different classes of dynamical phenomena. We can choose the architecture of computer control systems, so we can influence which dynamical phenomena the system will face, as in the example of offsetting the switching temperatures of a thermostat. To pursue qualitative theory of generic behavior for piecewise-smooth systems, we need to carefully define the class of possible models for the system under investigation. PSDS presents a valuable compendium of information about the bifurcations of different types of piecewise-smooth systems, but it stops short of completely specifying the mathematical context within which the bifurcation phenomena it discusses are generic. That leaves lots of interesting work to do in studying piecewise-smooth dynamical systems. PSDS is an excellent starting point where one can find extensive analysis of diverse examples.

JOHN GUCKENHEIMER
Cornell University

Basic Hypergeometric Series. Second Edition. *By George Gasper and Mizan Rahman.* Cambridge University Press, Cambridge, UK, 2004. $138.00. xxvi+428 pp., hardcover. ISBN 0-521-83357-4.

The first edition of the book under review appeared in 1990. It got a very favorable review by Jet Wimp in *SIAM Review*, 33 (1991), pp. 489–493, and I completely agree. I am using the book almost daily as a reference. The book has been doing very well since it appeared, although the volume, packed with formulas, is no obvious bedroom reading. For first and second editions together MathSciNet has now (beginning of 2008) 469 citations from references and 44 from reviews. Google Scholar gives 1634 citations.

With the second edition the book grew from xx+287 to xxvi+428 pages. The growth is mainly due to three new chapters and many added exercises. Of course, the errata to the first edition, which had grown into a long list [4], were incorporated into the second edition, and the list of references has been updated (now 48 pages with some 900 references).

The three new chapters, at the end of the book, are

- Chapter 9. Linear and Bilinear Generating Functions for Basic Orthogonal Polynomials;
- Chapter 10. q-Series in Two or More Variables;
- Chapter 11. Elliptic, Modular, and Theta Hypergeometric Series.

Chapters 9 and 10 are useful for the specialist. But Chapter 11 on the elliptic case is a welcome introduction to a new topic which is in very dynamic development, as will become clear below.

The first edition owed a lot of its popularity to the new readership coming from quantum group representations and mathematical physics. The authors, and more generally the special functions community, were refunded greatly for the service they had thus paid to their physics-oriented colleagues when Frenkel and Turaev [3] in 1997 introduced elliptic hypergeometric functions from their study of the elliptic $6j$-symbol, which is a solution of the Yang–Baxter equation for the fused eight-vertex model. Since then, a small but select circle has been infected by the elliptic hypergeometric disease, somewhat comparable with the q-disease which started in the late seventies. This stimulated the authors to study this new area themselves and to add a chapter on it to the second edition. This is certainly a good idea, and very useful for many readers. However, developments in this area are now so fast that this chapter may have appeared a little too early for it to become and remain the definitive introductory text to the subject. Good complementary reading to Chapter 11 is offered by Spiridonov's recent survey papers [8], [9].

Hypergeometric and q-hypergeometric functions tend to be hierarchically ordered, with the most complex functions with many parameters at the top, and with a descent by limit transitions, by which the functions degenerate and lose parameters. Just as ordinary hypergeometric series are written with the aid of the Pochhammer symbol, the q-hypergeometric series ($|q| < 1$) use the q-Pochhammer symbol $(a;q)_k := \prod_{j=0}^{k-1}(1-aq^j)$ (also well-defined for $k = \infty$). Then the ${}_r\phi_{r-1}$ q-hypergeometric series is given by

$$(1)\quad {}_r\phi_{r-1}\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_{r-1}\end{matrix};q,z\right) := \sum_{k=0}^{\infty} \frac{(a_1;q)_k\ldots(a_r;q)_k}{(b_1;q)_k\ldots(b_{r-1};q)_k(q;q)_k}\,z^k.$$

For general ${}_r\phi_s$ take s bottom parameters and insert on the right in the summand a factor $\left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{s-r+1}$. One might consider the infinite sequence of ${}_r\phi_{r-1}$'s, but they are dull. We want functions which admit deep transformation formulas and which give rise to (bi)orthogonal systems or kernels of explicitly invertible integral transforms. Then the top ones still look impressive, but their number of parameters remains modest.

In (q-)hypergeometric theory one may build up from the bottom to the top or follow a top-down approach. On the bottom one has, for instance, the (q-)binomial series

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\,z^k = (1-z)^{-a},$$
$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\,z^k = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

The bottom-up approach is the most usual, and also used (with a steep climb) by Gasper and Rahman.

Let us look at the top guys. For classical hypergeometric series $q=1$ they can be found in Bailey's beautiful booklet [1]. They are ${}_9F_8$ series of argument 1, *very well-poised* (i.e., upper parameters $a_1, a_2 = 1+\frac{1}{2}a_1, a_3,\ldots,a_9$, lower parameters $1+a_1-a_2,\ldots,1+a_1-a_9$) and 2-*balanced* (i.e., sum of lower parameters minus sum of upper parameters equals 2). In the *terminating* case (i.e., a_9 is a nonpositive integer $-n$) we have the Bailey–Whipple transformation formula and a system of biorthogonal rational functions which is already implicit in J. Wilson's Ph.D. thesis (1978). In the nonterminating case we have Bailey's four-term transformation formula.

In the q-case one has

$$(2)\quad {}_{10}\phi_9\left(\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f, g, h \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, qa/b, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h\end{matrix};q,q\right) \qquad (a^3q^2 = bcdefgh).$$

Again there are transformation formulas due to Bailey. The transformation in the terminating case ($h=q^{-n}$) together with the obvious symmetries generates a group of transformations isomorphic to the Weyl group of the exceptional root system E_6 (Lievens and Van der Jeugt [5]). Wilson [11] and Rahman [6] built five-parameter families of systems of biorthogonal rational functions from these ${}_{10}\phi_9$'s. A degenerate case of these systems yields the celebrated (${}_4\phi_3$) four-parameter q-Racah and Askey–Wilson polynomials. In the nonterminating case the "right" function is a suitable linear combination of two such ${}_{10}\phi_9$'s. Bailey's four-term transformation formula can be regarded as a symmetry of this function which again generates, together with the trivial symmetries, a symmetry group E_6 (see [5]).

For the elliptic hypergeometric series we replace the q-Pochhammer symbols $(a;q)_k$ in the q-hypergeometric series (1) by their elliptic analogues $(a;q,p)_k := \prod_{j=0}^{k-1}\theta(aq^j;p)$, where $|p|,|q|<1$ and $\theta(x;p) := (x;p)_\infty(px^{-1};p)_\infty$ (a modified theta function). Denote the resulting series by ${}_rE_{r-1}(a_1,\ldots,a_r;b_1,\ldots,b_{r-1};q,p;z)$. Ordinary and q-hypergeometric series $\sum_{k=0}^{\infty}c_k$ can be defined by the condition that c_{k+1}/c_k extends to a rational function in k or in q^k, respectively. In the elliptic case we require that c_{k+1}/c_k extends to a doubly periodic meromorphic function of k. The elliptic analogue of (1) satisfies this condition, provided the *elliptic balancing condition* $a_1\ldots a_r = qb_1\ldots b_{r-1}$ holds.

We obtain the "right" analogue of the terminating case of (2) as

$${}_{12}E_{11}(a_1,\ldots,a_{12};b_1,\ldots,b_{11};q,p;-1)$$

with $a_2 = qa_1^{\frac{1}{2}}$, $a_3 = -qa_1^{\frac{1}{2}}$, $a_4 = qa_1^{\frac{1}{2}}/p^{\frac{1}{2}}$, $a_5 = -qa_1^{\frac{1}{2}}/p^{\frac{1}{2}}$, $a_{12} = q^{-n}$, $b_i = qa_1/a_{i+1}$ such that $a_6\ldots a_{12} = q^2a_1^3$. For this function we have the Frenkel–Turaev [3] transformation formula. It implies a summation formula for a special ${}_{10}E_9$ (for which the book provides a new analytic proof, and next also for the transformation formula). From these special terminating ${}_{12}E_{11}$'s Spiridonov builds a five-parameter family of biorthogonal rational functions (see [8]). For the definition of a continuous analogue one needs Ruijsenaars's *elliptic* *gamma function*

$$\Gamma(z;p,q) := \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^jq^k} \qquad (|p|,|q| < 1).$$

Then Spiridonov builds a function $V(t_1, \ldots, t_8; p, q)$ $(t_1 \ldots t_8 = p^2q^2)$ given by an integral over a deformed unit circle of a quotient of products of elliptic gamma functions. For this function there is a group of transformations isomorphic to the Weyl group of E_7 (Rains [7]). From these functions a five-parameter generalized (continuous) biorthogonal system can be constructed (Spiridonov [10]).

Should one use the top-down or the bottom-up approach in the elliptic case? The answer is simple: top and bottom coincide. The special terminating ${}_{12}E_{11}$ and its continuous analogue seem to be the only possible hypergeometric functions on the elliptic level for which nontrivial transformation formulas exist. If one tries any degeneracy limit, while staying elliptic, nothing of interest remains. However, when one degenerates to the hyperbolic, trigonometric (q-), or rational (classical) case, then a full hierarchy is available (see [2]). The hyperbolic case, due to Ruijsenaars but not treated in the book, is of particular interest because it allows one to take $|q| = 1$. So one might imagine our universe being frozen in the far future in the elliptic state under E_7 symmetry, but with all information about a zoo of degeneracies still available, in case the sun might start shining again.

Much of current activity in special functions is on the multivariable generalizations associated with root systems, like Macdonald polynomials. This has already advanced to the elliptic level, for instance, for root system BC_n; see Rains [7]. The book gives some hints of these new developments in Chapter 10 (q-analogues of Appell hypergeometric series) and in section 11.6 (Rosengren's elliptic analogue of Milne's A_n summation formula).

What will users of the book look for? I guess that a large proportion will only use a limited number of summation and transformation formulas, without studying the proofs. Often they will be served well by Appendices II and III. It might therefore be a service to the community (but commercially not very wise) to post the PDFs of these appendices freely on the Web. Anyhow, the *Mathematica* files for these appendices are on Gasper's homepage. For those who want to study the proofs, it would have been helpful if every identity in the appendices had a link to the place in the book where it is proved. It is also on my wish list that the main identities for the elliptic would be included in the appendices.

Who will read the whole book? I think not many. And who would do all the exercises? I think almost nobody. The exercises are very numerous (296), very technical, and they often provide additional results of some importance. Many exercises have a reference to the literature.

The book is an impressive demonstration of the power mode of human formula manipulation. In history, Euler and Ramanujan are the best-known formula manipulators, but in every generation a few are active in the top class. The two authors of the book certainly belong to this elite group. For a while, in the twentieth century, this kind of work was considered to be old-fashioned. But there has been a renaissance during the last thirty years because of the very deep identities, orthogonal systems, and algebraic structures which arose, with applications in representation theory, combinatorics, probability theory, and physics. To my taste, the formula-manipulating approach is too dominating in the book, in neglect of other mathematical techniques which may offer in some cases a shorter and more elegant approach. For instance, a key formula in the chapter on elliptic hypergeometric functions is a three-term identity for products of four theta functions (nowadays often called Riemann's addition formula). This is handled in the book by Exercise 2.16(i), with a suggestion to derive it from Bailey's three-term transformation formula for very well-poised ${}_8\phi_7$'s. Compare this with the one-sentence sketch of the proof in [9, p. 3]: use Liouville's theorem for bounded analytic functions.

This book should be present in every mathematics library, and also on the shelf or desk of many researchers. Upgrade to the second edition is mandatory, for the elliptic chapter and the incorporated errata.

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Tom H. Koornwinder
University of Amsterdam

Modelling, Analysis and Optimization of Biosystems. *By Werner Krabs and Stefan Pickl.* Springer, Berlin, 2007. $109.00. xii+203 pp., hardcover. ISBN 978-3-540-71452-1.

The impact of mathematical modeling and analysis on biological applications has been significant. But the area of optimization is just beginning to be applied to biosystems, and this book illustrates some possible applications and techniques.

This book does not say who is the intended audience; it seems to be aimed at a mathematical audience instead of a biological audience. There are no biological data presented. An undergraduate student interested in population models and game theory could benefit from this book.

The first chapter deals with the modeling of interacting populations using systems of ordinary differential equations. There is a careful treatment of stability analysis using Jacobi matrices, eigenvalue analysis, Hurwitz's theorem, and Lyapunov's method. These results are presented in enough detail and with sufficient clarity that a student could learn about stability in systems with general growth functions. In section 1.4, the standard first order approximation of the derivative is used to derive a discretization of the continuous models; then stability is examined. Perhaps references to other ways of discretizing would have been helpful [1]. The "cobweb" method is illustrated for the Verhulst growth model. In the last section, determination of parameters from data is illustrated in a predator-prey discrete system.

The second chapter is the most interesting with a game theoretical model for one or two populations. The concepts of Nash and evolutionary stable equilibria are clearly explained. The dynamical treatments of two types of games are illustrated with concrete examples. The concept of bi-matrix-games is developed in the context of the "fight of sexes" in a population of animals. Careful proofs of verifying that a fixed point is a Nash equilibrium and examining asymptotic stability are given. A direct method for the calculation of Nash equilibrium is shown with tableau steps.

Four medical models of systems of ordinary differential equations are considered