On the recurrence relations connected with Elliott's formula, Appendix to "Error bursts in transmission channels"

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Keywords: recurrence relations

Abstract

A closed form is given for the expressions $G(m,n)$ and $B(m,n)$ of the previous paper. The resulting formula however is not suitable for efficient computations, except in some special cases, which are discussed at the end of this appendix.

Recall from (1) of the previous paper the following system of recurrence relations:

$$\begin{aligned} G(m,n) =& G(m,n-1)Qk + B(m,n-1)Pk \\ &+ G(m-1,n-1)Q(1-k) + B(m-1,n-1)P(1-k), \\ B(m,n) =& B(m,n-1)qh + G(m,n-1)ph \\ &+ B(m-1,n-1)q(1-h) + G(m-1,n-1)p(1-h) \end{aligned}$$

with initial conditions

$$\begin{aligned} &G(0,1) = k, \quad G(1,1) = 1-k, \\ &B(0,1) = h, \quad B(1,1) = 1-h, \\ &G(m,n) = B(m,n) = 0 \text{ if } m < 0 \text{ or } m > n. \end{aligned}$$

Computer Algebra in Industry
Edited by A. M. Cohen

We rewrite the system in matrix form:

$$v_{m,n} = X v_{m,n-1} + Y v_{m-1,n-1} \tag{1}$$

with initial conditions

$$v_{0,1} = f, \quad v_{1,1} = g,$$
$$v_{m,n} = 0 \text{ if } m < 0 \text{ or } m > n.$$

Here

$$v_{m,n} := \begin{pmatrix} G(m,n) \\ B(m,n) \end{pmatrix},$$
$$X := \begin{pmatrix} Qk & Pk \\ ph & qh \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} Q & P \\ p & q \end{pmatrix},$$
$$Y := \begin{pmatrix} Q(1-k) & P(1-k) \\ p(1-h) & q(1-h) \end{pmatrix} = \begin{pmatrix} 1-k & 0 \\ 0 & 1-h \end{pmatrix} \begin{pmatrix} Q & P \\ p & q \end{pmatrix},$$
$$f := \begin{pmatrix} k \\ h \end{pmatrix}, \quad g := \begin{pmatrix} 1-k \\ 1-h \end{pmatrix}.$$

Lemma 1. *Let $\ell = 1, 2, \ldots$ and $j = 0, 1, \ldots, \ell$. For a subset σ of $\{1, 2, \ldots, \ell\}$ of cardinality $|\sigma| = j$ and for $i = 1, \ldots, \ell$ put*

$$Z_{\sigma,i} := \begin{cases} Y & \text{if } i \in \sigma, \\ X & \text{if } i \notin \sigma. \end{cases}$$

Then

$$v_{m,n} = \sum_{j=0}^{\ell} \sum_{\substack{\sigma \subset \{1,\ldots,\ell\} \\ |\sigma| = j}} Z_{\sigma,1} Z_{\sigma,2} \cdots Z_{\sigma,\ell} \, v_{m-j,n-\ell}. \tag{2}$$

PROOF. By complete induction with respect to ℓ. For $\ell = 1$, (2) is a rewritten version of (1). From (1) and the case ℓ of (2) we get

$$\begin{aligned} v_{m,n} =& X \, v_{m,n-1} + Y \, v_{m-1,n-1} \\ =& \sum_{j=0}^{\ell} \sum_{\substack{\sigma \subset \{1,\ldots,\ell\} \\ |\sigma| = j}} X \, Z_{\sigma,1} \cdots Z_{\sigma,\ell} \, v_{m-j,n-\ell-1} \\ &+ \sum_{j=0}^{\ell} \sum_{\substack{\sigma \subset \{1,\ldots,\ell\} \\ |\sigma| = j}} Y \, Z_{\sigma,1} \cdots Z_{\sigma,\ell} \, v_{m-j-1,n-\ell-1} \\ =& \sum_{j=0}^{\ell} \sum_{\substack{\sigma \subset \{1,\ldots,\ell+1\} \\ |\sigma| = j, 1 \notin \sigma}} Z_{\sigma,1} \cdots Z_{\sigma,\ell+1} \, v_{m-j,n-\ell-1} \end{aligned}$$

$$+\sum_{j=1}^{\ell+1} \sum_{\substack{\sigma\subset\{1,\ldots,\ell+1\}\\ |\sigma|=j,1\in\sigma}} Z_{\sigma,1}\cdots Z_{\sigma,\ell+1}\, v_{m-j,n-\ell-1}$$

$$=\sum_{j=0}^{\ell+1} \sum_{\substack{\sigma\subset\{1,\ldots,\ell+1\}\\ |\sigma|=j}} Z_{\sigma,1}\cdots Z_{\sigma,\ell+1}\, v_{m-j,n-\ell-1}.$$

□

Theorem 2. *Let $n-1\ge m\ge 1$. Let $Z_{\sigma,i}$ be as in Lemma 1. Then*

$$v_{m,n} = \sum_{\substack{\sigma\subset\{1,\ldots,n-1\}\\ |\sigma|=m}} Z_{\sigma,1}\cdots Z_{\sigma,n-1}\, f + \sum_{\substack{\sigma\subset\{1,\ldots,n-1\}\\ |\sigma|=m-1}} Z_{\sigma,1}\cdots Z_{\sigma,n-1}\, g. \tag{3}$$

Proof. Because of the initial conditions for (1), the outer sum (for j) in (2) runs from $\max\{0, m-n+\ell\}$ to $\min\{\ell, m\}$. Now take $\ell = n-1$. Then the sum runs over $j = m-1, m$. □

The two cases for $v_{m,n}$ not covered by Theorem 2 are quite simple and follow immediately from (1) and the initial conditions:

$$v_{0,n} = X^{n-1}\, f, \tag{4}$$
$$v_{n,n} = Y^{n-1}\, g. \tag{5}$$

Now let A and D be invertible 2×2 matrices such that AXA^{-1} and DYD^{-1} are diagonal matrices. Then

$$v_{0,n} = A^{-1}\,(AXA^{-1})^{n-1}\,A\,f, \tag{6}$$
$$v_{n,n} = D^{-1}\,(DYD^{-1})^{n-1}\,D\,g \tag{7}$$

yield quite simple explicit expressions for the coordinates of $v_{0,n}$ and $v_{n,n}$. In fact, for $v_{0,n}$ this is the solution obtained in [1] in a quite different way.

In general, X and Y will not commute and can certainly not be simultaneously diagonalized. If they do commute then (3) yields

$$v_{m,n} = \binom{n-1}{m} X^{n-m-1}\, Y^m\, f + \binom{n-1}{m-1} X^{n-m}\, Y^{m-1}\, g.$$

If, moreover, there is a 2×2 invertible matrix A such that

$$AXA^{-1} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \quad AYA^{-1} = \begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix},$$

then

$$v_{m,n} = \binom{n-1}{m} A^{-1} \begin{pmatrix} \lambda_1^{n-m-1}\mu_1^m & 0 \\ 0 & \lambda_2^{n-m-1}\mu_2^m \end{pmatrix} A f + \binom{n-1}{m-1} A^{-1} \begin{pmatrix} \lambda_1^{n-m}\mu_1^{m-1} & 0 \\ 0 & \lambda_2^{n-m}\mu_2^{m-1} \end{pmatrix} A g, \tag{8}$$

which is a very simple explicit expression for $v_{m,n}$.

1. Reference

[1] E.O. Elliott, "Estimates of error rates for codes on burst-noise channels", *The Bell System Technical Journal*, (1963), 1977–1997.