Addition type formulas associated with dual product formulas

Tom Koornwinder

Korteweg-de Vries Institute, University of Amsterdam
T.H.Koornwinder@uva.nl
https://staff.fnwi.uva.nl/t.h.koornwinder/

Lecture on 11 August 2016 at the conference
Dunkl operators, special functions and harmonic analysis,
Universität Paderborn, 8–12 August, 2016

last modified: 15 August 2016
Oberwolfach conference *Special functions and group theory*,
March 14–18, 1983

Charles Dunkl
Tom Koornwinder  Dual addition formulas
Dick Askey
Alberto Grünbaum
\[
\prod_{i \neq j} \frac{1 - x_i x_j^{-1}}{(1 - x_i^{-1} x_j)^q} = \frac{(nk)!}{(k!)^n} \frac{(a_1 + \ldots + a_n)!}{a_1! \ldots \cdot a_n!}
\]
Francesco Calogero
Mizan Rahman
$\alpha : E'_x(\mathbb{R}) \rightarrow E'_x(\mathbb{R})$ is an

Product de convolution addition $\Delta$

$S \in E'_x(\mathbb{R}), f \in E'_x(\mathbb{R})$

$\langle S_f, T_x f \rangle$

$x \in \mathbb{R}, T_x f(y) = x_x x_y \left[ \int x_x x^{-1} f(y) \right]$

$\langle T_x f, y \rangle = \frac{1}{2} \left[ f(x+y) + f(x-y) \right]$

$\begin{cases} T_x f(y) = T_y f(x) \quad T_x f(x) = f(x) \\ D T_x f = T_x D f \end{cases}$

Khalifa Trimeche
Michel Mizony
This talk based on:

Tom H. Koornwinder,

*Dual addition formulas associated with dual product formulas,*

arXiv:1607.06053 [math.CA]
Addition formula for Legendre polynomials

Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, $P_n^{(\alpha, \beta)}(1) = \frac{(\alpha + 1)_n}{n!}$,

$$\int_{-1}^{1} P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) (1-x)^{\alpha} (1+x)^{\beta} \, dx = 0 \quad (m \neq n, \, \alpha, \beta > -1).$$

Legendre polynomials $P_n(x) := P_n^{(0,0)}(x)$, $P_n(1) = 1$.

Addition formula for Legendre polynomials:

$$P_n(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi)$$

$$= P_n(\cos \theta_1) P_n(\cos \theta_2) + 2 \sum_{k=1}^{n} \frac{(n - k)! (n + k)!}{2^{2k} (n!)^2}$$

$$\times (\sin \theta_1)^k P_{n-k}^{(k,k)}(\cos \theta_1) (\sin \theta_2)^k P_{n-k}^{(k,k)}(\cos \theta_2) \cos(k\phi).$$

Product formula for Legendre polynomials:

$$P_n(\cos \theta_1) P_n(\cos \theta_2) = \frac{1}{\pi} \int_{0}^{\pi} P_n(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi) \, d\phi.$$
Linearization formula for Legendre polynomials

\[ P_l(x)P_m(x) = \sum_{j=0}^{\text{min}(l,m)} \left( \frac{1}{2} \right)^j \frac{\left( \frac{1}{2} \right)_{l-j} \left( \frac{1}{2} \right)_{m-j} (l + m - j)!}{j! (l-j)! (m-j)! \left( \frac{3}{2} \right)_{l+m-j}} \right) \right) \times (2(l + m - 2j) + 1) P_{l+m-2j}(x). \]

Askey’s question

Find an addition type formula corresponding to this dual product formula for Legendre polynomials just as the addition formula for Legendre polynomials corresponds to the product formula for Legendre polynomials.

Compare with rewritten product formula:

\[ P_n(x)P_n(y) = \frac{1}{\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{P_n(z + xy)}{\sqrt{(1 - x^2)(1 - y^2) - z^2}} \sqrt{1-y^2} \sqrt{1-x^2} \ \text{dz}. \]

Corresponding addition formula expands \( P_n(z + xy) \) in terms of Chebyshev polynomials of dilated argument.
Chebyshev polynomials $T_n(\cos \theta) := \cos(n\theta)$,

\[ T_n(x) = \text{const. } P_{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x), \quad T_n(1) = 1. \]

Rewritten product formula:

\[ P_n(x)P_n(y) = \frac{1}{\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{\sqrt{1-y^2}}{\sqrt{1-y^2}} \frac{P_n(z + xy)}{\sqrt{(1 - x^2)(1 - y^2) - z^2}} \, dz. \]

Rewritten addition formula:

\[ P_n(z + xy) = P_n(x)P_n(y) + 2 \sum_{k=1}^{n} \frac{(n-k)! (n+k)!}{2^{2k} (n!)^2} \times (1-x^2)^{\frac{1}{2}k} P_{n-k}^{(k,k)}(x) (1-y^2)^{\frac{1}{2}k} P_{n-k}^{(k,k)}(y) T_k \left( \frac{z}{\sqrt{1 - x^2} \sqrt{1 - y^2}} \right). \]
Hint from a Hallnäs-Ruijsenaars product formula

Jacobi functions:

\[
\phi^{(\alpha,\beta)}_\lambda(t) := 2 F_1 \left( \frac{1}{2}(\alpha + \beta + 1 + i \lambda), \frac{1}{2}(\alpha + \beta + 1 - i \lambda); -\sinh^2 t \right).
\]

Transform pair for suitable \( f \) or \( g \) \((\alpha \geq \beta \geq -\frac{1}{2})\):

\[
\begin{cases}
g(\lambda) = \int_0^\infty f(t) \phi^{(\alpha,\beta)}_\lambda(t) (\sinh t)^{2\alpha+1} (\cosh t)^{2\beta+1} \, dt, \\
f(t) = \text{const.} \int_0^\infty g(\lambda) \phi^{(\alpha,\beta)}_\lambda(t) \frac{d\lambda}{|c(\lambda)|^2}.
\end{cases}
\]

Dual product formula for Jacobi functions \((\beta = -\frac{1}{2})\) by Hallnäs & Ruijsenaars (2015) reveals weight function for Wilson polynomials with parameters \( \pm i \lambda \pm i \mu + \frac{1}{2}\alpha + \frac{1}{4} \) (cases \( \alpha = 0 \) and \( \frac{1}{2} \) due to Mizony, 1976):

\[
\phi^{(\alpha,-\frac{1}{2})}_{2\lambda}(t) \phi^{(\alpha,-\frac{1}{2})}_{2\mu}(t) = \text{const.} \int_0^\infty \phi^{(\alpha,-\frac{1}{2})}_{2\nu}(t) \left| \frac{\Gamma(i\nu \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4})}{\Gamma(2i\nu)} \right|^2 \, d\nu.
\]
Linearization formula for Gegenbauer polynomials

Renormalized Jacobi polynomials $R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$.

Gegenbauer linearization formula (Rogers, 1895):

$$R_l^{(\alpha,\alpha)}(x) R_m^{(\alpha,\alpha)}(x) = \frac{l! \, m!}{(2\alpha + 1)_l (2\alpha + 1)_m} \sum_{j=0}^{\min(l,m)} \frac{l + m + \alpha + \frac{1}{2} - 2j}{\alpha + \frac{1}{2}}$$

$$\times \frac{(\alpha + \frac{1}{2})_j (\alpha + \frac{1}{2})_{l-j} (\alpha + \frac{1}{2})_{m-j} (2\alpha + 1)_{l+m-j}}{j! \, (l-j)! \, (m-j)! \, (\alpha + \frac{3}{2})_{l+m-j}} R_{l+m-2j}^{(\alpha,\alpha)}(x)$$

$$= \sum_{j=0}^{m} \frac{w_{\alpha-\frac{1}{2},\alpha-\frac{1}{2},-m-1,-l-\alpha-\frac{1}{2}}(j)}{h_{0; \alpha-\frac{1}{2},\alpha-\frac{1}{2},-m-1,-l-\alpha-\frac{1}{2}}} R_{l+m-2j}^{(\alpha,\alpha)}(x) \quad (l \geq m, \alpha > -\frac{1}{2}),$$

where $w_{\alpha,\beta,\gamma,\delta}(x)$ are the weights for the Racah polynomials

$R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) \quad (\gamma = -N - 1, n = 0, 1, \ldots, N)$. 
Racah polynomials

\[ R_n(x(x + γ + δ + 1); α, β, γ, δ) \]
\[ := 4F_3 \left( \begin{array}{c} -n, n + α + β + 1, -x, x + γ + δ + 1 \\ α + 1, β + δ + 1, γ + 1 \end{array} ; 1 \right), \; γ = -N - 1, \]
\[ \sum_{x=0}^{N} (R_m R_n)(x(x + γ + δ + 1); α, β, γ, δ) w_{α,β,γ,δ}(x) = h_{n;α,β,γ,δ} \delta m, n, \]
\[ w_{α,β,γ,δ}(x) = \frac{γ + δ + 1 + 2x}{γ + δ + 1} \]
\[ \times \frac{(α + 1)_x(β + δ + 1)_x(γ + 1)_x(γ + δ + 1)_x}{(-α + γ + δ + 1)_x(-β + γ + 1)_x(δ + 1)_x x!}, \]
\[ \frac{h_{n;α,β,γ,δ}}{h_{0;α,β,γ,δ}} = \frac{α + β + 1}{α + β + 2n + 1} \frac{(β + 1)_n(α + β - γ + 1)_n(α - δ + 1)_n n!}{(α + 1)_n(α + β + 1)_n(β + δ + 1)_n(γ + 1)_n}, \]
\[ h_{0;α,β,γ,δ} = \sum_{x=0}^{N} w_{α,β,γ,δ}(x) = \frac{(α + β + 2)_N(−δ)_N}{(α - δ + 1)_N(β + 1)_N}. \]
Racah coefficients of $R^{(\alpha,\alpha)}_{l+m-2j}(x)$

$$R^{(\alpha,\alpha)}_{l}(x)R^{(\alpha,\alpha)}_{m}(x) = \sum_{j=0}^{m} \frac{w_{\alpha-\frac{1}{2},\alpha-\frac{1}{2},-m-1,-l-\alpha-\frac{1}{2}}(j)}{h_0;\alpha-\frac{1}{2},\alpha-\frac{1}{2},-m-1,-l-\alpha-\frac{1}{2}} R^{(\alpha,\alpha)}_{l+m-2j}(x)$$

$(l \geq m, \alpha > -\frac{1}{2})$. More generally evaluate

$$S^{\alpha}_{n}(l, m) := \sum_{j=0}^{m} w_{\alpha-\frac{1}{2},\alpha-\frac{1}{2},-m-1,-l-\alpha-\frac{1}{2}}(j) R^{(\alpha,\alpha)}_{l+m-2j}(x)$$

$$\times R_{n}(j(j-l-m-\alpha-\frac{1}{2}); \alpha - \frac{1}{2}, \alpha - \frac{1}{2}, -m - 1, -l - \alpha - \frac{1}{2}).$$

By Racah Rodrigues formula, summation by parts, and a Gegenbauer difference formula we get

$$S^{\alpha}_{n}(l, m) = \frac{(2\alpha + 1)_{l+n}(2\alpha + 1)_{m+n}(\alpha + \frac{1}{2})_{l+m}}{2^{2n}(\alpha + \frac{1}{2})_{l}(\alpha + \frac{1}{2})_{m}(2\alpha + 1)_{l+m}(\alpha + 1)_{n}^{2}} (x^2 - 1)^n$$

$$\times R^{(\alpha+n,\alpha+n)}_{l-n}(x) R^{(\alpha+n,\alpha+n)}_{m-n}(x).$$

Then Fourier-Racah inversion gives:
Theorem (Dual addition formula for Gegenbauer polynomials)

\[ R_{l+m-2j}^{(\alpha,\alpha)}(x) = \sum_{n=0}^{m} \frac{\alpha + n}{\alpha + \frac{1}{2} n} \frac{(-1)^n(-m)_n(2\alpha + 1)_n}{2^{2n}(\alpha + 1)^2_n n!} \]
\[ \times (x^2 - 1)^n R_{l-n}^{(\alpha+n,\alpha+n)}(x) R_{m-n}^{(\alpha+n,\alpha+n)}(x) \]
\[ \times R_n(j(j - l - m - \alpha - \frac{1}{2}); \alpha - \frac{1}{2}, \alpha - \frac{1}{2}, -m - 1, -l - \alpha - \frac{1}{2}) \]
\[ (l \geq m, \, j = 0, 1, \ldots, m). \]

Compare with addition formula for Gegenbauer polynomials:

\[ R_n^{(\alpha,\alpha)}(xy + z) = \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha + k}{\alpha + \frac{1}{2} k} \frac{(n + 2\alpha + 1)_k(2\alpha + 1)_k}{2^{2k}(\alpha + 1)_k^2} (1 - x^2)^{\frac{1}{2} k} \]
\[ \times R_{n-k}^{(\alpha+k,\alpha+k)}(x)(1 - y^2)^{\frac{1}{2} k} R_{n-k}^{(\alpha+k,\alpha+k)}(y) R_{k}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})} \left( \frac{z}{\sqrt{1 - x^2} \sqrt{1 - y^2}} \right). \]
The two addition formulas have the common specialization

\[ 1 = \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha + k}{\alpha + \frac{1}{2} k} \frac{(n + 2\alpha + 1)_k (2\alpha + 1)_k}{2^{2k}(\alpha + 1)_k^2} (1-x^2)^k \left( R_{n-k}^{(\alpha+k,\alpha+k)}(x) \right)^2. \]

This implies that

\[ |R_{n}^{(\alpha,\alpha)}(x)| \leq 1 \quad (-1 \leq x \leq 1, \quad \alpha > -\frac{1}{2}). \]
Degenerate linearization formula

Let $l \geq m$. Gegenbauer linearization formula:

\[
R_l^{(\alpha,\alpha)}(x) R_m^{(\alpha,\alpha)}(x) = \frac{l! \ m!}{(2\alpha + 1)_l (2\alpha + 1)_m} \sum_{j=0}^{m} \frac{l + m + \alpha + \frac{1}{2} - 2j}{\alpha + \frac{1}{2}} \\
\times \frac{\alpha + \frac{1}{2}) j (\alpha + \frac{1}{2})_l (\alpha + \frac{1}{2})_m (2\alpha + 1)_{l+m-j}}{j! (l-j)! (m-j)! (\alpha + \frac{3}{2})_{l+m-j}} \ R_{l+m-2j}^{(\alpha,\alpha)}(x)
\]

Assume $x > 1$, divide by $R_l^{(\alpha,\alpha)}(x)$ and use

\[
\lim_{l \to \infty} \frac{R_{l+m-2j}^{(\alpha,\alpha)}(x)}{R_l^{(\alpha,\alpha)}(x)} = (x + (x^2 - 1)^{\frac{1}{2}})^{m-2j} \quad (x > 1).
\]

We obtain:

\[
R_m^{(\alpha,\alpha)}(x) = \frac{m!}{(2\alpha + 1)_m} \sum_{j=0}^{m} \frac{\alpha + \frac{1}{2}) j (\alpha + \frac{1}{2})_m (2\alpha + 1)_{l+m-j}}{j! (m-j)!} (x+(x^2 - 1)^{\frac{1}{2}})^{m-2j}.
\]
Let \( l \geq m \). Gegenbauer dual addition formula:

\[
R_{l+m-2j}^{(\alpha,\alpha)}(x) = \sum_{n=0}^{m} \frac{\alpha + n}{\alpha + \frac{1}{2}n} \frac{(-l)_n(-m)_n(2\alpha + 1)_n}{2^n(\alpha + 1)_n^n n!} \times (x^2 - 1)^n R_{l-n}^{(\alpha+n,\alpha+n)}(x) R_{m-n}^{(\alpha+n,\alpha+n)}(x) \times R_n(j(j - l - m - \alpha - \frac{1}{2}); \alpha - \frac{1}{2}, \alpha - \frac{1}{2}, -m - 1, -l - \alpha - \frac{1}{2}).
\]

Assume \( x > 1 \), divide by \( R_{l}^{(\alpha,\alpha)}(x) \) and use asymptotics as \( l \to \infty \) of \( R_{l}^{(\alpha,\alpha)}(x) \). We obtain:

\[
(x + (x^2 - 1)^{\frac{1}{2}})^{m-2j} = \sum_{n=0}^{m} \frac{\alpha + n}{\alpha + \frac{1}{2}n} \frac{(-1)_n(-m)_n(2\alpha + 1)_n}{2^n(\alpha + 1)_n^n n!} \times (x^2 - 1)^{\frac{1}{2}n} R_{m-n}^{(\alpha+n,\alpha+n)}(x) Q_n(j; \alpha - \frac{1}{2}, \alpha - \frac{1}{2}, m),
\]

where \( Q_n \) is a Hahn polynomial.
Find dual addition formula for \( q \)-ultraspherical polynomials. Linearization formula also due to Rogers (1895). Probably \( q \)-Racah polynomials will pop up.

Find addition-type formula on a higher level which gives as limit cases for ultraspherical polynomials both the addition formula and the dual addition formula.

Find group theoretic interpretation of dual addition formula, for instance for \( \alpha = \frac{1}{2} \) in connection with SU(2).

Further perspective
Happy birthday to Charles next month!