In this informal note I answer two questions posed to me by P. J. Larcombe. Sections 1 and 2 give the answers to the two questions, as I mailed them to Larcombe on July 2 and 8, 2000, respectively. I conclude with a short comment in Section 3.

1. The answer to Larcombe’s first question (note of July 2, 2000)

In an email of June 20, 2000 P. J. Larcombe conjectured that

\[
\lim_{n \to \infty} {}_3F_2 \left[ \begin{array}{c}
-n, \frac{1}{2}, \frac{1}{2} \\
\frac{1}{2} - n, \frac{1}{2} - n
\end{array} ; -1 \right] = 2. \quad (1.1)
\]

I will give a proof of (1.1). Note that the terms of the (terminating well-poised) \( {}_3F_2 \)-series on the left remain invariant under reversion of the direction of summation. Thus we can write

\[
{}_3F_2 \left[ \begin{array}{c}
-n, \frac{1}{2}, \frac{1}{2} \\
\frac{1}{2} - n, \frac{1}{2} - n
\end{array} ; -1 \right] = \sum_{k=0}^{\infty} c_{n,k},
\]

where

\[
c_{n,k} = \begin{cases}
2 \frac{(-n)_k \left( \frac{1}{2} \right)_k \left( \frac{1}{2} \right)_k (-1)^k}{\frac{1}{2} - n)_k \left( \frac{1}{2} - n \right)_k k!} & \text{if } k < \frac{1}{2} n, \\
(-n)_k \frac{\left( \frac{1}{2} \right)_k \left( \frac{1}{2} \right)_k (-1)^k}{\frac{1}{2} - n)_k \left( \frac{1}{2} - n \right)_k k!} & \text{if } k = \frac{1}{2} n \text{ (occurs only if } n \text{ is even),} \\
0 & \text{if } k > \frac{1}{2} n.
\end{cases}
\]

Now I will prove (1.1) by dominated convergence. I use that \( \lim_{n \to \infty} c_{n,k} = 2 \) if \( k = 0 \) and \( = 0 \) otherwise, and that \( 0 \leq c_{n,k} \leq 4 \cdot \left( \frac{1}{2} \right)^k \). The last inequality follows because, for \( k \leq \frac{1}{2} n \), we have:

\[
\frac{(-n)_k \left( \frac{1}{2} \right)_k \left( \frac{1}{2} \right)_k (-1)^k}{\left( \frac{1}{2} - n \right)_k \left( \frac{1}{2} - n \right)_k k!} \\
\leq \frac{n}{n - k + \frac{1}{2}} \left( \frac{k - \frac{1}{2}}{n - \frac{1}{2}} \right)^k \leq 2 \cdot \left( \frac{1}{2} \right)^k.
\]
2. The answer to Larcombe’s second question (note of July 8, 2000)
In an email of July 4, 2000 P. J. Larcombe communicated that
\[
\lim_{n \to \infty} 2^{n-1} 3F_2 \left[ \frac{-n, -\frac{1}{2}(n-1), -\frac{1}{2}n}{\frac{1}{2}-n, \frac{1}{2}-n}; 1 \right] = 1. \quad (2.1)
\]
He asked for an independent proof.
I will show in this note that for nonnegative integer \( m \) we have
\[
3F_2 \left[ \frac{-2m, -m + \frac{1}{2}, -m}{\frac{1}{2} - 2m, \frac{1}{2} - 2m}; 1 \right] = \frac{2^{-2m+1} \Gamma(m + \frac{1}{2})^4}{\Gamma(m + \frac{1}{4})^2 \Gamma(m + \frac{3}{4})^2} \times 3F_2 \left[ \frac{-m, \frac{1}{2}, \frac{1}{2}}{-m + \frac{1}{2}, m + 1}; 1 \right], \quad (2.2)
\]
and
\[
3F_2 \left[ \frac{-2m - 1, -m - \frac{1}{2}, -m}{\frac{1}{2} - 2m, \frac{1}{2} - 2m}; 1 \right] = \frac{2^{-2m-1} \Gamma(m + \frac{1}{2})^2 \Gamma(m + \frac{3}{2})^2}{(m+1) \Gamma(m + \frac{1}{4})^2 \Gamma(m + \frac{5}{4})^2} \times 3F_2 \left[ \frac{-m, \frac{1}{2}, \frac{1}{2}}{-m + \frac{1}{2}, m + 2}; 1 \right]. \quad (2.3)
\]
If we now use that
\[
\lim_{m \to \infty} m^{b-a} \frac{\Gamma(m+a)}{\Gamma(m+b)} = 1,
\]
see for instance [1, (1.4.3)], then (2.1) will follow from (2.2) and (2.3) if we can show that the two \( 3F_2(1) \) expressions on the right-hand side of (2.2) and (2.3) tend to 1 as \( m \to \infty \). This last result can be shown by writing these \( 3F_2(1) \) expressions as \( \sum_{k=0}^{\infty} c_{m,k} \) where \( c_{m,k} = 0 \) if \( k > m \) and where \( c_{m,k} \) for \( k \leq m \) is given by
\[
c_{m,k} = \frac{\frac{1}{2} \ldots (k - \frac{1}{2})}{(m-k + \frac{1}{2}) \ldots (m - \frac{1}{2})} \frac{(m-k+1) \ldots m}{(m-j+1) \ldots (m+j+k)} \frac{(\frac{1}{2})_k}{k!}.
\]
Here \( j = 0 \) for (2.2) and \( j = 1 \) for (2.3). So for \( k \leq m < 2k \) we have
\[
0 \leq c_{m,k} \leq \frac{(m-k+1) \ldots m}{(m+1) \ldots (m+k)} \leq \frac{k \ldots (2k-1)}{(2k) \ldots (3k-1)} \leq \frac{(\frac{3}{2})^k}{k!},
\]
and for \( m \geq 2k \) we have
\[
0 \leq c_{m,k} \leq \frac{\frac{1}{2} \ldots (k - \frac{1}{2})}{(m-k + \frac{1}{2}) \ldots (m - \frac{1}{2})} \leq \frac{\frac{1}{2} \ldots (k - \frac{1}{2})}{(k + \frac{1}{2}) \ldots (2k - \frac{1}{2})} \leq \frac{(1)^k}{k!}.
\]
Hence for all \( k \) we have \( 0 \leq c_{m,k} \leq \frac{(\frac{3}{2})^k}{k!} \), independently of \( m \). Since \( \lim_{m \to \infty} c_{m,k} = \delta_{k,0} \), the desired result follows by dominated convergence.
It remains to prove (2.2) and (2.3). For the proof of (2.2) first revert the order of summation on the left-hand side of (2.1) and next apply the transformation formula
\[
3F_2 \left[ \frac{a, b, c}{d, e}; 1 \right] = \frac{\Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(e-a) \Gamma(d+e-a-b-c)} \times 3F_2 \left[ \frac{a, d-b, d-c}{d, e-b-c}; 1 \right] \quad (2.4)
\]
(see for instance [1, Corollary 3.3.5]), and use the duplication formula \( \Gamma(2z) \Gamma(\frac{1}{2}) = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) \) (see for instance [1, (1.5.1)]). This yields the identities

\[
\begin{align*}
3F_2 \left[ \begin{array}{c}
-2m, -m + \frac{1}{2}, -m \\
\frac{1}{2} - 2m, \frac{1}{2} - 2m
\end{array} ; 1 \right] \\
= (-1)^m (m + 1)_m (\frac{1}{2})_m \frac{(m + 1/2)_m}{((m + 1/2)_m)^2} 3F_2 \left[ \begin{array}{c}
-m, m + \frac{1}{2}, m + \frac{1}{2} \\
- m + 1, \frac{1}{2}
\end{array} ; 1 \right] \\
= \frac{(m + 1)_m (\frac{1}{2})_m}{((m + 1/2)_m)^2} 3F_2 \left[ \begin{array}{c}
-m, \frac{1}{2}, \frac{1}{2} \\
- m + 1, m + 1
\end{array} ; 1 \right] \\
= 2^{2m+1} \frac{\Gamma(m + \frac{1}{2})^4}{\Gamma(m + \frac{1}{4})^2 \Gamma(m + \frac{3}{4})^2} 3F_2 \left[ \begin{array}{c}
-m, \frac{1}{2}, \frac{1}{2} \\
- m + \frac{1}{2}, m + 1
\end{array} ; 1 \right].
\end{align*}
\]

For the derivation of (2.3) we have a similar string of identities:

\[
\begin{align*}
3F_2 \left[ \begin{array}{c}
-2m - 1, -m - \frac{1}{2}, -m \\
\frac{1}{2} - 2m, -\frac{1}{2} - 2m
\end{array} ; 1 \right] \\
= (-1)^m (m + 2)_m (\frac{3}{2})_m \frac{(m + \frac{3}{2})_m}{((m + \frac{3}{2})_m)^2} 3F_2 \left[ \begin{array}{c}
-m, m + \frac{3}{2}, m + \frac{3}{2} \\
- m + 1, \frac{3}{2}
\end{array} ; 1 \right] \\
= \frac{(m + 2)_m (\frac{3}{2})_m}{((m + \frac{3}{2})_m)^2} 3F_2 \left[ \begin{array}{c}
-m, \frac{1}{2}, \frac{1}{2} \\
- m + 1, m + 2
\end{array} ; 1 \right] \\
= 2^{2m} \frac{\Gamma(m + \frac{1}{2}) \Gamma(m + \frac{1}{4})^2 \Gamma(m + \frac{3}{4})^2}{(m + 1) \Gamma(m + \frac{1}{4})^2 \Gamma(m + \frac{3}{4})^2} 3F_2 \left[ \begin{array}{c}
-m, \frac{1}{2}, \frac{1}{2} \\
- m + \frac{1}{2}, m + 2
\end{array} ; 1 \right].
\end{align*}
\]

3. Concluding remarks

Formulas (1.1) and (2.1) were earlier obtained in a quite different way by Larcombe et al. in [2]. It was pointed out by Larcombe and French in [3] that the two results are related by the identity

\[
2^n 3F_2 \left[ \begin{array}{c}
-\frac{1}{2}n, -n, -\frac{1}{2}(n - 1) \\
\frac{1}{2} - n, \frac{1}{2} - n
\end{array} ; 1 \right] = 3F_2 \left[ \begin{array}{c}
-n, \frac{1}{2}, \frac{1}{2} \\
\frac{1}{2} - n, \frac{1}{2} - n
\end{array} ; -1 \right],
\]

which is a special case of an identity of Whipple (see formula (7.3) in [5] and formula (9.5) in [4]).

Larcombe and French are preparing a paper, where the above sketchy proofs will be given in more detail and where a precise reference will be given for the dominated convergence theorem in the context of infinite series (equivalent to Tannery’s theorem).
References


