see [AAR, (6.1.4)].

14.1 Askey–Wilson

Symmetry The Askey–Wilson polynomials $p_n(x;a,b,c,d\,|\,q)$ are symmetric in a,b,c,d.
This follows from the orthogonality relation (14.1.2) together with the value of its coefficient of x^n given in (14.1.5b). Alternatively, combine (14.1.1) with [GR, (III.15)].
As a consequence, it is sufficient to give generating function (14.1.13). Then the generating functions (14.1.14), (14.1.15) will follow by symmetry in the parameters.

Basic hypergeometric representation In addition to (14.1.1) we have (in notation (19)):

$$p_n(\cos\theta;a,b,c,d\,|\,q)=\frac{(ae^{-i\theta},be^{-i\theta},ce^{-i\theta},de^{-i\theta};q)_n}{(e^{-2i\theta};q)_n}\,e^{in\theta}\\ \times {}_8W_7\big(q^{-n}e^{2i\theta};ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta},q^{-n};q,q^{2-n}/(abcd)\big). \quad (129)$$

This follows from (14.1.1) by combining (III.15) and (III.19) in [GR]. It is also given in [513, (4.2)], but be aware for some slight errors. The symmetry in a,b,c,d is evident from (129).

Special value and different notation

$$p_n\big(\tfrac12(a+a^{-1});a,b,c,d\,|\,q\big)=a^{-n}\,(ab,ac,ad;q)_n\,, \quad (130)$$

and similarly for arguments $\frac12(b+b^{-1})$, $\frac12(c+c^{-1})$ and $\frac12(d+d^{-1})$ by symmetry of p_n in a,b,c,d. Formula (130) is an immediate consequence of (14.1.1).

We will also write

$$R_n(z;a,b,c,d\,|\,q):=\frac{p_n(\frac12(z+z^{-1});a,b,c,d\,|\,q)}{p_n(\frac12(a+a^{-1});a,b,c,d\,|\,q)}={}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right). \quad (131)$$

Here there is no longer full symmetry in a,b,c,d, only in b,c,d.

Trivial symmetry From (14.1.1) we see [72, (1.34)]

$$\begin{aligned}&p_n(x;a,b,c,d\,|\,q)=(-1)^np_n(-x;-a,-b,-c,-d\,|\,q),\\ &R_n(z;a,b,c,d\,|\,q)=R_n(-z;-a,-b,-c,-d\,|\,q).\end{aligned} \quad (132)$$

Duality Define parameters $\tilde a,\tilde b,\tilde c,\tilde d$ in terms of a,b,c,d by

$$\tilde a=(q^{-1}abcd)^{\frac12},\quad \tilde b=ab/\tilde a,\quad \tilde c=ac/\tilde a,\quad \tilde d=ad/\tilde a. \quad (133)$$

Jumping from one branch to the other branch in the square root in the formula for $\tilde a$ implies that $\tilde a,\tilde b,\tilde c,\tilde d$ move to $-\tilde a,-\tilde b,-\tilde c,-\tilde d$. Repetition of the parameter transformation recovers the original parameters up to a possible common multiplication of a,b,c,d by -1, while the branch choice for $\tilde a$ is irrelevant:

$$a=\big(q^{-1}\tilde a\tilde b\tilde c\tilde d\big)^{\frac12},\quad b=\tilde a\tilde b/a,\quad c=\tilde a\tilde c/a,\quad d=\tilde a\tilde d/a. \quad (134)$$

From (131) we have the duality relation

$$R_n\big(aq^m;a,b,c,d\,|\,q\big)=R_m\big(\tilde aq^n;\tilde a,\tilde b,\tilde c,\tilde d\,|\,q\big)\qquad (m,n\in\mathbb{Z}_{\ge0}). \quad (135)$$

By (132) both sides of (135) are invariant under common multiplication by -1 of a,b,c,d, respectively $\tilde a,\tilde b,\tilde c,\tilde d$.

Orthogonality relation The conditions on the parameters in (14.1.2) can be slightly relaxed: Let $|a|,|b|,|c|,|d|\le 1$ such that pairwise products of a,b,c,d are not equal to 1 and such that non-real parameters occur in complex conjugate pairs.

In fact, the only possible cases which then offend the condition $|a|,|b|,|c|,|d|<1$ are that either precisely one parameter has absolute value 1 and equals 1 or -1, or precisely two parameter values have absolute value 1, one equal to 1 and the other equal to -1. Then the weight fucntion will not cause a singularity by its factors $1\pm e^{i\theta}$ and $1\pm e^{-i\theta}$ in the denominator, since these are compensated by the factors $1-e^{2i\theta}$ and $1-e^{-2i\theta}$ in the numerator.

The orthogonality (14.1.3) involving discrete terms can be given for more general parameter values as in [72, Theorem 2.5]. There a,b,c,d are real or occur in complex conjugate pairs if non-real, and pairwise products have absolute value ≤ 1 but are not equal to 1.

Re: (14.1.5) Let

$$p_n(x):=\frac{p_n(x;a,b,c,d\,|\,q)}{2^n(abcdq^{n-1};q)_n}=x^n+\widetilde{k}_nx^{n-1}+\cdots\,. \tag{136}$$

Then

$$\widetilde{k}_n=-\frac{(1-q^n)(a+b+c+d-(abc+abd+acd+bcd)q^{n-1})}{2(1-q)(1-abcdq^{2n-2})}\,. \tag{137}$$

This follows because $\tilde{k}_n-\tilde{k}_{n+1}$ equals the coefficient $\frac12\big(a+a^{-1}-(A_n+C_n)\big)$ of $p_n(x)$ in (14.1.5).

q-Difference equation The q-difference operator acting on $P_n(z)$ on the right-hand side of (14.1.7), gives, when acting on $Q_n(z):=(az,az^{-1};q)_\infty$, the result

$$\begin{aligned}q^{-n}(1-q^n)(1-abcdq^{n-1})Q_n(z)-q^{-n}(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-q^n)Q_{n-1}(z)\\ =A(z)Q_n(qz)-\big(A(z)+A(z^{-1})\big)Q_n(z)+A(z^{-1})Q_n(q^{-1}z).\end{aligned} \tag{138}$$

This formula is implicit in [K36]. Use there (3.1) with the Askey–Wilson parameters (7.15) and (7.8), and combine it with (14.1.7).

Generating functions Rahman [449, (4.1), (4.9)] gives:

$$\begin{aligned}&\sum_{n=0}^\infty\frac{(abcdq^{-1};q)_na^n}{(ab,ac,ad,q;q)_n}\,t^n\,p_n(\cos\theta;a,b,c,d\,|\,q)\\
&=\frac{(abcdtq^{-1};q)_\infty}{(t;q)_\infty}\,{}_6\phi_5\left(\begin{matrix}(abcdq^{-1})^{\frac12},-(abcdq^{-1})^{\frac12},(abcd)^{\frac12},-(abcd)^{\frac12},ae^{i\theta},ae^{-i\theta}\\ ab,ac,ad,abcdtq^{-1},qt^{-1}\end{matrix};q,q\right)\\
&+\frac{(abcdq^{-1},abt,act,adt,ae^{i\theta},ae^{-i\theta};q)_\infty}{(ab,ac,ad,t^{-1},ate^{i\theta},ate^{-i\theta};q)_\infty}\\
&\times{}_6\phi_5\left(\begin{matrix}t(abcdq^{-1})^{\frac12},-t(abcdq^{-1})^{\frac12},t(abcd)^{\frac12},-t(abcd)^{\frac12},ate^{i\theta},ate^{-i\theta}\\ abt,act,adt,abcdt^2q^{-1},qt\end{matrix};q,q\right)\quad(|t|<1).\end{aligned} \tag{139}$$

In the limit (140) the first term on the right-hand side of (139) tends to the left-hand side of (9.1.15), while the second term tends formally to 0. The special case $ad=bc$ of (139) was earlier given in [236, (4.1), (4.6)].

Limit relations

Askey–Wilson $\longrightarrow$ Wilson

Instead of (14.1.21) we can keep a polynomial of degree n while the limit is approached:

$$\lim_{q\to1}\frac{p_n(1-\frac12 x(1-q)^2;q^a,q^b,q^c,q^d\,|\,q)}{(1-q)^{3n}}=W_n(x;a,b,c,d). \tag{140}$$

For the proof first derive the corresponding limit for the monic polynomials by comparing (14.1.5) with (9.4.4).

Askey–Wilson $\longrightarrow$ Continuous Hahn

Instead of (14.4.15) we can keep a polynomial of degree n while the limit is approached:

$$\lim_{q\uparrow1}\frac{p_n\big(\cos\phi-x(1-q)\sin\phi;q^ae^{i\phi},q^be^{i\phi},q^{\overline{a}}e^{-i\phi},q^{\overline{b}}e^{-i\phi}\,|\,q\big)}{(1-q)^{2n}}$$
$$=(-2\sin\phi)^n\,n!\,p_n(x;a,b,\overline{a},\overline{b})\qquad(0<\phi<\pi). \tag{141}$$

Here the right-hand side has a continuous Hahn polynomial (9.4.1). For the proof first derive the corresponding limit for the monic polynomials by comparing (14.1.5) with (9.1.5). In fact, define the monic polynomial

$$\widetilde{p}_n(x):=\frac{p_n\big(\cos\phi-x(1-q)\sin\phi;q^ae^{i\phi},q^be^{i\phi},q^{\overline{a}}e^{-i\phi},q^{\overline{b}}e^{-i\phi}\,|\,q\big)}{(-2(1-q)\sin\phi)^n\,(abcdq^{n-1};q)_n}\,.$$

Then it follows from (14.1.5) that

$$x\,\widetilde{p}_n(x)=\widetilde{p}_{n+1}(x)+\frac{(1-q^a)e^{i\phi}+(1-q^{-a})e^{-i\phi}+\widetilde{A}_n+\widetilde{C}_n}{2(1-q)\sin\phi}\,\widetilde{p}_n(x)+\frac{\widetilde{A}_{n-1}\widetilde{C}_n}{(1-q)^2\sin^2\phi}\,\widetilde{p}_{n-1}(x),$$

where $\widetilde{A}_n$ and $\widetilde{C}_n$ are as given after (14.1.3) with a,b,c,d replaced by $q^ae^{i\phi},q^be^{i\phi},q^{\overline{a}}e^{-i\phi},q^{\overline{b}}e^{-i\phi}$. Then the recurrence equation for $\widetilde{p}_n(x)$ tends for $q\uparrow1$ to the recurrence equation (9.4.4) with $c=\overline{a}$, $d=\overline{b}$.

Askey–Wilson $\longrightarrow$ Meixner–Pollaczek

Instead of (14.9.15) we can keep a polynomial of degree n while the limit is approached:

$$\lim_{q\uparrow1}\frac{p_n\big(\cos\phi-x(1-q)\sin\phi;q^\lambda e^{i\phi},0,q^\lambda e^{-i\phi},0\,|\,q\big)}{(1-q)^n}=n!\,P_n^{(\lambda)}(x;\pi-\phi)\quad(0<\phi<\pi). \tag{142}$$

Here the right-hand side has a Meixner–Pollaczek polynomial (9.7.1). For the proof first derive the corresponding limit for the monic polynomials by comparing (14.1.5) with (9.7.4). In fact, define the monic polynomial

$$\widetilde{p}_n(x):=\frac{p_n\big(\cos\phi-x(1-q)\sin\phi;q^\lambda e^{i\phi},0,q^\lambda e^{-i\phi},0\,|\,q\big)}{(-2(1-q)\sin\phi)^n}\,.$$

Then it follows from (14.1.5) that

$$x\,\widetilde{p}_n(x) = \widetilde{p}_{n+1}(x) + \frac{(1-q^\lambda)e^{i\phi} + (1-q^{-\lambda})e^{-i\phi} + \widetilde{A}_n + \widetilde{C}_n}{2(1-q)\sin\phi}\,\widetilde{p}_n(x) + \frac{\widetilde{A}_{n-1}\widetilde{C}_n}{(1-q)^2\sin^2\phi}\,\widetilde{p}_{n-1}(x),$$

where $\widetilde{A}_n$ and $\widetilde{C}_n$ are as given after (14.1.3) with a,b,c,d replaced by $q^\lambda e^{i\phi}, 0, q^\lambda e^{-i\phi}, 0$. Then the recurrence equation for $\widetilde{p}_n(x)$ tends for $q \uparrow 1$ to the recurrence equation (9.7.4).

References See also Koornwinder [K21].

14.2 q-Racah

Symmetry

$$R_n(x;\alpha,\beta,q^{-N-1},\delta\,|\,q) = \frac{(\beta q, \alpha\delta^{-1}q;q)_n}{(\alpha q, \beta\delta q;q)_n}\,\delta^n\,R_n(\delta^{-1}x;\beta,\alpha,q^{-N-1},\delta^{-1}\,|\,q). \tag{143}$$

This follows from (14.2.1) combined with [GR, (III.15)].
In particular,

$$R_n(x;\alpha,\beta,q^{-N-1},-1\,|\,q) = \frac{(\beta q, -\alpha q;q)_n}{(\alpha q, -\beta q;q)_n}\,(-1)^n\,R_n(-x;\beta,\alpha,q^{-N-1},-1\,|\,q), \tag{144}$$

and

$$R_n(x;\alpha,\alpha,q^{-N-1},-1\,|\,q) = (-1)^n\,R_n(-x;\alpha,\alpha,q^{-N-1},-1\,|\,q), \tag{145}$$

Trivial symmetry Clearly from (14.2.1):

$$R_n(x;\alpha,\beta,\gamma,\delta\,|\,q) = R_n(x;\beta\delta,\alpha\delta^{-1},\gamma,\delta\,|\,q) = R_n(x;\gamma,\alpha\beta\gamma^{-1},\alpha,\gamma\delta\alpha^{-1}\,|\,q). \tag{146}$$

For $\alpha = q^{-N-1}$ this shows that the three cases $\alpha q = q^{-N}$ or $\beta\delta q = q^{-N}$ or $\gamma q = q^{-N}$ of (14.2.1) are not essentially different.

Duality It follows from (14.2.1) that

$$R_n(q^{-y}+\gamma\delta q^{y+1};q^{-N-1},\beta,\gamma,\delta\,|\,q) = R_y(q^{-n}+\beta q^{n-N};\gamma,\delta,q^{-N-1},\beta\,|\,q) \quad (n,y=0,1,\ldots,N). \tag{147}$$

14.3 Continuous dual q-Hahn

The continuous dual q-Hahn polynomials are the special case $d=0$ of the Askey–Wilson polynomials:

$$p_n(x;a,b,c\,|\,q) := p_n(x;a,b,c,0\,|\,q).$$

Hence all formulas in §14.3 are specializations for $d=0$ of formulas in §14.1.

14.4 Continuous q-Hahn

The continuous q-Hahn polynomials are the special case of Askey–Wilson polynomials with parameters $ae^{i\phi}, be^{i\phi}, ae^{-i\phi}, be^{-i\phi}$:

$$p_n(x;a,b,\phi\,|\,q) := p_n(x;ae^{i\phi},be^{i\phi},ae^{-i\phi},be^{-i\phi}\,|\,q).$$

In [72, (4.29)] and [GR, (7.5.43)] (who write $p_n(x;a,b\,|\,q)$, $x=\cos(\theta+\phi)$) and in [KLS, §14.4] (who writes $p_n(x;a,b,c,d;q)$, $x=\cos(\theta+\phi)$) the parameter dependence on ϕ is incorrectly omitted.

Since all formulas in §14.4 are specializations of formulas in §14.1, there is no real need to give these specializations explicitly. In particular, the limit (14.4.15) is in fact a limit from Askey–Wilson to continuous Hahn. See also (141).

14.5 Big q-Jacobi

Different notation See p.442, Remarks:

$$P_n(x;a,b,c,d;q) := P_n(qac^{-1}x;a,b,-ac^{-1}d;q) = {}_3\phi_2\left(\begin{matrix} q^{-n},q^{n+1}ab,qac^{-1}x \\ qa,-qac^{-1}d \end{matrix};q,q\right). \tag{148}$$

Furthermore,

$$P_n(x;a,b,c,d;q) = P_n(\lambda x;a,b,\lambda c,\lambda d;q), \tag{149}$$

$$P_n(x;a,b,c;q) = P_n(-q^{-1}c^{-1}x;a,b,-ac^{-1},1;q) \tag{150}$$

Orthogonality relation (equivalent to (14.5.2), see also [K22, (2.42), (2.41), (2.36), (2.35)]). Let $c,d>0$ and either $a\in(-c/(qd),1/q)$, $b\in(-d/(cq),1/q)$ or $a/c=-\overline{b}/d\notin\mathbb{R}$. Then

$$\int_{-d}^{c} P_m(x;a,b,c,d;q)P_n(x;a,b,c,d;q)\,\frac{(qx/c,-qx/d;q)_\infty}{(qax/c,-qbx/d;q)_\infty}\,d_qx = h_n\,\delta_{m,n}\,, \tag{151}$$

where

$$\frac{h_n}{h_0} = q^{\frac12 n(n-1)}\left(\frac{q^2a^2d}{c}\right)^n \frac{1-qab}{1-q^{2n+1}ab}\,\frac{(q,qb,-qbc/d;q)_n}{(qa,qab,-qad/c;q)_n} \tag{152}$$

and

$$h_0 = (1-q)c\,\frac{(q,-d/c,-qc/d,q^2ab;q)_\infty}{(qa,qb,-qbc/d,-qad/c;q)_\infty}\,. \tag{153}$$

Other hypergeometric representation and asymptotics

$$P_n(x;a,b,c,d;q) = \frac{(-qbd^{-1}x;q)_n}{(-q^{-n}a^{-1}cd^{-1};q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},q^{-n}b^{-1},cx^{-1} \\ qa,-q^{-n}b^{-1}dx^{-1} \end{matrix};q,q\right) \tag{154}$$

$$= (qac^{-1}x)^n\,\frac{(qb,cx^{-1};q)_n}{(qa,-qac^{-1}d;q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},q^{-n}a^{-1},-qbd^{-1}x \\ qb,q^{1-n}c^{-1}x \end{matrix};q,-q^{n+1}ac^{-1}d\right) \tag{155}$$

$$= (qac^{-1}x)^n\,\frac{(qb,q;q)_n}{(-qac^{-1}d;q)_n}\sum_{k=0}^{n}\frac{(cx^{-1};q)_{n-k}}{(q,qa;q)_{n-k}}\,\frac{(-qbd^{-1}x;q)_k}{(qb,q;q)_k}\,(-1)^kq^{\frac12 k(k-1)}(-dx^{-1})^k. \tag{156}$$

Formula (154) follows from (148) by [GR, (III.11)] and next (155) follows by series inversion [GR, Exercise 1.4(ii)]. Formulas (154) and (156) are also given in [Ism, (18.4.28), (18.4.29)]. It follows from (155) or (156) that (see [298, (1.17)] or [Ism, (18.4.31)])

$$\lim_{n\to\infty}(qac^{-1}x)^{-n}P_n(x;a,b,c,d;q)=\frac{(cx^{-1},-dx^{-1};q)_\infty}{(-qac^{-1}d,qa;q)_\infty}\,, \tag{157}$$

uniformly for x in compact subsets of $\mathbb{C}\backslash\{0\}$. (Exclusion of the spectral points $x=cq^m,dq^m$ ($m=0,1,2,\ldots$), as was done in [298] and [Ism], is not necessary. However, while (157) yields 0 at these points, a more refined asymptotics at these points is given in [298] and [Ism].) For the proof of (157) use that

$$\lim_{n\to\infty}(qac^{-1}x)^{-n}P_n(x;a,b,c,d;q)=\frac{(qb,cx^{-1};q)_n}{(qa,-qac^{-1}d;q)_n}\,{}_1\phi_1\left(\begin{matrix}-qbd^{-1}x\\ qb\end{matrix};q,-dx^{-1}\right), \tag{158}$$

which can be evaluated by [GR, (II.5)]. Formula (158) follows formally from (155), and it follows rigorously, by dominated convergence, from (156).

Symmetry (see [K22, §2.5] and combine with (148)).

$$\frac{P_n(x;a,b,c,d;q)}{P_n(-d/(qb);a,b,c,d;q)}=P_n(-x;b,a,d,c;q)=P_n(x;-bcd^{-1},-ac^{-1}d,c,d;q). \tag{159}$$

In particular (*symmetric big q-Jacobi polynomials*),

$$P_n(-x;a,a,1,1;q)=(-1)^nP_n(x;a,a,1,1;q). \tag{160}$$

Special values

$$P_n(c/(qa);a,b,c,d;q)=1, \tag{161}$$

$$P_n(-d/(qb);a,b,c,d;q)=\left(-\frac{ad}{bc}\right)^n\frac{(qb,-qbc/d;q)_n}{(qa,-qad/c;q)_n}\,, \tag{162}$$

$$P_n(c;a,b,c,d;q)=q^{\frac12 n(n+1)}\left(\frac{ad}{c}\right)^n\frac{(-qbc/d;q)_n}{(-qad/c;q)_n}\,, \tag{163}$$

$$P_n(-d;a,b,c,d;q)=q^{\frac12 n(n+1)}(-a)^n\frac{(qb;q)_n}{(qa;q)_n}\,. \tag{164}$$

Recurrence relation See (14.5.3). For $n=1,2,\ldots$:

$$\begin{aligned}qac^{-1}xP_n(x;a,b,c,d;q)&=A_nP_{n+1}(x;a,b,c,d;q)\\&\quad+(1-A_n-C_n)P_n(x;a,b,c,d;q)+C_nP_{n-1}(x;a,b,c,d;q),\end{aligned} \tag{165}$$

where

$$A_n=\frac{(1-q^{n+1}a)(1-q^{n+1}ab)(1+q^{n+1}ac^{-1}d)}{(1-q^{2n+1}ab)(1-q^{2n+2}ab)}\,,$$

$$C_n=q^{n+1}a^2c^{-1}d\,\frac{(1-q^n)(1+q^nbcd^{-1})(1-q^nb)}{(1-q^{2n}ab)(1-q^{2n+1}ab)}\,.$$

For $n=0$:

$$qac^{-1}xP_0(x;a,b,c,d;q) = \frac{(1-qa)(1+qac^{-1}d)}{1-q^2ab}\,P_1(x;a,b,c,d;q) + \frac{qa(c-d-q(bc-ad))}{c(1-q^2ab)}\,P_0(x;a,b,c,d;q). \tag{166}$$

In (165) we have $1-A_n-C_n=0$ for $n=1,2,\ldots$ if $a=b$, $c=d$ or $ab=1$, $acd^{-1}=1$. In (166) the last term on the right vanishes if $a=b$, $c=d$, but not if $ab=1$, $acd^{-1}=1$, $a\neq 1$.

So for symmetric big q-Jacobi polynomials we have

$$qaxP_n(x;a,a,1,1;q) = \frac{1-q^{n+1}a^2}{1-q^{2n+1}a^2}\,P_{n+1}(x;a,a,1,1;q) + q^{n+1}a^2\,\frac{1-q^n}{1-q^{2n+1}a^2}\,P_{n-1}(x;a,a,1,1;q). \tag{167}$$

Equivalently,

$$xp_n(x) = \frac{1-q^{n+1}a^2}{1-q^{2n+1}a^2}\,p_{n+1}(x) + \frac{q^{n-1}(1-q^n)}{1-q^{2n+1}a^2}\,p_{n-1}(x), \tag{168}$$

where $p_n(x) = (qa)^{-n}P_n(x;a,a,1,1;q)$.

Second order q-difference equation (see (14.5.5). Let $P_n(x) = P_n(x;a,b,c,d;q)$.

$$(q^{-n}-1)(1-q^{n+1}ab)P_n(x) = qabx^{-2}(x-q^{-1}a^{-1}c)(x+q^{-1}b^{-1}d)(P_n(qx)-P_n(x)) + x^{-2}(x-c)(x+d)(P_n(q^{-1}x)-P_n(x)). \tag{169}$$

Quadratic transformations (see [K22, (2.48), (2.49)] and (200)).
These express big q-Jacobi polynomials $P_m(x;a,a,1,1;q)$ in terms of little q-Jacobi polynomials (see §14.12).

$$P_{2n}(x;a,a,1,1;q) = \frac{p_n(x^2;q^{-1},a^2;q^2)}{p_n((qa)^{-2};q^{-1},a^2;q^2)}\,, \tag{170}$$

$$P_{2n+1}(x;a,a,1,1;q) = \frac{qax\,p_n(x^2;q,a^2;q^2)}{p_n((qa)^{-2};q,a^2;q^2)}\,. \tag{171}$$

Hence, by (14.12.1), [GR, Exercise 1.4(ii)] and (200),

$$P_n(x;a,a,1,1;q) = \frac{(qa^2;q^2)_n}{(qa^2;q)_n}\,(qax)^n\,{}_2\phi_1\left(\begin{matrix}q^{-n},q^{-n+1}\\ q^{-2n+1}a^{-2}\end{matrix};q^2,(ax)^{-2}\right) \tag{172}$$

$$= \frac{(q;q)_n}{(qa^2;q)_n}\,(qa)^n\sum_{k=0}^{[\frac12 n]}(-1)^kq^{k(k-1)}\,\frac{(qa^2;q^2)_{n-k}}{(q^2;q^2)_k\,(q;q)_{n-2k}}\,x^{n-2k}. \tag{173}$$

q-Chebyshev polynomials In (148), with $c = d = 1$, the cases $a = b = q^{-\frac{1}{2}}$ and $a = b = q^{\frac{1}{2}}$ can be considered as q-analogues of the Chebyshev polynomials of the first and second kind, respectively (§9.8.2) because of the limit (14.5.17). The quadratic relations (170), (171) can also be specialized to these cases. The definition of the q-Chebyshev polynomials may vary by normalization and by dilation of argument. They were considered in [K4]. By [24, p.279] and (170), (171), the *Al-Salam-Ismail polynomials* $U_n(x; a, b)$ (q-dependence suppressed) in the case $a = q$ can be expressed as q-Chebyshev polynomials of the second kind:

$$U_n(x, q, b) = (q^{-3}b)^{\frac{1}{2}n}\,\frac{1-q^{n+1}}{1-q}\,P_n(b^{-\frac{1}{2}}x; q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1, 1; q).$$

Similarly, by [K8, (5.4), (5.1), (5.3)] and (170), (171), Cigler's q-Chebyshev polynomials $T_n(x, s, q)$ and $U_n(x, s, q)$ can be expressed in terms of the q-Chebyshev cases of (148):

$$\begin{aligned} T_n(x, s, q) &= (-s)^{\frac{1}{2}n}\,P_n((-qs)^{-\frac{1}{2}}x; q^{-\frac{1}{2}}, q^{-\frac{1}{2}}, 1, 1; q),\\ U_n(x, s, q) &= (-q^{-2}s)^{\frac{1}{2}n}\,\frac{1-q^{n+1}}{1-q}\,P_n((-qs)^{-\frac{1}{2}}x; q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1, 1; q). \end{aligned}$$

Limit to Discrete q-Hermite I

$$\lim_{a\to 0} a^{-n}\,P_n(x; a, a, 1, 1; q) = q^n\,h_n(x; q). \tag{174}$$

Here $h_n(x; q)$ is given by (14.28.1). For the proof of (174) use (154).

Pseudo big q-Jacobi polynomials Let $a, b, c, d \in \mathbb{C}$, $z_+ > 0$, $z_- < 0$ such that $\frac{(ax,bx;q)_\infty}{(cx,dx;q)_\infty} > 0$ for $x \in z_-q^{\mathbb{Z}} \cup z_+q^{\mathbb{Z}}$. Then $(ab)/(qcd) > 0$. Assume that $(ab)/(qcd) < 1$. Let N be the largest nonnegative integer such that $q^{2N} > (ab)/(qcd)$. Then

$$\int_{z_-q^{\mathbb{Z}}\cup z_+q^{\mathbb{Z}}} P_m(cx; c/b, d/a, c/a; q)\,P_n(cx; c/b, d/a, c/a; q)\,\frac{(ax, bx; q)_\infty}{(cx, dx; q)_\infty}\,d_qx = h_n\delta_{m,n} \quad (m, n = 0, 1, \ldots, N), \tag{175}$$

where

$$\frac{h_n}{h_0} = (-1)^n \left(\frac{c^2}{ab}\right)^n q^{\frac{1}{2}n(n-1)}q^{2n}\,\frac{(q, qd/a, qd/b; q)_n}{(qcd/(ab), qc/a, qc/b; q)_n}\,\frac{1-qcd/(ab)}{1-q^{2n+1}cd/(ab)} \tag{176}$$

and

$$h_0 = \int_{z_-q^{\mathbb{Z}}\cup z_+q^{\mathbb{Z}}} \frac{(ax, bx; q)_\infty}{(cx, dx; q)_\infty}\,d_qx = (1-q)z_+\,\frac{(q, a/c, a/d, b/c, b/d; q)_\infty}{(ab/(qcd); q)_\infty}\,\frac{\theta(z_-/z_+, cdz_-z_+; q)}{\theta(cz_-, dz_-, cz_+, dz_+; q)}\,. \tag{177}$$

See Groenevelt & Koelink [K19, Prop. 2.2]. Formula (177) was first given by Slater [K34, (5)] as an evaluation of a sum of two ${}_2\psi_2$ series. The same formula is given in Slater [471, (7.2.6)] and in [GR, Exercise 5.10], but in both cases with the same slight error, see [K19, 2nd paragraph after Lemma 2.1] for correction. The theta function is given by (20). Note that

$$P_n(cx; c/b, d/a, c/a; q) = P_n(-q^{-1}ax; c/b, d/a, -a/b, 1; q). \tag{178}$$

In [K17] the weights of the pseudo big q-Jacobi polynomials occur in certain measures on the space of N-point configurations on the so-called extended Gelfand-Tsetlin graph.

Limit relations

Pseudo big q-Jacobi $\longrightarrow$ Discrete Hermite II

$$\lim_{a\to\infty} i^n q^{\frac12 n(n-1)} P_n(q^{-1}a^{-1}ix; a, a, 1, 1; q) = \widetilde{h}_n(x; q). \tag{179}$$

For the proof use (173) and (235). Note that $P_n(q^{-1}a^{-1}ix; a, a, 1, 1; q)$ is obtained from the right-hand side of (178) by replacing a, b, c, d by $-ia^{-1}, ia^{-1}, i, -i$.

Pseudo big q-Jacobi $\longrightarrow$ Pseudo Jacobi

$$\lim_{q\uparrow 1} P_n(iq^{\frac12(-N-1+i\nu)}x; -q^{-N-1}, -q^{-N-1}, q^{-N+i\nu-1}; q) = \frac{P_n(x; \nu, N)}{P_n(-i; \nu, N)}. \tag{180}$$

Here the big q-Jacobi polynomial on the left-hand side equals $P_n(cx; c/b, d/a, c/a; q)$ with $a = iq^{\frac12(N+1-i\nu)}$, $b = -iq^{\frac12(N+1+i\nu)}$, $c = iq^{\frac12(-N-1+i\nu)}$, $d = -iq^{\frac12(-N-1-i\nu)}$.

14.7 Dual q-Hahn

Orthogonality relation More generally we have (14.7.2) with positive weights in any of the following cases: (i) $0 < \gamma q < 1$, $0 < \delta q < 1$; (ii) $0 < \gamma q < 1$, $\delta < 0$; (iii) $\gamma < 0$, $\delta > q^{-N}$; (iv) $\gamma > q^{-N}$, $\delta > q^{-N}$; (v) $0 < q\gamma < 1$, $\delta = 0$. This also follows by inspection of the positivity of the coefficient of $p_{n-1}(x)$ in (14.7.4). Case (v) yields Affine q-Krawtchouk in view of (14.7.13).

Symmetry

$$R_n(x; \gamma, \delta, N \,|\, q) = \frac{(\delta^{-1}q^{-N}; q)_n}{(\gamma q; q)_n} \left(\gamma\delta q^{N+1}\right)^n R_n(\gamma^{-1}\delta^{-1}q^{-1-N}x; \delta^{-1}q^{-N-1}, \gamma^{-1}q^{-N-1}, N \,|\, q). \tag{181}$$

This follows from (14.7.1) combined with [GR, (III.11)].

14.8 Al-Salam–Chihara

Standardization and notation The definition (14.8.1) by q-hypergeometric representation follows the convention of [72, p.25] that $Q_n(x; a, b \,|\, q) = p_n(x; a, b, 0, 0 \,|\, q)$, where $p_n(x; a, b, c, d \,|\, q)$ is the Askey–Wilson polynomial (14.1.1). In [Ism, (15.1.6)] these polynomials are notated $p_n(x; a, b \,|\, q)$, equal to $a^n/(ab; q)_n$ times $Q_n(x; a, b \,|\, q)$ as in (14.8.1).

Symmetry The Al-Salam–Chihara polynomials $Q_n(x; a, b \,|\, q)$ are symmetric in a, b.

This follows from the orthogonality relation (14.8.2) together with the value of its coefficient of x^n given in (14.8.5b).

Orthogonality relation Just as in Section 14.1 the condition $|a|, |b| < 1$ on the parameters in (14.8.2) can be slightly relaxed into $|a|, |b| \le 1$, $ab \ne 1$.

q^{-1}-Al-Salam–Chihara

Re: (14.8.1) For $x \in \mathbb{Z}_{\geq 0}$:

$$
\begin{aligned}
Q_n(\tfrac{1}{2}(aq^{-x}+a^{-1}q^x);a,b\,|\,q^{-1}) &= (-1)^n b^n q^{-\frac{1}{2}n(n-1)}\,\big((ab)^{-1};q\big)_n \\
&\qquad\times {}_3\phi_1\left(\begin{matrix} q^{-n}, q^{-x}, a^{-2}q^x \\ (ab)^{-1} \end{matrix}; q, q^n ab^{-1}\right) \qquad (182)\\
&= (-ab^{-1})^x\, q^{-\frac{1}{2}x(x+1)}\,\frac{(qba^{-1};q)_x}{(a^{-1}b^{-1};q)_x}\,{}_2\phi_1\left(\begin{matrix} q^{-x}, a^{-2}q^x \\ qba^{-1} \end{matrix}; q, q^{n+1}\right) \qquad (183)\\
&= (-ab^{-1})^x\, q^{-\frac{1}{2}x(x+1)}\,\frac{(qba^{-1};q)_x}{(a^{-1}b^{-1};q)_x}\,p_x(q^n; ba^{-1},(qab)^{-1};q). \qquad (184)
\end{aligned}
$$

Formula (182) follows from the first identity in (14.8.1). Next (183) follows from [GR, (III.8)]. Finally (184) gives the little q-Jacobi polynomials (14.12.1). See also [79, §3] and [K9, §3].

Orthogonality

$$
\begin{aligned}
\sum_{x=0}^{\infty} \frac{(1-q^{2x}a^{-2})(a^{-2},(ab)^{-1};q)_x}{(1-a^{-2})(q,bqa^{-1};q)_x}\,(ba^{-1})^x q^{x^2}\,(Q_mQ_n)(\tfrac{1}{2}(aq^{-x}+a^{-1}q^x);a,b\,|\,q^{-1})\\
= \frac{(qa^{-2};q)_\infty}{(ba^{-1}q;q)_\infty}\,(q,(ab)^{-1};q)_n\,(ab)^n q^{-n^2}\,\delta_{m,n}. \qquad (185)
\end{aligned}
$$

The constraints for having positive weights in (185) are $(ab)^{-1} < 1$, $0 < qa^{-1}b < 1$. Equivalently, we are in one of the following cases:

1. $a, b > 0$, $ab > 1$, $qa^{-1}b < 1$.
2. $a, b < 0$, $ab > 1$, $qa^{-1}b < 1$.
3. $a = \mathrm{i}a_0$, $b = \mathrm{i}b_0$, $a_0, b_0 > 0$, $qa_0^{-1}b_0 < 1$.
4. $a = -\mathrm{i}a_0$, $b = -\mathrm{i}b_0$, $a_0, b_0 > 0$, $qa_0^{-1}b_0 < 1$.

Formula (185) with constraints follows from (184) together with (14.12.2) and the completeness of the orthogonal system of the little q-Jacobi polynomials, See also [79, §3]. An alternative proof is given in [64]. There combine (3.82) with (3.81), (3.67), (3.40).

Normalized recurrence relation

$$
xp_n(x) = p_{n+1}(x) + \tfrac{1}{2}(a+b)q^{-n}p_n(x) + \tfrac{1}{4}(q^{-n}-1)(abq^{-n+1}-1)p_{n-1}(x), \qquad (186)
$$

where

$$
Q_n(x;a,b\,|\,q^{-1}) = 2^n p_n(x).
$$

Limit to Big q^{-1}-Hermite In (184) and (185) replace (a,b) by $(\mathrm{i}b^{-\frac{1}{2}}, \mathrm{i}ab^{-\frac{1}{2}})$ with $0 < aq < 1$ and $b > 0$. Then let $b \downarrow 0$. By (14.8.17) and (14.12.14) we arrive at big q^{-1}-Hermite polynomials as duals of q-Bessel polynomials.

14.9 q-Meixner–Pollaczek

The q-Meixner–Pollaczek polynomials are the special case of Askey–Wilson polynomials with parameters $ae^{i\phi}, 0, ae^{-i\phi}, 0$:

$$P_n(x;a,\phi\,|\,q) := \frac{1}{(q;q)_n}\,p_n(x;ae^{i\phi},0,ae^{-i\phi},0\,|\,q) \quad (x=\cos(\theta+\phi)).$$

In [KLS, §14.9] the parameter dependence on ϕ is incorrectly omitted.

Since all formulas in §14.9 are specializations of formulas in §14.1, there is no real need to give these specializations explicitly. See also (142).

There is an error in [KLS, (14.9.6), (14.9.8)]. Read $x=\cos(\theta+\phi)$ instead of $x=\cos\theta$.

14.10 Continuous q-Jacobi

Symmetry

$$P_n^{(\alpha,\beta)}(-x\,|\,q) = (-1)^n q^{\frac12(\alpha-\beta)n}\,P_n^{(\beta,\alpha)}(x\,|\,q). \tag{187}$$

This follows from (132) and (14.1.19).

14.10.1 Continuous q-ultraspherical / Rogers

Re: (14.10.17)

$$C_n(\cos\theta;\beta\,|\,q) = \frac{(\beta^2;q)_n}{(q;q)_n}\,\beta^{-\frac12 n}\,{}_4\phi_3\left(\begin{matrix} q^{-\frac12 n}, \beta q^{\frac12 n}, \beta^{\frac12}e^{i\theta}, \beta^{\frac12}e^{-i\theta} \\ -\beta, \beta^{\frac12}q^{\frac14}, -\beta^{\frac12}q^{\frac14}\end{matrix};q^{\frac12},q^{\frac12}\right), \tag{188}$$

see [GR, (7.4.13), (7.4.14)].

Special value (see [63, (3.23)])

$$C_n\big(\tfrac12(\beta^{\frac12}+\beta^{-\frac12});\beta\,|\,q\big) = \frac{(\beta^2;q)_n}{(q;q)_n}\,\beta^{-\frac12 n}. \tag{189}$$

Re: (14.10.21) (another q-difference equation). Let $C_n[e^{i\theta};\beta\,|\,q] := C_n(\cos\theta;\beta\,|\,q)$.

$$\frac{1-\beta z^2}{1-z^2}\,C_n[q^{\frac12}z;\beta\,|\,q] + \frac{1-\beta z^{-2}}{1-z^{-2}}\,C_n[q^{-\frac12}z;\beta\,|\,q] = (q^{-\frac12 n}+q^{\frac12 n}\beta)\,C_n[z;\beta\,|\,q], \tag{190}$$

see [351, (6.10)].

Re: (14.10.23) This can also be written as

$$C_n[q^{\frac12}z;\beta\,|\,q] - C_n[q^{-\frac12}z;\beta\,|\,q] = q^{-\frac12 n}(\beta-1)(z-z^{-1})C_{n-1}[z;q\beta\,|\,q]. \tag{191}$$

Two other shift relations follow from the previous two equations:

$$(\beta+1)C_n[q^{\frac12}z;\beta\,|\,q]=(q^{-\frac12 n}+q^{\frac12 n}\beta)C_n[z;\beta\,|\,q]+q^{-\frac12 n}(\beta-1)(z-\beta z^{-1})C_{n-1}[z;q\beta\,|\,q], \tag{192}$$

$$(\beta+1)C_n[q^{-\frac12}z;\beta\,|\,q]=(q^{-\frac12 n}+q^{\frac12 n}\beta)C_n[z;\beta\,|\,q]+q^{-\frac12 n}(\beta-1)(z^{-1}-\beta z)C_{n-1}[z;q\beta\,|\,q]. \tag{193}$$

Trigonometric representation (see p.473, Remarks, first formula)

$$C_n(\cos\theta;\beta\,|\,q)=\sum_{k=0}^{n}\frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}\,e^{i(n-2k)\theta}\,. \tag{194}$$

Limit for $q\downarrow -1$ (see [63, pp. 74–75]). By (194) and (81) we obtain

$$\lim_{q\uparrow 1}C_{2m}(x;-q^\lambda\,|\,-q)=C_m^{\frac12(\lambda+1)}(2x^2-1)+C_{m-1}^{\frac12(\lambda+1)}(2x^2-1),$$
$$\lim_{q\uparrow 1}C_{2m+1}(x;-q^\lambda\,|\,-q)=2x\,C_m^{\frac12(\lambda+1)}(2x^2-1).$$

By (77) and [HTF2, 10.6(36)] this can be rewritten as

$$\lim_{q\uparrow 1}C_{2m}(x;-q^\lambda\,|\,-q)=\frac{(\lambda)_m}{(\frac12\lambda)_m}\,P_m^{(\frac12\lambda,\frac12\lambda-1)}(2x^2-1), \tag{195}$$

$$\lim_{q\uparrow 1}C_{2m+1}(x;-q^\lambda\,|\,-q)=2\,\frac{(\lambda+1)_m}{(\frac12\lambda+1)_m}\,x\,P_m^{(\frac12\lambda,\frac12\lambda)}(2x^2-1). \tag{196}$$

By (68) the limits (195), (196) imply that

$$\lim_{q\uparrow 1}C_n(x;-q^\lambda\,|\,-q)=\text{const.}\,S_n^{(\frac12\lambda,\frac12\lambda-1)}(x), \tag{197}$$

where the right-hand side gives a one-parameter subclass of the generalized Gegenbauer polynomial. Note that in [K16, Section 7.1] the generalized Gegenbauer polynomials are also observed as fitting in the $q=-1$ Askey scheme, but the limit (197) is not observed there.

14.11 Big q-Laguerre

Symmetry The big q-Laguerre polynomials $P_n(x;a,b;q)$ are symmetric in a,b.

This follows from (14.11.1). As a consequence, it is sufficient to give generating function (14.11.11). Then the generating function (14.1.12) will follow by symmetry in the parameters.

14.12 Little q-Jacobi

Notation Here the little q-Jacobi polynomial is denoted by $p_n(x;a,b;q)$ instead of $p_n(x;a,b\,|\,q)$.

Basic Hypergeometric Representation In addition to (14.12.1) we have (see [K22, (2.46)])

$$p_n(x;a,b;q) = (-qb)^{-n} q^{-\frac{1}{2}n(n-1)} \frac{(qb;q)_n}{(qa;q)_n} \, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, qbx \\ qb, 0 \end{matrix}; q, q\right). \tag{198}$$

Special values (see [K22, §2.4]).

$$p_n(0;a,b;q) = 1, \tag{199}$$

$$p_n(q^{-1}b^{-1};a,b;q) = (-qb)^{-n}\, q^{-\frac{1}{2}n(n-1)} \, \frac{(qb;q)_n}{(qa;q)_n}\,, \tag{200}$$

$$p_n(1;a,b;q) = (-a)^n \, q^{\frac{1}{2}n(n+1)} \, \frac{(qb;q)_n}{(qa;q)_n}\,. \tag{201}$$

14.14 Quantum q-Krawtchouk

q-Hypergeometric representation For $n = 0, 1, \ldots, N$ (see (14.14.1) and use (18)):

$$K_n^{\mathrm{qtm}}(y;p,N;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, y \\ q^{-N} \end{matrix}; q, pq^{n+1}\right) \tag{202}$$

$$= (pyq^{N+1};q)_n \, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-N}/y, 0 \\ q^{-N}, q^{-N-n}/(py) \end{matrix}; q, q\right). \tag{203}$$

Special values By (202) and [GR, (II.4)]:

$$K_n^{\mathrm{qtm}}(1;p,N;q) = 1, \qquad K_n^{\mathrm{qtm}}(q^{-N};p,N;q) = (pq;q)_n. \tag{204}$$

By (203) and (204) we have the self-duality

$$\frac{K_n^{\mathrm{qtm}}(q^{x-N};p,N;q)}{K_n^{\mathrm{qtm}}(q^{-N};p,N;q)} = \frac{K_x^{\mathrm{qtm}}(q^{n-N};p,N;q)}{K_x^{\mathrm{qtm}}(q^{-N};p,N;q)} \qquad (n, x \in \{0, 1, \ldots, N\}). \tag{205}$$

By (204) and (205) we have also

$$K_N^{\mathrm{qtm}}(q^{-x};p,N;q) = (pq^N;q^{-1})_x \qquad (x \in \{0, 1, \ldots, N\}). \tag{206}$$

Limit for $q \to 1$ to Krawtchouk (see (14.14.14) and Section 9.11):

$$\lim_{q\to 1} K_n^{\mathrm{qtm}}(1 + (1-q)x;p,N;q) = K_n(x;p^{-1},N), \tag{207}$$

$$\lim_{q\to 1} K_n^{\mathrm{qtm}}(q^{-x};p,N;q) = K_n(x;p^{-1},N). \tag{208}$$

Quantum q^{-1}-Krawtchouk By (202), (204), (17) and (211) (see also p.496, second formula):

$$\frac{K_n^{\rm qtm}(y;p,N;q^{-1})}{K_n^{\rm qtm}(q^N;p,N;q^{-1})} = \frac{1}{(pq^{-1};q^{-1})_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n}, y^{-1} \\ q^{-N} \end{matrix};q,pyq^{-N}\right) \tag{209}$$
$$= K_n^{\rm Aff}(q^{-N}y;p^{-1},N;q). \tag{210}$$

Rewrite (210) as

$$K_m^{\rm qtm}(1+(1-q^{-1})qx;p^{-1},N;q^{-1}) = ((pq)^{-1};q^{-1})_n\,K_n^{\rm Aff}\Big(1+(1-q)q^{-N}\big(\tfrac{1-q^N}{1-q}-x\big);p,N;q\Big).$$

In view of (207) and (216) this tends to (98) as $q\to 1$.

The orthogonality relation (14.14.2) holds with positive weights for $q>1$ if $p>q^{-1}$.

History The origin of the name of the quantum q-Krawtchouk polynomials is by their interpretation as matrix elements of irreducible corepresentations of (the quantized function algebra of) the quantum group $SU_q(2)$ considered with respect to its quantum subgroup $U(1)$. The orthogonality relation and dual orthogonality relation of these polynomials are an expression of the unitarity of these corepresentations. See for instance [343, Section 6].

14.16 Affine q-Krawtchouk

q-Hypergeometric representation For $n=0,1,\ldots,N$ (see (14.16.1)):

$$K_n^{\rm Aff}(y;p,N;q) = \frac{1}{(p^{-1}q^{-1};q^{-1})_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-N}y^{-1} \\ q^{-N} \end{matrix};q,p^{-1}y\right) \tag{211}$$
$$= {}_3\phi_2\left(\begin{matrix} q^{-n}, y, 0 \\ q^{-N}, pq \end{matrix};q,q\right). \tag{212}$$

Self-duality By (212):

$$K_n^{\rm Aff}(q^{-x};p,N;q) = K_x^{\rm Aff}(q^{-n};p,N;q) \qquad (n,x\in\{0,1,\ldots,N\}). \tag{213}$$

Special values By (211) and [GR, (II.4)]:

$$K_n^{\rm Aff}(1;p,N;q)=1, \qquad K_n^{\rm Aff}(q^{-N};p,N;q) = \frac{1}{((pq)^{-1};q^{-1})_n}. \tag{214}$$

By (214) and (213) we have also

$$K_N^{\rm Aff}(q^{-x};p,N;q) = \frac{1}{((pq)^{-1};q^{-1})_x}. \tag{215}$$

Limit for $q\to 1$ to Krawtchouk (see (14.16.14) and Section 9.11):

$$\lim_{q\to 1} K_n^{\rm Aff}(1+(1-q)x;p,N;q) = K_n(x;1-p,N), \tag{216}$$
$$\lim_{q\to 1} K_n^{\rm Aff}(q^{-x};p,N;q) = K_n(x;1-p,N). \tag{217}$$

A relation between quantum and affine q-Krawtchouk
By (202), (211), (214) and (213) we have for $x \in \{0,1,\dots,N\}$:

$$K^{\mathrm{qtm}}_{N-n}(q^{-x};p^{-1}q^{-N-1},N;q) = \frac{K^{\mathrm{Aff}}_x(q^{-n};p,N;q)}{K^{\mathrm{Aff}}_x(q^{-N};p,N;q)} \tag{218}$$

$$= \frac{K^{\mathrm{Aff}}_n(q^{-x};p,N;q)}{K^{\mathrm{Aff}}_N(q^{-x};p,N;q)}\,. \tag{219}$$

Formula (218) is given in [K3, formula after (12)] and [K15, (59)]. In view of (208) and (217) formula (219) has (99) as a limit case for $q \to 1$.

Affine q^{-1}-Krawtchouk By (211), (214), (17) and (202) (see also p.505, first formula):

$$\frac{K^{\mathrm{Aff}}_n(y;p,N;q^{-1})}{K^{\mathrm{Aff}}_n(q^N;p,N;q^{-1})} = {}_2\phi_1\left(\begin{matrix} q^{-n},\, q^{-N}y \\ q^{-N}\end{matrix};q,p^{-1}q^{n+1}\right) \tag{220}$$

$$= K^{\mathrm{qtm}}_n(q^{-N}y;p^{-1},N;q). \tag{221}$$

Formula (221) is equivalent to (210). Just as for (210), it tends after suitable substitutions to (98) as $q \to 1$.

The orthogonality relation (14.16.2) holds with positive weights for $q > 1$ if $0 < p < q^{-N}$.

History The affine q-Krawtchouk polynomials were considered by Delsarte [161, Theorem 11], [K11, (16)] in connection with certain association schemes. He called these polynomials generalized Krawtchouk polynomials. (Note that the ${}_2\phi_2$ in [K11, (16)] is in fact a ${}_3\phi_2$ with one upper parameter equal to 0.) Next Dunkl [186, Definition 2.6, Section 5.1] reformulated this as an interpretation as spherical functions on certain Chevalley groups. He called these polynomials q-Kratchouk polynomials. The current name *affine q-Krawtchouk polynomials* was introduced by Stanton [488, (4.13)]. He chose this name because, in [488, pp. 115–116] the polynomials arise in connection with an affine action of a group G on a space X. Here X is the set of $(v-n)\times n$ matrices over $\mathrm{GF}(q)$. Let G be the group of block matrices $\begin{pmatrix} A & 0 \\ SA & B\end{pmatrix}$, where $A \in \mathrm{GL}_n(q)$, $B \in \mathrm{GL}_{v-n}(q)$ and $S \in X$. Then G acts on X by $\begin{pmatrix} A & 0 \\ SA & B\end{pmatrix}\cdot T = BTA^{-1} + S$.

14.17 Dual q-Krawtchouk

Symmetry

$$K_n(x;c,N\,|\,q) = c^n\,K_n(c^{-1}x;c^{-1},N\,|\,q). \tag{222}$$

This follows from (14.17.1) combined with [GR, (III.11)].
In particular,

$$K_n(x;-1,N\,|\,q) = (-1)^n\,K_n(-x;-1,N\,|\,q). \tag{223}$$

14.20 Little q-Laguerre / Wall

Notation Here the little q-Laguerre polynomial is denoted by $p_n(x;a;q)$ instead of $p_n(x;a\,|\,q)$.

Re: (14.20.11) The right-hand side of this generating function converges for $|xt| < 1$. We can rewrite the left-hand side by use of the transformation

$$ {}_2\phi_1\left(\begin{matrix} 0,0 \\ c \end{matrix}; q, z\right) = \frac{1}{(z;q)_\infty}\, {}_0\phi_1\left(\begin{matrix} - \\ c \end{matrix}; q, cz\right). $$

Then we obtain:

$$ (t;q)_\infty\, {}_2\phi_1\left(\begin{matrix} 0,0 \\ aq \end{matrix}; q, xt\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{\frac{1}{2}n(n-1)}}{(q;q)_n}\, p_n(x;a;q)\, t^n \qquad (|xt| < 1). \tag{224} $$

Expansion of x^n

Divide both sides of (224) by $(t;q)_\infty$. Then coefficients of the same power of t on both sides must be equal. We obtain:

$$ x^n = (a;q)_n \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\, q^{nk}\, p_k(x;a;q). \tag{225} $$

Quadratic transformations

Little q-Laguerre polynomials $p_n(x;a;q)$ with $a = q^{\pm\frac{1}{2}}$ are related to discrete q-Hermite I polynomials $h_n(x;q)$:

$$ p_n(x^2;q^{-1};q^2) = \frac{(-1)^n q^{-n(n-1)}}{(q;q^2)_n}\, h_{2n}(x;q), \tag{226} $$

$$ x p_n(x^2;q;q^2) = \frac{(-1)^n q^{-n(n-1)}}{(q^3;q^2)_n}\, h_{2n+1}(x;q). \tag{227} $$

14.21 q-Laguerre

Notation Here the q-Laguerre polynomial is denoted by $L_n^\alpha(x;q)$ instead of $L_n^{(\alpha)}(x;q)$.

Orthogonality relation

(14.21.2) can be rewritten with simplified right-hand side:

$$ \int_0^\infty L_m^\alpha(x;q)\, L_n^\alpha(x;q)\, \frac{x^\alpha}{(-x;q)_\infty}\, dx = h_n\, \delta_{m,n} \qquad (\alpha > -1) \tag{228} $$

with

$$ \frac{h_n}{h_0} = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}, \qquad h_0 = -\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\, \frac{\pi}{\sin(\pi\alpha)}\,. \tag{229} $$

The expression for h_0 (which is Askey's q-gamma evaluation [K1, (4.2)]) should be interpreted by continuity in α for $\alpha \in \mathbb{Z}_{\geq 0}$. Explicitly we can write

$$ h_n = q^{-\frac{1}{2}\alpha(\alpha+1)}\, (q;q)_\alpha\, \log(q^{-1}) \qquad (\alpha \in \mathbb{Z}_{\geq 0}). \tag{230} $$

Expansion of x^n

$$x^n = q^{-\frac{1}{2}n(n+2\alpha+1)}\,(q^{\alpha+1};q)_n \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k}\, q^k\, L_k^{\alpha}(x;q). \tag{231}$$

This follows from (225) by the equality given in the Remark at the end of §14.20. Alternatively, it can be derived in the same way as (225) from the generating function (14.21.14).

Quadratic transformations

q-Laguerre polynomials $L_n^{\alpha}(x;q)$ with $\alpha = \pm\frac{1}{2}$ are related to discrete q-Hermite II polynomials $\widetilde{h}_n(x;q)$:

$$L_n^{-1/2}(x^2;q^2) = \frac{(-1)^n q^{2n^2-n}}{(q^2;q^2)_n}\,\widetilde{h}_{2n}(x;q), \tag{232}$$

$$xL_n^{1/2}(x^2;q^2) = \frac{(-1)^n q^{2n^2+n}}{(q^2;q^2)_n}\,\widetilde{h}_{2n+1}(x;q). \tag{233}$$

These follows from (226) and (227), respectively, by applying the equalities given in the Remarks at the end of §14.20 and §14.28.

14.27 Stieltjes-Wigert

An alternative weight function

The formula on top of p.547 should be corrected as

$$w(x) = \frac{\gamma}{\sqrt{\pi}}\, x^{-\frac{1}{2}} \exp(-\gamma^2 \ln^2 x), \quad x > 0, \quad \text{with} \quad \gamma^2 = -\frac{1}{2\ln q}\,. \tag{234}$$

For w the weight function given in [Sz, §2.7] the right-hand side of (234) equals const. $w(q^{-\frac{1}{2}}x)$. See also [DLMF, §18.27(vi)].

14.28 Discrete q-Hermite I

History Discrete q Hermite I polynomials (not yet with this name) first occurred in Hahn [261], see there p.29, case V and the q-weight $\pi(x)$ given by the second expression on line 4 of p.30. However note that on the line on p.29 dealing with case V, one should read $k^2 = q^{-n}$ instead of $k^2 = -q^n$. Then, with the indicated substitutions, [261, (4.11), (4.12)] yield constant multiples of $h_{2n}(q^{-1}x;q)$ and $h_{2n+1}(q^{-1}x;q)$, respectively, due to the quadratic transformations (226), (227) together with (4.20.1).

14.29 Discrete q-Hermite II

Basic hypergeometric representation (see (14.29.1))

$$\widetilde{h}_n(x;q) = x^n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-n+1} \\ 0 \end{matrix}; q^2, -q^2x^{-2}\right). \tag{235}$$

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