Orthogonal polynomials in several variables potentially useful in pde

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A system of orthogonal polynomials (OP's) $\{p_n\}_{n=0}^{\infty}$ on $\mathbb{R}$ with respect to a positive measure μ on $\mathbb{R}$ is called *classical* if there is a second order differential operator L such that $Lp_n = \lambda_n p_n$ $(n = 0, 1, 2, \ldots)$ for certain eigenvalues λ_n. By a theorem of Bochner [1] there are three families of classical OP's (up to an affine transformation of the argument of the OP):

1. Hermite: $p_n = H_n$, $d\mu(x) = e^{-x^2}\,dx$ on $\mathbb{R}$,
   $(Lf)(x) = \frac{1}{2}f''(x) - xf'(x)$, $\lambda_n = -n$.
2. Laguerre: $p_n = L_n^{\alpha}$, $d\mu(x) = x^{\alpha}e^{-x}\,dx$ on $[0,\infty)$, $\alpha > -1$,
   $(Lf)(x) = xf''(x) + (\alpha + 1 - x)f'(x)$, $\lambda_n = -n$.
3. Jacobi: $p_n = P_n^{(\alpha,\beta)}$, $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx$ on $[-1,1]$, $\alpha, \beta > -1$,
   $(Lf)(x) = (1-x^2)f''(x) + (\beta-\alpha-(\alpha+\beta+2)x)f'(x)$, $\lambda_n = -n(n+\alpha+\beta+1)$.

Let μ be a positive measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}} |x^{\alpha}|\,d\mu(x) < \infty$ $(\alpha \in (\mathbb{Z}_{\geq 0})^d)$ and the support of μ has nonempty interior. Let $\mathcal{P}_n$ consist of all polynomials p of degree $\leq n$ such that $\int_{\mathbb{R}^d} pq\,d\mu = 0$ for all polynomials q of degree $< n$. Then $\mathcal{P}_n$ has the same dimension $\binom{n+d-1}{n}$ as the space of homogeneous polynomials of degree n in d variables. Furthermore, the spaces $\mathcal{P}_n$ $(n = 0, 1, 2, \ldots)$ are mutually orthogonal in $L^2(\mu)$. We call $\{\mathcal{P}_n\}_{n=0}^{\infty}$ a *system of orthogonal polynomials* with respect to the measure μ.

As a refinement of this notion we may choose an orthogonal basis $\{p_{\alpha}\}_{\alpha_1+\cdots+\alpha_d=n}$ for each space $\mathcal{P}_n$, and call the polynomials p_{α} orthogonal polynomials. Of course, there are many ways to choose such orthogonal bases.

A system $\{\mathcal{P}_n\}$ of orthogonal polynomials in d variables is called *classical* if there is a second order pdo L acting on the space of polynomials such that $\mathcal{P}_n$ is an eigenspace of L for a certain eigenvalue λ_n $(n = 0, 1, 2, \ldots)$. As a refinement there may be, apart from $L = L_1$, $d-1$ further pdo's $L_2, \ldots, L_d$ such that $L_1, L_2, \ldots, L_d$ commute, are self-adjoint with respect to μ, and have one-dimensional joint eigenspaces. Then we have OP's p_{α} with $L_j p_{\alpha} = \lambda_{\alpha}^{(j)} p_{\alpha}$.

It was shown by Krall & Sheffer [8] and Kwon, Lee & Littlejohn [9] that there are five families of classical orthogonal polynomials in 2 variables, as follows:

1. $d\mu(x,y) = e^{-x^2-y^2}\,dx\,dy$ on $\mathbb{R}^2$, $L = \frac{1}{2}(\partial_{xx} + \partial_{yy}) - x\partial_x - y\partial_y$, $\lambda_n = -n$.
2. $d\mu(x,y) = x^{\alpha}y^{\beta}e^{-x-y}\,dx\,dy$ on $[0,\infty) \times [0,\infty)$, $\alpha, \beta > -1$,
   $L = x\partial_{xx} + y\partial_{yy} + (1 + \alpha - x)\partial_x + (1 + \alpha - y)\partial_y$, $\lambda_n = -n$.
3. $d\mu(x,y) = y^{\beta}e^{-x^2-y}\,dx\,dy$ on $\mathbb{R} \times [0,\infty)$, $\beta > -1$,
   $L = \frac{1}{2}\partial_{xx} + y\partial_{yy} - x\partial_x + (1 + \beta - y)\partial_y$, $\lambda_n = -n$.
4. $d\mu(x,y) = x^{\alpha}y^{\beta}(1-x-y)^{\gamma}\,dx\,dy$ on $\{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0,\ x + y \leq 1\}$, $\alpha, \beta, \gamma > -1$, $L = x(1-x)\partial_{xx} + y(1-y)\partial_{yy} - 2xy\partial_{xy} + \big(\alpha + 1 - (\alpha + \beta + \gamma + 3)x\big)\partial_x + \big(\beta + 1 - (\alpha + \beta + \gamma + 3)y\big)\partial_y$, $\lambda_n = -n(n + \alpha + \beta + \gamma + 2)$.

5. $d\mu(x,y) = (1-x^2-y^2)^\alpha\,dx\,dy$ on $\{(x,y)\in\mathbb{R}^2 \mid (x^2+y^2\le 1\}$, $\alpha>-1$, $L = (1-x^2)\partial_{xx} + (1-y^2)\partial_{yy} - 2xy\partial_{xy} - (2\alpha+3)(x\partial_x + y\partial_y)$, $\lambda_n = -n(n+2\alpha+2)$.

Orthogonal bases $\{p_{n,k}\}_{k=0,1,\ldots,n}$ for $\mathcal{P}_n$ $(n=0,1,2,\ldots)$ in these five cases can be obtained by Gram-Schmidt orthogonalization of the monomials $1, x, y, x^2, xy, y^2, \ldots, x^n, x^{n-1}y, \ldots, x^{n-k}y^k, \ldots$. The resulting polynomials are as follows.

1. $p_{n,k}(x,y) = H_{n-k}(x)H_k(y)$.
2. $p_{n,k}(x,y) = L^\alpha_{n-k}(x)L^\beta_k(y)$.
3. $p_{n,k}(x,y) = H_{n-k}(x)L^\beta_k(y)$.
4. $p_{n,k}(x,y) = P^{(\alpha,\beta+\gamma+2k+1)}_{n-k}(1-2x)\,(1-x)^k P^{(\beta,\gamma)}_k\big(1-2y/(1-x)\big)$.
5. $p_{n,k}(x,y) = P^{(\alpha+k+\frac12,\alpha+k+\frac12)}_{n-k}(x)\,(1-x^2)^{k/2} P^{(\alpha,\alpha)}_k\big(y/\sqrt{1-x^2}\,\big)$.

The expansions in monomials of these polynomials $p_{n,k}$ do not involve all monomials $x^{m-j}y^j$ with (m,j) equal or less than (n,k) in the lexicographic ordering. For classes 1, 2 and 3 $p_{n,k}(x,y)$ only contains monomials $x^{m-j}y^j$ with $m-j\le n-k$ and $j\le k$. For classes 4 and 5 $p_{n,k}(x,y)$ only contains monomials $x^{m-j}y^j$ with $m\le n$ and $j\le k$. Furthermore, in these five cases there is a second order differental operator L_2 commuting with L which has the $p_{n,k}$ as eigenfunctions with eigenvalue only depending on k.

The OP's $p_{n,k}$ for case 4 (on the triangular region), as explicitly given above, were introduced by Proriol [10] in 1967. They were mentioned in the survey paper by Koornwinder [7] in 1975. Their special case $\alpha=\beta=\gamma=0$ (constant weight function) was rediscovered by Dubiner [2] in 1991, who was motivated by applications to finite elements. Dubiner's paper was much quoted in this context. For a while, the special functions and finite elements communities were not aware that they had a joint interest. But in 2000 Hesthaven & Teng [4] referred to Proriol's paper, while later Karniadakis & Sherwin in their book [6] had ample references to papers on special functions. Conversely, in 2001 Dunkl & Xu referred in their book [3] to Dubiner's paper.

Another important orthogonal system for case 5 on the disk is as follows.

$$R^\alpha_{m,n}(z) := \text{const.}\begin{cases} P^{(\alpha,m-n)}_n(2|z|^2-1)z^{m-n}, & m\ge n,\\ P^{(\alpha,n-m)}_m(2|z|^2-1)\overline{z}^{n-m}, & n\ge m \end{cases}$$
$$((m,n)\in(\mathbb{Z}_{\ge0})^2,\ z\in\mathbb{C},\ \alpha>-1).$$

Then $R^\alpha_{m,n}(z) = \text{const.}\,z^m\overline{z}^n + \text{polynomial in } z,\overline{z}$ of lower degree. and

$$\int_{x^2+y^2<1} R^\alpha_{m,n}(x+iy)\,\overline{R^\alpha_{k,l}(x+iy)}\,(1-x^2-y^2)^\alpha\,dx\,dy = 0 \quad ((m,n)\ne(k,l)).$$

For $\alpha = 0$ these polynomials are called *Zernike polynomials*. They were introduced by Zernike [11] in 1934 for applications in optics and are still much used there. The polynomials $R^{\alpha}_{m,n}$ for general α first occurred in Zernike & Brinkman [12].

For numerical applications it is important that Jacobi polynomials can be approximated by polynomials which are orthogonal on finitely many equidistant points. These are the *Hahn polynomials* $Q_n(x;\alpha,\beta,N)$ $(n = 0,1,\ldots,N)$ satisfying

$$\sum_{x=0}^{N} (Q_n Q_m)(x;\alpha,\beta,N) \binom{\alpha+x}{x} \binom{\beta+N-x}{N-x} = 0 \quad (n \neq m).$$

The approximation is: $\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = \text{const.}\, P_n^{(\alpha,\beta)}(1-2x)$.

From the Hahn polynomials we can build polynomials (Karlin & McGregor [5])

$$Q_{n,k}(x,y;\alpha,\beta,\gamma,N) := Q_{n-k}(x;\alpha,\beta{+}\gamma{+}2k{+}1,N{-}k)\binom{N-x}{k} Q_k(y;\beta,\gamma,N{-}x)$$

which are orthogonal on the set $\{(x,y) \in \mathbb{Z}^2 \mid x,y \geq 0,\ x+y \leq N\}$ with respect to the weights

$$w(x,y;\alpha,\beta,\gamma,N) := \binom{\alpha+x}{x}\binom{\beta+y}{y}\binom{\gamma+N-x-y}{N-x-y}.$$

They approximate the polynomials of class 4 on the triangle:

$$\lim_{N\to\infty} Q_{n,k}(Nx,Ny;\alpha,\beta,\gamma,N) = \text{const.}\, p^{\alpha,\beta,\gamma}_{n,k}(x,y),$$

which looks promising for applications.

References

[1] S. Bochner, *Über Sturm-Liouvillesche Polynomsysteme*, Math. Z. **29** (1929), 730–736.

[2] M. Dubiner, *Spectral methods on triangles and other domains*, J. Sci. Comput. **6** (1991), 345–390.

[3] Ch. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, Cambridge University Press, 2001.

[4] J. S. Hesthaven and C. H. Teng, *Stable spectral methods on tetrahedral elements*, SIAM J. Sci. Comput. **21** (2000), 2352–2380.

[5] S. Karlin & J. McGregor, *Linear growth models with many types and multidimensional Hahn polynomials*, in *Theory and application of special functions*, R. A. Askey (ed.), Academic Press, 1975, pp. 261–288.

[6] G. E. Karniadakis & S. J. Sherwin, *Spectral/hp element methods for computational fluid dynamics*, Oxford University Press, 2005, second ed.

[7] T. H. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, in *Theory and application of special functions*, R. A. Askey (ed.), Academic Press, 1975, pp. 435–495.

[8] H. L. Krall and I. M. Sheffer, *Orthogonal polynomials in two variables*, Ann. Mat. Pura Appl. (4) **76** (1967), 325–376.

[9] K. H. Kwon, J. K. Lee and L. L. Littlejohn, *Orthogonal polynomial eigenfunctions of second-order partial differential equations*, Trans. Amer. Math. Soc. **353** (2001), 3629–3647.

[10] J. Proriol, *Sur une famille de polynomes à deux variables orthogonaux dans un triangle*, C. R. Acad. Sci. Paris **245** (1957), 2459–2461.

[11] F. Zernike, *Beugungstheorie des Schneidenverfahrens und seiner verbesserten Form, der Phasenkontrastmethode*, Physica **1** (1934), 689–704.

[12] F. Zernike and H. C. Brinkman, *Hypersphärische Funktionen und die in sphärischen Bereichen orthogonalen Polynome*, Proc. Royal Acad. Amsterdam **38** (1935), 161–170.

Implementation and Efficiency of Discontinuous Galerkin Spectral Element Methods for Fluid Flow Problems

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Discontinuous Galerkin spectral element methods (DGSEMs) are high order methods with many features to make them attractive for use to compute highly accurate solutions to fluid flow and wave propagation problems. They are geometrically flexible, like finite element methods, and can be used in arbitrarily complex geometries. They are designed so that one increases the number of degrees of freedom either by increasing the order of approximation or by increasing the number of elements. The result is an approximation that can be both exponentially convergent in the polynomial order of the approximation and high order in the element size. The methods have been shown to have exponentially small dissipation and dispersion errors, which makes them ideal for wave propagation problems. The approximations are highly localized, making boundary conditions and parallelization easy to implement. Finally, the DGSEMs are robust, at least when compared to strong form spectral methods.

Conventional wisdom, however, states that DG spectral element methods are: (i) Too hard to implement and (ii) Less efficient than other methods, especially compact finite difference methods. We can show that, as usual, conventional wisdom is not necessarily correct. We will describe an efficient and simple to implement form of the spectral element method, and examine strategies for reducing its issues with stiffness. We will also compare the approximation with an optimized compact finite difference method to discuss the issue of relative efficiency.

We solve problems of compressible flow, approximating flows modeled by a system of conservation laws

$$\vec{q}_t + \nabla \cdot \vec{f} = 0, \tag{1}$$

with fluxes

$$\vec{f} = \vec{f}^i + \vec{f}^v \tag{2}$$

for either inviscid problems modeled by the Euler equations of gas-dynamics or viscous problems modeled by the compressible Navier-Stokes equations

The development of a DGSEM approximation has the following steps: The domain of interest is decomposed into multiple elements, which can be arbitrarily complex. Each element is mapped onto a reference element, on which a strong form of the equations still applies, namely

$$\tilde{q}_t + \nabla \cdot \tilde{f} = 0,$$

where

$$J\vec{a}^i = J\nabla\xi^i = \vec{a}_j \times \vec{a}_k = \frac{\partial \vec{X}}{\partial \xi^j} \times \frac{\partial \vec{X}}{\partial \xi^k} \quad (i,j,k)\ cyclic$$