Addition type formulas associated with dual product formulas

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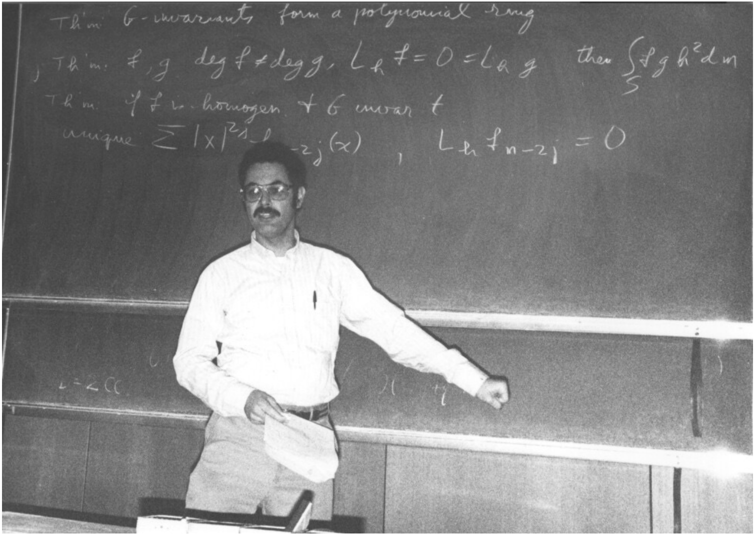

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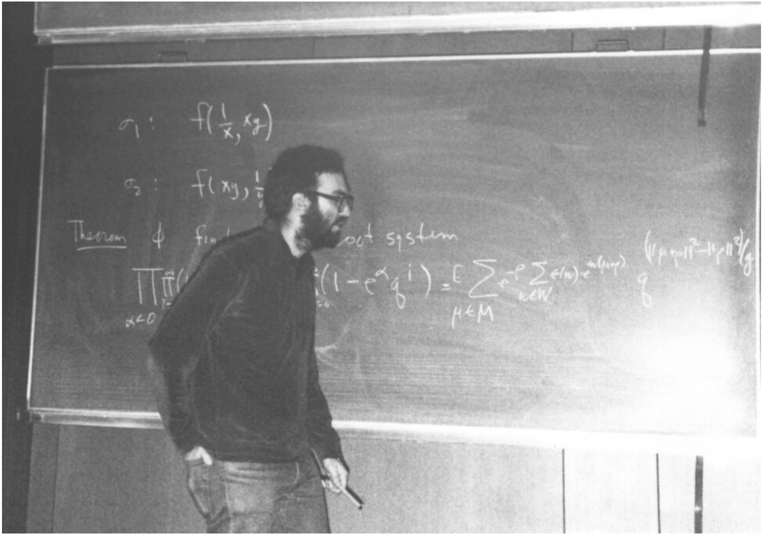

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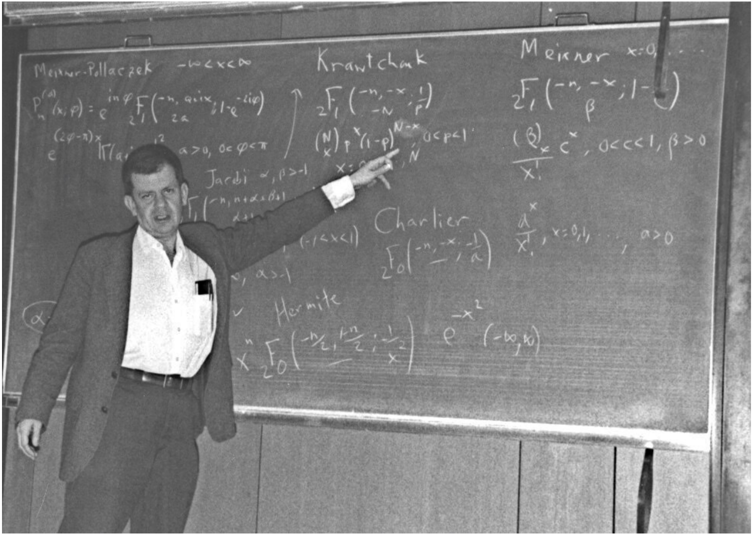

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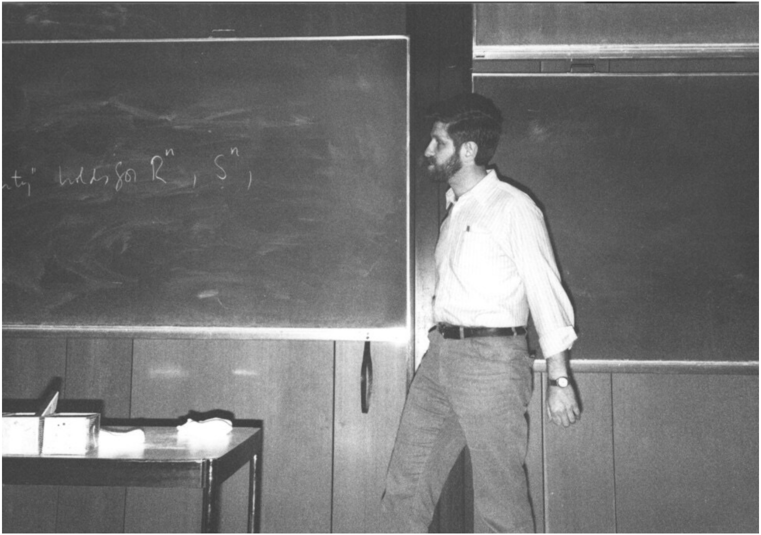

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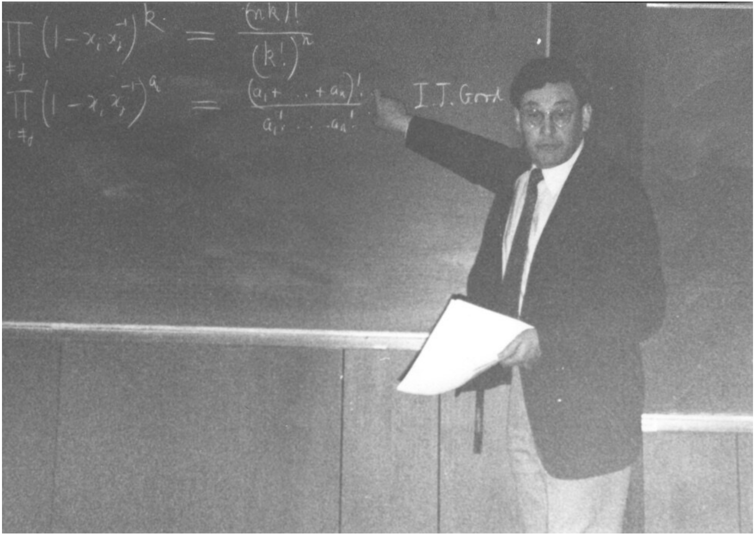

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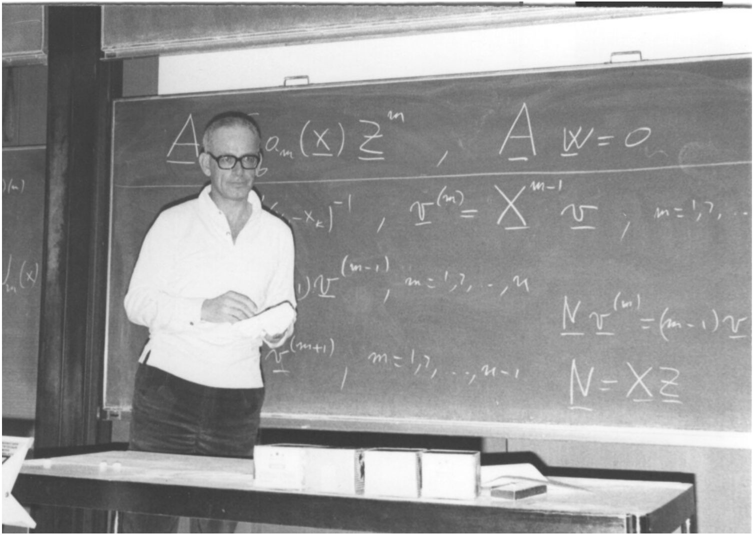
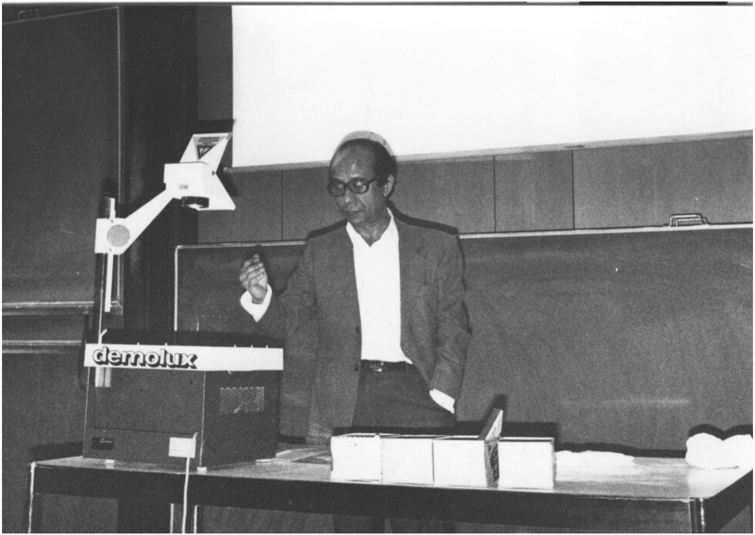


Francesco Calogero



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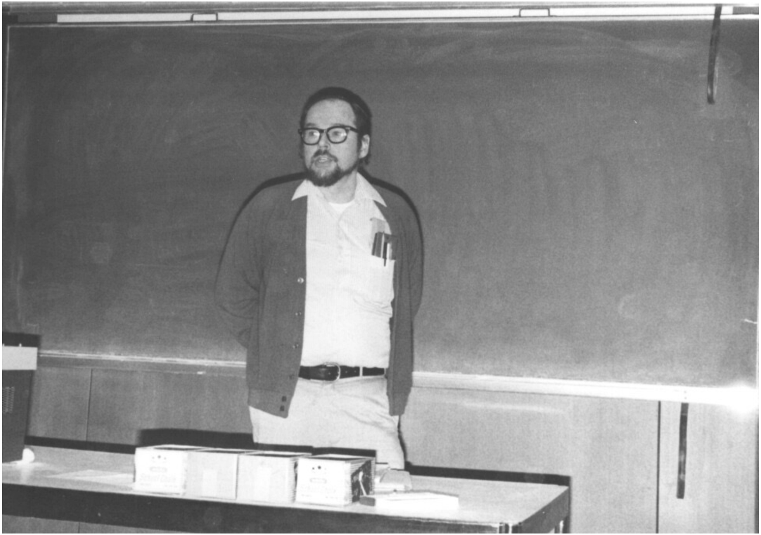
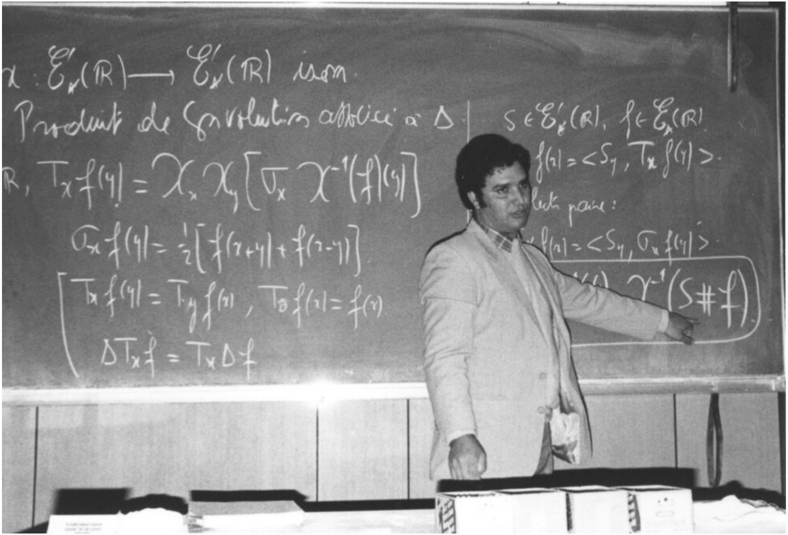
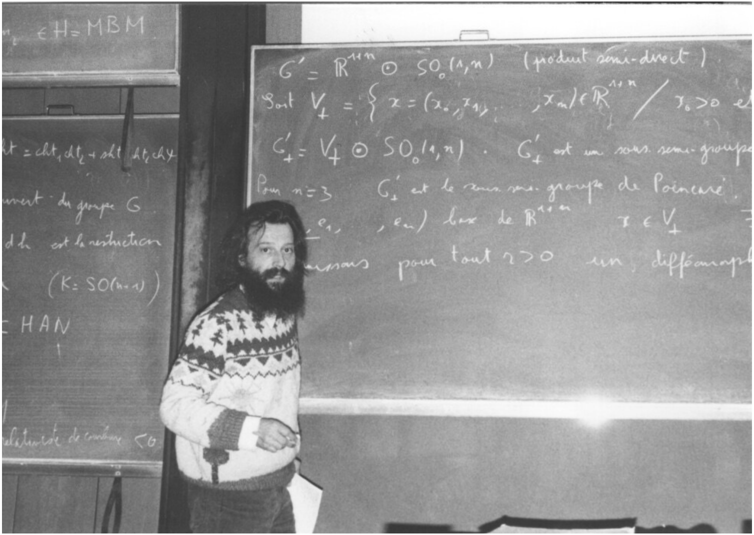


Willard Miller



Khalifa Trimeche



Michel Mizony

This talk based on:

Tom H. Koornwinder,

Dual addition formulas associated with dual product formulas,

`arXiv:1607.06053 [math.CA]`

Addition formula for Legendre polynomials

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\quad P_n^{(\alpha,\beta)}(1) = \dfrac{(\alpha+1)_n}{n!}$,

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha (1+x)^\beta\,dx = 0 \quad (m \neq n,\ \alpha,\beta > -1).$$

Legendre polynomials $P_n(x) := P_n^{(0,0)}(x)$, $\quad P_n(1) = 1$.

Addition formula for Legendre polynomials:

$$\begin{aligned} &P_n(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi) \\ &\quad = P_n(\cos\theta_1)P_n(\cos\theta_2) + 2\sum_{k=1}^{n} \frac{(n-k)!\,(n+k)!}{2^{2k}(n!)^2} \\ &\qquad \times (\sin\theta_1)^k P_{n-k}^{(k,k)}(\cos\theta_1)\,(\sin\theta_2)^k P_{n-k}^{(k,k)}(\cos\theta_2)\cos(k\phi). \end{aligned}$$

Product formula for Legendre polynomials:

$$P_n(\cos\theta_1)P_n(\cos\theta_2) = \frac{1}{\pi}\int_0^\pi P_n(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi)\,d\phi.$$

Linearization formula for Legendre polynomials

$$P_l(x)P_m(x) = \sum_{j=0}^{\min(l,m)} \frac{(\frac{1}{2})_j(\frac{1}{2})_{l-j}(\frac{1}{2})_{m-j}(l+m-j)!}{j!\,(l-j)!\,(m-j)!\,(\frac{3}{2})_{l+m-j}} \times \big(2(l+m-2j)+1\big)\,P_{l+m-2j}(x).$$

Askey's question

Find an addition type formula corresponding to this dual product formula for Legendre polynomials just as the addition formula for Legendre polynomials corresponds to the product formula for Legendre polynomials.

Compare with rewritten product formula:

$$P_n(x)P_n(y) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}\,\sqrt{1-y^2}}^{\sqrt{1-x^2}\,\sqrt{1-y^2}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}}\,dz.$$

Corresponding addition formula expands $P_n(z+xy)$ in terms of Chebyshev polynomials of dilated argument.

Rewritten addition formula for Legendre polynomials

Chebyshev polynomials $T_n(\cos\theta) := \cos(n\theta)$,

$$T_n(x) = \text{const.}\, P_n^{(-\frac{1}{2},-\frac{1}{2})}(x), \quad T_n(1) = 1.$$

Rewritten product formula:

$$P_n(x)P_n(y) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}\sqrt{1-y^2}}^{\sqrt{1-x^2}\sqrt{1-y^2}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}}\,dz.$$

Rewritten addition formula:

$$P_n(z+xy) = P_n(x)P_n(y) + 2\sum_{k=1}^{n} \frac{(n-k)!\,(n+k)!}{2^{2k}(n!)^2}$$

$$\times(1-x^2)^{\frac{1}{2}k} P_{n-k}^{(k,k)}(x)\,(1-y^2)^{\frac{1}{2}k} P_{n-k}^{(k,k)}(y)\, T_k\left(\frac{z}{\sqrt{1-x^2}\sqrt{1-y^2}}\right).$$

Hint from a Hallnäs-Ruijsenaars product formula

Jacobi functions:

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\begin{matrix}\frac12(\alpha+\beta+1+i\lambda),\frac12(\alpha+\beta+1-i\lambda)\\ \alpha+1\end{matrix};-\sinh^2 t\right).$$

Transform pair for suitable f or g $\quad(\alpha\ge\beta\ge-\frac12)$:

$$\begin{cases} g(\lambda)=\displaystyle\int_0^\infty f(t)\,\phi_\lambda^{(\alpha,\beta)}(t)\,(\sinh t)^{2\alpha+1}\,(\cosh t)^{2\beta+1}\,dt,\\ f(t)=\mathrm{const.}\displaystyle\int_0^\infty g(\lambda)\,\phi_\lambda^{(\alpha,\beta)}(t)\,\frac{d\lambda}{|c(\lambda)|^2}\,. \end{cases}$$

Dual product formula for Jacobi functions ($\beta=-\frac12$) by Hallnäs & Ruijsenaars (2015) reveals weight function for Wilson polynomials with parameters $\pm i\lambda\pm i\mu+\frac12\alpha+\frac14$ (cases $\alpha=0$ and $\frac12$ due to Mizony, 1976): $\quad \phi_{2\lambda}^{(\alpha,-\frac12)}(t)\,\phi_{2\mu}^{(\alpha,-\frac12)}(t)$

$$=\mathrm{const.}\int_0^\infty \phi_{2\nu}^{(\alpha,-\frac12)}(t)\left|\frac{\Gamma\left(i\nu\pm i\lambda\pm i\mu+\frac12\alpha+\frac14\right)}{\Gamma(2i\nu)}\right|^2 d\nu.$$

Linearization formula for Gegenbauer polynomials

Renormalized Jacobi polynomials $R_n^{(\alpha,\beta)}(x) := \dfrac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$.

Gegenbauer linearization formula (Rogers, 1895):

$$R_l^{(\alpha,\alpha)}(x)R_m^{(\alpha,\alpha)}(x) = \frac{l!\,m!}{(2\alpha+1)_l(2\alpha+1)_m}\sum_{j=0}^{\min(l,m)}\frac{l+m+\alpha+\frac12-2j}{\alpha+\frac12}$$

$$\times\frac{(\alpha+\frac12)_j(\alpha+\frac12)_{l-j}(\alpha+\frac12)_{m-j}(2\alpha+1)_{l+m-j}}{j!\,(l-j)!\,(m-j)!\,(\alpha+\frac32)_{l+m-j}}\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$

$$=\sum_{j=0}^{m}\frac{w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)}{h_{0;\,\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}}\,R_{l+m-2j}^{(\alpha,\alpha)}(x)\quad(l\ge m,\ \alpha>-\tfrac12),$$

where $w_{\alpha,\beta,\gamma,\delta}(x)$ are the weights for the Racah polynomials $R_n\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)\quad(\gamma=-N-1,\,n=0,1,\ldots,N)$.

Racah polynomials

$$R_n\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)$$
$$:= {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix};1\right), \quad \gamma=-N-1,$$
$$\sum_{x=0}^{N}(R_mR_n)\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)\,w_{\alpha,\beta,\gamma,\delta}(x) = h_{n;\alpha,\beta,\gamma,\delta}\delta_{m,n}\,,$$

$$w_{\alpha,\beta,\gamma,\delta}(x) = \frac{\gamma+\delta+1+2x}{\gamma+\delta+1} \times \frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x(\delta+1)_x\,x!}\,,$$

$$\frac{h_{n;\,\alpha,\beta,\gamma,\delta}}{h_{0;\,\alpha,\beta,\gamma,\delta}} = \frac{\alpha+\beta+1}{\alpha+\beta+2n+1}\,\frac{(\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n\,n!}{(\alpha+1)_n(\alpha+\beta+1)_n(\beta+\delta+1)_n(\gamma+1)_n},$$

$$h_{0;\,\alpha,\beta,\gamma,\delta} = \sum_{x=0}^{N} w_{\alpha,\beta,\gamma,\delta}(x) = \frac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N}\,.$$

Racach coefficients of $R_{l+m-2j}^{(\alpha,\alpha)}(x)$

$$R_l^{(\alpha,\alpha)}(x)R_m^{(\alpha,\alpha)}(x)=\sum_{j=0}^{m}\frac{w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)}{h_{0;\,\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}}\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$

$(l\ge m,\ \alpha>-\frac12)$. More generally evaluate

$$S_n^\alpha(l,m):=\sum_{j=0}^{m}w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$
$$\times R_n\big(j(j-l-m-\alpha-\tfrac12);\,\alpha-\tfrac12,\alpha-\tfrac12,-m-1,-l-\alpha-\tfrac12\big).$$

By Racah Rodrigues formula, summation by parts, and a Gegenbauer difference formula we get

$$S_n^\alpha(l,m)=\frac{(2\alpha+1)_{l+n}(2\alpha+1)_{m+n}(\alpha+\frac12)_{l+m}}{2^{2n}(\alpha+\frac12)_l(\alpha+\frac12)_m(2\alpha+1)_{l+m}(\alpha+1)_n^2}\,(x^2-1)^n$$
$$\times R_{l-n}^{(\alpha+n,\alpha+n)}(x)\,R_{m-n}^{(\alpha+n,\alpha+n)}(x).$$

Then Fourier-Racah inversion gives:

Dual Gegenbauer addition formula

Theorem (Dual addition formula for Gegenbauer polynomials)

$$R_{l+m-2j}^{(\alpha,\alpha)}(x) = \sum_{n=0}^{m} \frac{\alpha+n}{\alpha+\frac{1}{2}n} \, \frac{(-l)_n(-m)_n(2\alpha+1)_n}{2^{2n}(\alpha+1)_n^2\, n!}$$
$$\times (x^2-1)^n \, R_{l-n}^{(\alpha+n,\alpha+n)}(x) \, R_{m-n}^{(\alpha+n,\alpha+n)}(x)$$
$$\times R_n\big(j(j-l-m-\alpha-\tfrac{1}{2});\, \alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}, -m-1, -l-\alpha-\tfrac{1}{2}\big)$$
$$(l \geq m,\ j = 0, 1, \ldots, m).$$

Compare with addition formula for Gegenbauer polynomials:

$$R_n^{(\alpha,\alpha)}(xy+z) = \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha+k}{\alpha+\frac{1}{2}k} \, \frac{(n+2\alpha+1)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2} \, (1-x^2)^{\frac{1}{2}k}$$
$$\times R_{n-k}^{(\alpha+k,\alpha+k)}(x)(1-y^2)^{\frac{1}{2}k} R_{n-k}^{(\alpha+k,\alpha+k)}(y) R_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}\left(\frac{z}{\sqrt{1-x^2}\,\sqrt{1-y^2}}\right).$$

A common specialization

The two addition formulas have the common specialization

$$1 = \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha + k}{\alpha + \frac{1}{2}k} \, \frac{(n+2\alpha+1)_k (2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2} \, (1-x^2)^k \big(R_{n-k}^{(\alpha+k,\alpha+k)}(x) \big)^2.$$

This implies that

$$|R_n^{(\alpha,\alpha)}(x)| \le 1 \quad (-1 \le x \le 1,\ \alpha > -\tfrac{1}{2}).$$

Degenerate linearization formula

Let $l \geq m$. Gegenbauer linearization formula:

$$R_l^{(\alpha,\alpha)}(x)R_m^{(\alpha,\alpha)}(x) = \frac{l!\, m!}{(2\alpha+1)_l(2\alpha+1)_m} \sum_{j=0}^{m} \frac{l+m+\alpha+\frac{1}{2}-2j}{\alpha+\frac{1}{2}}$$

$$\times \frac{(\alpha+\frac{1}{2})_j(\alpha+\frac{1}{2})_{l-j}(\alpha+\frac{1}{2})_{m-j}(2\alpha+1)_{l+m-j}}{j!\,(l-j)!\,(m-j)!\,(\alpha+\frac{3}{2})_{l+m-j}} R_{l+m-2j}^{(\alpha,\alpha)}(x)$$

Assume $x > 1$, divide by $R_l^{(\alpha,\alpha)}(x)$ and use

$$\lim_{l\to\infty} \frac{R_{l+m-2j}^{(\alpha,\alpha)}(x)}{R_l^{(\alpha,\alpha)}(x)} = \left(x+(x^2-1)^{\frac{1}{2}}\right)^{m-2j} \quad (x>1).$$

We obtain:

$$R_m^{(\alpha,\alpha)}(x) = \frac{m!}{(2\alpha+1)_m} \sum_{j=0}^{m} \frac{(\alpha+\frac{1}{2})_j(\alpha+\frac{1}{2})_{m-j}}{j!\,(m-j)!} \left(x+(x^2-1)^{\frac{1}{2}}\right)^{m-2j}.$$

Degenerate dual addition formula

Let $l \geq m$. Gegenbauer dual addition formula:

$$R^{(\alpha,\alpha)}_{l+m-2j}(x) = \sum_{n=0}^{m} \frac{\alpha+n}{\alpha+\frac{1}{2}n}\,\frac{(-l)_n(-m)_n(2\alpha+1)_n}{2^{2n}(\alpha+1)_n^2\,n!}$$

$$\times\,(x^2-1)^n\,R^{(\alpha+n,\alpha+n)}_{l-n}(x)\,R^{(\alpha+n,\alpha+n)}_{m-n}(x)$$

$$\times\,R_n\big(j(j-l-m-\alpha-\tfrac{1}{2});\,\alpha-\tfrac{1}{2},\alpha-\tfrac{1}{2},-m-1,-l-\alpha-\tfrac{1}{2}\big).$$

Assume $x > 1$, divide by $R^{(\alpha,\alpha)}_l(x)$ and use asymptotics as $l \to \infty$ of $R^{(\alpha,\alpha)}_l(x)$. We obtain:

$$\big(x+(x^2-1)^{\frac{1}{2}}\big)^{m-2j} = \sum_{n=0}^{m} \frac{\alpha+n}{\alpha+\frac{1}{2}n}\,\frac{(-1)^n(-m)_n(2\alpha+1)_n}{2^n(\alpha+1)_n\,n!}$$

$$\times\,(x^2-1)^{\frac{1}{2}n}\,R^{(\alpha+n,\alpha+n)}_{m-n}(x)\,Q_n\big(j;\,\alpha-\tfrac{1}{2},\alpha-\tfrac{1}{2},m\big),$$

whre Q_n is a Hahn polynomial.

Further perspective

1. Find dual addition formula for q-ultraspherical polynomials. Linearization formula also due to Rogers (1895). Probably q-Racah polynomials will pop up.
2. Find addition-type formula on a higher level which gives as limit cases for ultraspherical polynomials both the addition formula and the dual addition formula.
3. Find group theoretic interpretation of dual addition formula, for instance for $\alpha = \frac{1}{2}$ in connection with SU(2).

Happy birthday to Charles next month!