On a product expansion of two Legendre functions conjectured by S. de Haro and A. C. Petkou

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This note is a slightly extended version of Appendix E in the paper *Instantons and conformal holography* by S. de Haro and A. C. Petkou [3].

Let P_ν^μ be a *Legendre function* as defined by [2, 3.2(3)]:

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} {}_2F_1\left(\begin{matrix}-\nu, \nu+1\\ 1-\mu\end{matrix}; \tfrac{1}{2}(1-z)\right) \quad (\mu \notin \mathbb{Z}_{>0},\ z \in \mathbb{C}\backslash(-\infty, 1]). \tag{1}$$

Let C_n^λ be a *Gegenbauer polynomial* as defined by [2, 10.9(20)]:

$$C_n^\lambda(x) := \frac{(2\lambda)_n}{n!} {}_2F_1\left(\begin{matrix}-n, n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix}; \tfrac{1}{2}(1-x)\right) \quad (n \in \mathbb{Z}_{\geq 0},\ \lambda > 0). \tag{2}$$

Since (see [1, (2.3.14)])

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = \frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1\left(\begin{matrix}a, b\\ a+b-c+1\end{matrix}; 1-z\right) \quad (a \in \mathbb{Z}_{\leq 0}), \tag{3}$$

and, in particular,

$${}_2F_1\left(\begin{matrix}a, b\\ \frac{1}{2}(a+b+1)\end{matrix}; z\right) = (-1)^a\, {}_2F_1\left(\begin{matrix}a, b\\ \frac{1}{2}(a+b+1)\end{matrix}; 1-z\right) \quad (a \in \mathbb{Z}_{\leq 0}), \tag{4}$$

we see that (2) gives a polynomial of degree n in z which is even or odd according to whether n is even or odd.

We have the two following remarkable identities:

$$P_{\lambda-1}^{n+\lambda}(z) = \frac{2^\lambda (-1)^n n!}{\Gamma(1-\lambda)} (z^2-1)^{-\frac{1}{2}\lambda} C_n^\lambda\left(\frac{z}{\sqrt{z^2-1}}\right) \qquad (z \in \mathbb{C}\backslash[-1,1],\ \lambda \in (0,\infty)\backslash\mathbb{Z},\ n \in \mathbb{Z}_{\geq 0}), \tag{5}$$

$$\begin{aligned} P_{\lambda-1}^{-n-\lambda}(z) &= \frac{\Gamma(\frac{1}{2}-\lambda)}{2^\lambda \pi^{\frac{1}{2}} n!} (z^2-1)^{-\frac{1}{2}\lambda} {}_2F_1\left(\begin{matrix}-n, n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix}; \tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right)\\ &+ \frac{2^{\lambda-1}\Gamma(\lambda-\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(2\lambda+n)} (z^2-1)^{\frac{1}{2}\lambda-\frac{1}{2}} {}_2F_1\left(\begin{matrix}-n-2\lambda+1, n+1\\ -\lambda+\frac{3}{2}\end{matrix}; \tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right)\\ &\qquad (z \in \mathbb{C}\backslash[-1,1],\ \lambda \in (0,\infty)\backslash(\mathbb{Z}+\tfrac{1}{2}),\ n \in \mathbb{Z}_{\geq 0}). \end{aligned} \tag{6}$$

The first term on the right-hand side of (6), multiplied by $(-1)^n(z^2-1)^{\frac{1}{2}(\lambda+n)}$, is an even polynomial in z. If moreover $\lambda\in\mathbb{Z}_{>0}$ then the second term on the right-hand side of (6), multiplied by $(-1)^n(z^2-1)^{\frac{1}{2}(\lambda+n)}$, is an odd polynomial in z. Hence

$$\begin{aligned}\tilde{P}^{n+\lambda}_{\lambda-1}(z):=&P^{-n-\lambda}_{\lambda-1}(z)+(-1)^nP^{-n-\lambda}_{\lambda-1}(-z)\\ =&\frac{\Gamma(\frac{1}{2}-\lambda)}{2^{\lambda-1}\pi^{\frac{1}{2}}n!}\,(z^2-1)^{-\frac{1}{2}\lambda}\,{}_2F_1\left(\begin{matrix}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix};\tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right)\\ =&\frac{(-1)^\lambda 2^\lambda\Gamma(\lambda)}{\Gamma(2\lambda+n)}\,(z^2-1)^{-\frac{1}{2}\lambda}\,C^\lambda_n\left(\frac{z}{\sqrt{z^2-1}}\right)\\ &\qquad(z\in\mathbb{C}\backslash[-1,1],\ \lambda\in\mathbb{Z}_{>0},\ n\in\mathbb{Z}_{\ge0})\end{aligned}\tag{7}$$

and

$$\begin{aligned}\overline{P}^{n+\lambda}_{\lambda-1}(z):=&P^{-n-\lambda}_{\lambda-1}(z)+(-1)^{n+1}P^{-n-\lambda}_{\lambda-1}(-z)\\ =&\frac{2^\lambda\Gamma(\lambda-\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(2\lambda+n)}\,(z^2-1)^{\frac{1}{2}\lambda-\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}-n-2\lambda+1,n+1\\ -\lambda+\frac{3}{2}\end{matrix};\tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right)\\ &\qquad(z\in\mathbb{C}\backslash[-1,1],\ \lambda\in\mathbb{Z}_{>0},\ n\in\mathbb{Z}_{\ge0}).\end{aligned}\tag{8}$$

For the proofs of (5) and (6) use [2, 3.2(26)] together with the transformation formula

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)=(1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\ c\end{matrix};z\right).\tag{9}$$

Then

$$\begin{aligned}P^\mu_\nu(z)=&\frac{\Gamma(-\frac{1}{2}-\nu)}{(2\pi)^{\frac{1}{2}}\Gamma(-\nu-\mu)}\,\frac{(z-\sqrt{z^2-1})^{\nu+\frac{1}{2}}}{(z^2-1)^{\frac{1}{4}}}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}+\mu,\frac{1}{2}-\mu\\ \frac{3}{2}+\nu\end{matrix};\tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right)\\ &+\frac{\Gamma(\frac{1}{2}+\nu)}{(2\pi)^{\frac{1}{2}}\Gamma(1+\nu-\mu)}\,\frac{(z-\sqrt{z^2-1})^{-\nu-\frac{1}{2}}}{(z^2-1)^{\frac{1}{4}}}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}+\mu,\frac{1}{2}-\mu\\ \frac{1}{2}-\nu\end{matrix};\tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right)\\ =&\frac{\Gamma(-\frac{1}{2}-\nu)}{2^{1+\nu}\pi^{\frac{1}{2}}\Gamma(-\nu-\mu)}\,(z^2-1)^{-\frac{1}{2}\nu-\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}\nu-\mu+1,\nu+\mu+1\\ \frac{3}{2}+\nu\end{matrix};\tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right)\\ &+\frac{2^\nu\Gamma(\frac{1}{2}+\nu)}{\pi^{\frac{1}{2}}\Gamma(1+\nu-\mu)}\,(z^2-1)^{\frac{1}{2}\nu}\,{}_2F_1\left(\begin{matrix}-\nu-\mu,-\nu+\mu\\ \frac{1}{2}-\nu\end{matrix};\tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right).\end{aligned}\tag{10}$$

If $\nu:=\lambda-1$, $\mu:=n+\lambda$ and $n\in\mathbb{Z}_{\ge0}$ then $1+\nu-\mu=-n$, hence the second term in (10) vanishes, so

$$P^{n+\lambda}_{\lambda-1}(z)=\frac{\Gamma(\frac{1}{2}-\lambda)}{2^\lambda\pi^{\frac{1}{2}}\Gamma(-n-2\lambda+1)}\,(z^2-1)^{-\frac{1}{2}\lambda}\,{}_2F_1\left(\begin{matrix}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix};\tfrac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right).\tag{11}$$

By (2) and the duplication formula for the Gamma function we obtain (5). Formula (6) is the case $\nu:=\lambda-1$, $\mu:=-n-\lambda$ of (10).

We make further specializations of formulas (5), (7) and (8):

$$P_{\frac{3}{2}}^{l+\frac{1}{2}}(z) = 3\sqrt{\frac{2}{\pi}}\,(-1)^l(l-2)!\,(z^2-1)^{-\frac{5}{4}}\,C_{l-2}^{\frac{5}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right) \qquad (z\in\mathbb{C}\backslash[-1,1],\ l\in\mathbb{Z}_{\ge 2}), \tag{12}$$

$$\tilde{P}_2^{j+1}(z) = -\,\frac{16}{(j+3)!}\,(z^2-1)^{-\frac{3}{2}}\,C_{j-2}^{3}\left(\frac{z}{\sqrt{z^2-1}}\right) \qquad (z\in\mathbb{C},\ j\in\mathbb{Z}_{\ge 2}), \tag{13}$$

$$\overline{P}_2^{j+1}(z) = \frac{6}{(j+3)!}\,(z^2-1)\;{}_2F_1\left(\begin{matrix}-j-3,\,j-1\\ -\frac{3}{2}\end{matrix};\frac{1}{2}\left(1-\frac{z}{\sqrt{z^2-1}}\right)\right) \qquad (z\in\mathbb{C}\backslash[-1,1],\ j\in\mathbb{Z}_{\ge 2}). \tag{14}$$

Formulas (12) and (14) give more convenient expressions for the special functions occurring in the following conjectured identity in [3, (98)]:

$$P_{\frac{3}{2}}^{l+\frac{1}{2}}(z)\,P_{\frac{3}{2}}^{l'+\frac{1}{2}}(z) = \sum_{j=2}^{l+l'} d_{j,l,l'}\,\tilde{P}_2^{j+1}(z) \qquad (l,l'\in\mathbb{Z}_{\ge 2}). \tag{15}$$

By substitution of (12) and (14) in (15) we see that (15) will indeed hold with summation only over $j=2,\ldots,l+l'$ with $l+l'-j$ even and with the coefficients $d_{j,l,l'}$ uniquely determined by the expansion

$$\tfrac{9}{8}(-1)^{l+l'-1}(l-2)!\,(l'-2)!\,(x^2-1)\,C_{l-2}^{\frac{5}{2}}(x)\,C_{l'-2}^{\frac{5}{2}}(x) = \sum_{\substack{j=2,\ldots,l+l'\\ l+l'-j \text{ even}}} \frac{d_{j,l,l'}}{(j+3)!}\,C_{j-2}^{3}(x). \tag{16}$$

Indeed, on the left we have a polynomial of degree $l+l'-2$ in x, even or odd according to whether $l+l'$ is even or odd.

In principle, the coefficients $d_{j,l,l'}$ can be computed from the known analytic expressions for the coefficients in Dougall's linearization formula

$$C_m^{\beta}(x)C_l^{\beta}(x) = \sum_{k=0}^{\min(m,l)} a_{\beta}(k,l,m)C_{l+m-2k}^{\beta}(x) \tag{17}$$

(see [1, Theorem 6.8.2]), in Gegenbauer's connection formula

$$C_n^{\beta}(x) = \sum_{k=0}^{[\frac{1}{2}n]} b_{\beta,\gamma}(n,k)C_{n-2k}^{\gamma}(x) \tag{18}$$

(see [1, Theorem 7.1.4']), and in the recurrence relation

$$x^2C_n^{\gamma}(x) = c_{\gamma}(n,n+2)\,C_{n+2}^{\gamma}(x) + c_{\gamma}(n,n)\,C_n^{\gamma}(x) + c_{\gamma}(n,n-2)\,C_{n-2}^{\gamma}(x) \tag{19}$$

(iterate the recurrence relation [1, (6.4.16)]). However, it seems improbable that the resulting expression for the coefficients $d_{j,l,l'}$ can be reduced to an analytic expression not involving a sum.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, 1999.

[2] A. Erdélyi, *Higher Trancendental Functions, Vols. I, II*, McGrawHill, 1953.

[3] S. de Haro and A. C. Petkou, *Instantons and conformal holography*, J. High Energy Phys. 12 (2006), paper 076; arXiv:hep-th/0606276v3.