A nonsymmetric version of Okounkov's *BC*-type interpolation Macdonald polynomials

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This talk is dedicated to **Grigori Olshanski** on the occasion of his seventieth birthday.
By his 1997 papers with Okounkov on shifted Schur functions and shifted Jack polynomials he was one of the pioneers on interpolation polynomials in connection with multivariate orthogonal polynomials associated wih root systems.

(q-)Pochhammer symbols:

Let $q \in \mathbb{C}$, $0 < |q| < 1$, $n \in \mathbb{Z}_{\geq 0}$.

$$
\begin{aligned}
(a)_n &:= a(a+1)\ldots(a+n-1), \\
(a;q)_n &:= (1-a)(1-qa)\ldots(1-q^{n-1}a), \\
(a_1,\ldots,a_k;q)_n &:= (a_1;q)_n\ldots(a_k;q)_n.
\end{aligned}
$$

$$
\begin{aligned}
(a;q)_\infty &:= \prod_{j=0}^{\infty}(1-q^j a), \\
(a_1,\ldots,a_k;q)_\infty &:= (a_1;q)_\infty\ldots(a_k;q)_\infty.
\end{aligned}
$$

Interpolation:

1. $$x(x-1)\dots(x-n+1) = (-1)^n(-x)_n$$

   is the unique monic polynomial of degree n which vanishes at $x = 0, 1, \dots, n-1$.

2. $$(x-1)(x-q)\dots(x-q^{n-1}) = x^n(x^{-1};q)_n$$

   is the unique monic polynomial of degree n which vanishes at $x = 1, q, \dots, q^{n-1}$.

3. $$\prod_{j=0}^{n-1}(z+z^{-1}-aq^j-a^{-1}q^{-j}) = \frac{(az, az^{-1};q)_n}{(-1)^n\, q^{\frac{1}{2}n(n-1)}\, a^n}$$

   is the unique monic symmetric Laurent polynomial of degree n which vanishes on $a, aq, \dots, aq^{n-1}$ (and their inverses) .

Askey–Wilson polynomials:

$$R_n(z;a,b,c,d\,|\,q)=\frac{p_n(\frac12(z+z^{-1});a,b,c,d\,|\,q)}{p_n(\frac12(a+a^{-1});a,b,c,d\,|\,q)}$$

$$:={}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ab,ac,ad\end{matrix};q,q\right)$$

$$=\sum_{k=0}^n\frac{(q^{-n},q^{n-1}abcd;q)_k\,(az,az^{-1};q)_k}{q^{-k}\,(ab,ac,ad,q;q)_k}.$$

Orthogonality. $0<q<1$; $|a|,|b|,|c|,|d|\le 1$ with pairwise products of $a,b,c,d\ne 1$, and non-real a,b,c,d in complex conjugate pairs.

$$\Delta_+(z)=\Delta_+(z;a,b,c,d,;q):=\frac{(z^2;q)_\infty}{(az,bz,cz,dz;q)_\infty},$$

$$\Delta(z):=\Delta_+(z)\Delta_+(z^{-1}).$$

$$\int_{|z|=1}R_n(z)\,R_m(z)\,\Delta(z)\,\frac{dz}{z}=0\qquad\text{if } n\ne m.$$

Dual parameters $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$:
$\widetilde{a} := (q^{-1}abcd)^{\frac{1}{2}}, \quad \widetilde{a}\,\widetilde{b} = ab, \quad \widetilde{a}\,\widetilde{c} = ac, \quad \widetilde{a}\,\widetilde{d} = ad.$
Then

$$(q^{-n}, q^{n-1}abcd; q)_k = \left(\widetilde{a}\,(\widetilde{a}q^n), \widetilde{a}\,(\widetilde{a}q^n)^{-1}; q\right)_k.$$

Hence

$$R_n(z; a, b, c, d \,|\, q) = \sum_{k=0}^{n} \frac{\left(\widetilde{a}\,(\widetilde{a}q^n), \widetilde{a}\,(\widetilde{a}q^n)^{-1}; q\right)_k (az, az^{-1}; q)_k}{q^{-k}\,(ab, ac, ad, q; q)_k}\,,$$

from which the **duality relation**

$$R_n(a^{-1}q^{-m}; a, b, c, d \,|\, q) = R_m\left(\widetilde{a}^{-1}q^{-n}; \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \,|\, q\right) \quad (m, n \in \mathbb{Z}_{\geq 0}),$$

since both sides are equal to

$$\sum_{k=0}^{\min(m,n)} \frac{\left(q^{-n}, \widetilde{a}^2 q^n; q\right)_k (q^{-m}, a^2 q^m; q)_k}{q^{-k}\,(ab, ac, ad, q; q)_k}\,.$$

Limit of Askey–Wilson polynomials

$$R_n(az;a,b,c,d\,|\,q) = \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd;q)_k\,(a^2z, z^{-1};q)_k\,q^k}{(ab,ac,ad,q;q)_k}\,.$$

For $a \to \infty$ the classical q-binomial formula

$$\begin{aligned} z^n &= \sum_{k=0}^{n} \frac{(q^{-n}, z^{-1};q)_k}{(q;q)_k}\,(-1)^k q^{-\frac{1}{2}k(k-1)}\,(q^n z)^k \\ &= \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}\,(z-1)(z-q)\ldots(z-q^{k-1}). \end{aligned}$$

For $q \to 1$ the classical binomial formula

$$z^n = \sum_{k=0}^{n} \binom{n}{k} (z-1)^k.$$

Partitions:
$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.
Λ_n^+ is the set of all such partitions.
λ has *weight* $|\lambda| := \lambda_1 + \cdots + \lambda_n$.
$\delta := (n-1, n-2, \ldots, 0)$.

Dominance and inclusion partial ordering. For $\lambda, \mu \in \Lambda_n^+$:

$$\mu \leq \lambda \quad \text{iff} \quad \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \quad (i = 1, \ldots, n);$$
$$\mu \subseteq \lambda \quad \text{iff} \quad \mu_i \leq \lambda_i \quad (i = 1, \ldots, n).$$

Symmetrized monomials. For $\lambda \in \Lambda_n^+$:

$$m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu, \qquad \widetilde{m}_\lambda(x) := \sum_{\mu \in W_n \lambda} x^\mu \quad (W_n := S_n \ltimes (\mathbb{Z}_2)^n).$$

These are symmetric polynomials and symmetric Laurent polynomials, respectively.

Macdonald weight function $(0 < q, t < 1)$:

$$\Delta_+(x) = \Delta_+(x;q,t) := \prod_{1\le i<j\le n} \frac{(x_i x_j^{-1};q)_\infty}{(t\,x_i x_j^{-1};q)_\infty}\,,$$

$$\Delta(x) := \Delta_+(x)\Delta_+(x^{-1}).$$

Macdonald polynomials. Define for $\lambda \in \Lambda_n^+$

$$P_\lambda(x;q,t) = P_\lambda(x) = \sum_{\mu\le\lambda} u_{\lambda,\mu} m_\mu(x), \quad u_{\lambda,\lambda} = 1,$$

such that

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu < \lambda.$$

(Note that $P_\mu(x^{-1}) = \overline{P_\mu(x)}$ for $x \in \mathbb{T}^n \subset \mathbb{C}^n$.)

Then $P_\lambda(x)$ is homogeneous of degree $|\lambda|$ in x and there is full orthogonality:

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu \ne \lambda.$$

Koornwinder weight function. $0 < q, t < 1$,
$|a|, |b|, |c|, |d| \le 1$ with pairwise products of $a, b, c, d \neq 1$ and non-real a, b, c, d in complex conjugate pairs.

$$\Delta_+(x) = \Delta_+(x; q, t; a, b, c, d)$$
$$:= \prod_{j=1}^{n} \frac{(x_j^2; q)_\infty}{(ax_j, bx_j, cx_j, dx_j; q)_\infty} \prod_{1 \le i < j \le n} \frac{(x_i x_j, x_i x_j^{-1}; q)_\infty}{(t x_i x_j, t x_i x_j^{-1}; q)_\infty},$$

$\Delta(x) := \Delta_+(x)\Delta_+(x^{-1})$. For $n = 1$ no t: Askey–Wilson case.

Koornwinder polynomials (1992). Define for $\lambda \in \Lambda_n^+$

$$P_\lambda(x; q, t; a, b, c, d) = P_\lambda(x) = \sum_{\mu \le \lambda} u_{\lambda,\mu} \widetilde{m}_\mu(x), \quad u_{\lambda,\lambda} = 1,$$

such that

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x)\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu < \lambda.$$

Then full orthogonality (for $\mu \neq \lambda$).

5-parameter generalization of 3-parameter Macdonald BC_n polynomials.

A-type interpolation polynomials (Kostant & Sahi, Sahi, Knop, Knop & Sahi, 1991–1997)

Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $0 < |\tau_1| < |\tau_2| < \cdots < |\tau_n| < 1$.
Let $q, t \in \mathbb{C}$, $0 < |q|, |t| < 1$.
For $\lambda \in \Lambda_n^+$ define $P_\lambda^{\mathrm{ip,A}}(x; q, \tau)$ as the unique symmetric polynomial of degree $|\lambda| = \lambda_1 + \cdots + \lambda_n$ such that its term involving x^λ has coefficient 1 and

$$P_\lambda^{\mathrm{ip,A}}(q^\mu \tau; q, \tau) = 0 \quad \text{if } \mu \in \Lambda_n^+, |\mu| \le |\lambda|, \mu \ne \lambda.$$

In the *principal specialization* $\tau := t^\delta = (t^{n-1}, t^{n-2}, \ldots, 1)$ moreover the following two properties hold:

Extra-vanishing property:
$P_\lambda^{\mathrm{ip,A}}(q^\mu t^\delta; q, t^\delta) = 0$ if $\mu \in \Lambda_n^+$ and $\lambda \not\subseteq \mu$.

Expansion in terms of Macdonald polynomials:

$$P_\lambda^{\mathrm{ip,A}}(x; q, t^\delta) = \sum_{\mu \subseteq \lambda} b_{\lambda,\mu} P_\mu(x; q, t), \quad b_{\lambda,\lambda} = 1.$$

Hence $\quad r^{-|\lambda|} P_\lambda^{\mathrm{ip,A}}(rx; q, t^\delta) \to P_\lambda(x; q, t) \quad$ as $r \to \infty$.

BC-type interpolation polynomials (Okounkov, 1998)

τ, q, t as before, and $a \in \mathbb{C}$, $0 < |a| < 1$.
For $\lambda \in \Lambda_n^+$ define $P_\lambda^{\mathrm{ip,B}}(x; q, \tau)$ as the unique W_n-invariant Laurent polynomial of degree $|\lambda|$ such that its term involving x^λ has coefficient 1 and

$$P_\lambda^{\mathrm{ip,B}}(q^\mu \tau; q, \tau) = 0 \quad \text{if } \mu \in \Lambda_n^+,\ |\mu| \le |\lambda|,\ \mu \ne \lambda.$$

In the *principal specialization* $\tau := t^\delta a$ we have moreover:

Extra-vanishing property:
$P_\lambda^{\mathrm{ip,B}}(q^\mu t^\delta a; q, t^\delta a) = 0$ if $\mu \in \Lambda_n^+$ and $\lambda \not\subseteq \mu$.

Sahi

Knop

Okounkov

Limit formulas

$$\lim_{r\to\infty} r^{-|\lambda|} P_\lambda(rx; q, t; a, b, c, d) = P_\lambda(x; q, t),$$
$$\lim_{r\to\infty} r^{-|\lambda|} P_\lambda^{\mathrm{ip,B}}(rx; q, t^\delta a) = P_\lambda(x; q, t),$$
$$\lim_{r\to\infty} r^{-|\lambda|} P_\lambda^{\mathrm{ip,A}}(rx; q, t^\delta) = P_\lambda(x; q, t).$$

So Koornwinder polynomials and BC-type and A-type interpolation polynomials all have the Macdonald polynomials as highest degree part.

$$\lim_{a\to\infty} a^{-|\lambda|} P_\lambda(ax; q, t; a, b, c, d) = P_\lambda(x; q, t),$$
$$\lim_{a\to\infty} a^{-|\lambda|} P_\lambda^{\mathrm{ip,B}}(ax; q, t^\delta a) = P_\lambda^{\mathrm{ip,A}}(x; q, t^\delta).$$

Binomial formulas.

For Koornwinder polynomials (Okounkov, 1998; recall that $\widetilde{a} := (q^{-1}abcd)^{\frac{1}{2}}$):

$$\frac{P_\lambda(x;q,t;a,b,c,d)}{P_\lambda(t^\delta a;q,t;a,b,c,d)} = \sum_{\mu\subseteq\lambda} \frac{P_\mu^{\mathrm{ip,B}}(q^\lambda t^\delta \tilde{a};q,t^\delta\tilde{a})}{P_\mu^{\mathrm{ip,B}}(q^\mu t^\delta \tilde{a};q,t^\delta\tilde{a})}\,\frac{P_\mu^{\mathrm{ip,B}}(x;q,t^\delta a)}{P_\mu(t^\delta a;a,b,c,d,t;q)}\,.$$

For $\lambda = (\lambda_1, 0, \ldots, 0)$ the first factor on the right becomes elementary.

For Macdonald polynomials (Okounkov, 1997):

$$\frac{P_\lambda(x;q,t)}{P_\lambda(t^\delta;q,t)} = \sum_{\mu\subseteq\lambda} \frac{P_\mu^{\mathrm{ip,A}}(q^\lambda t^\delta;q,t^\delta)}{P_\mu^{\mathrm{ip,A}}(q^\mu t^\delta;q,t^\delta)}\,\frac{P_\mu^{\mathrm{ip,A}}(x;q,t^\delta)}{P_\mu(t^\delta;q,t)}\,.$$

Okounkov (1997) gives also a binomial formula for A-type interpolation polynomials.

Formulas are related by limit relations. Formulas imply duality.

BC-type interpolation polynomials for $n = 2$ (K, 2015)

From Okounkov's (1998) combinatorial formula:

$$P^{\mathrm{ip,B}}_{m_1,m_2}(x_1,x_2;q,t^\delta a) = \frac{q^{-\frac12 m_1(m_1-1)-\frac12 m_2(m_2-1)}}{(-t)^{m_1-m_2}a^{m_1+m_2}}\,(t,q^{2m_2}ta^2;q)_{m_1-m_2}$$
$$\times (ax_1,ax_1^{-1},ax_2,ax_2^{-1};q)_{m_2}\sum_{\substack{j,k\ge0\\ j+k\le m_1-m_2}}\frac{(q^{-m_1+m_2};q)_{j+k}q^{j+k}}{(q^{2m_2}ta^2;q)_{j+k}}$$
$$\times\frac{(q^{m_2}ax_1,q^{m_2}ax_1^{-1};q)_j}{(q^{1-m_1+m_2}t^{-1},q;q)_j}\,\frac{(q^{m_2}ax_2,q^{m_2}ax_2^{-1};q)_k}{(q^{1-m_1+m_2}t^{-1},q;q)_k}$$
$$=\frac{q^{-\frac12 m_1(m_1-1)-\frac12 m_2(m_2-1)}}{(-t)^{m_1-m_2}a^{m_1+m_2}}\,(ax_1,ax_1^{-1},ax_2,ax_2^{-1};q)_{m_2}$$
$$\times(q^{m_2}tax_1,q^{m_2}tax_1^{-1};q)_{m_1-m_2}$$
$$\times\;{}_4\phi_3\left(\begin{matrix}q^{-m_1+m_2},t,q^{m_2}ax_2,q^{m_2}ax_2^{-1}\\ q^{1-m_1+m_2}t^{-1},q^{m_2}tax_1,q^{m_2}tax_1^{-1}\end{matrix};q,q\right).$$

Limit for $q \to 1$ of BC-type binomial formula:

On the left (Heckman–Opdam) BC-type Jacobi polynomials.

In first factor on the right BC-type interpolation polynomials for $q = 1$. In second factor on the right Jack polynomials.

For $n = 2$ both factors on the right have explicit expressions, and the formula was derived quite differently in 1978 by Sprinkhuizen & K.

Nonsymmetric Macdonald polynomials
$E_\lambda(x;q,t) \quad (\lambda \in \Lambda_n := (\mathbb{Z}_{\geq 0})^n)$ and
nonsymmetric Koornwinder polynomials
$E_\lambda(x;a,b,c,d,t;q) \quad (\lambda \in \mathbb{Z}^n)$ can be defined as eigenfunctions (polynomials or Laurent polynomials, respectively) of suitable q-difference-reflection operators (generalized Dunkl operators) coming from the polynomial representation of a suitable double affine Hecke algebra.

Expansion of symmetric polynomials in terms of non-symmetric polynomials $(\lambda \in \Lambda_n^+)$:

$$P_\lambda = \sum_{\mu \in S_n \lambda} a_{\lambda,\mu} E_\mu \quad \text{(Macdonald polynomials)},$$

$$P_\lambda = \sum_{\mu \in W_n \lambda} b_{\lambda,\mu} E_\mu \quad \text{(Koornwinder polynomials)}$$

for suitable coefficients $a_{\lambda,\mu}$ and $b_{\lambda,\mu}$.

The one-variable cases:
In Macdonald case just z^n (trivial Weyl group, no distinction between symmetric and non-symmetric).

Nonsymmetric Askey–Wilson polynomials
(Sahi, 1999; Noumi & Stokman, 2004)

$$E_n(z;a,b,c,d;q) :=$$
$$R_n(z;a,b,c,d;q) - \frac{q^{1-n}(1-q^n)(1-q^{n-1}cd)}{(1-qab)(1-ab)(1-ac)(1-ad)}$$
$$\times\, az^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d;q), \quad n \ge 0,$$
$$E_{-n}(z;a,b,c,d;q) :=$$
$$R_n(z;a,b,c,d;q) - \frac{q^{1-n}(1-q^n ab)(1-q^{n-1}abcd)}{(1-qab)(1-ab)(1-ac)(1-ad)}$$
$$\times\, b^{-1}z^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d;q), \quad n \ge 1.$$

The symmetric Askey–Wilson polynomials are symmetrizations of the nonsymmetric Askey–Wilson polynomials.

Nonsymmetric interpolation

Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $0 < |\tau_1| < |\tau_2| < \cdots < |\tau_n| < 1$.

The case A_{n-1} (Sahi, Knop).
Simple roots $e_1 - e_2, \ldots, e_{n-1} - e_n$. $\alpha \in \Lambda_n$, $|\alpha| := \sum_{i=1}^d |\alpha_i|$.
Let w_α be shortest element in S_n such that $w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$.
Then, for $i < j$, $w_\alpha^{-1}(i) < w_\alpha^{-1}(j)$ iff $\alpha_i \geq \alpha_j$.
Let $0 < |q| < 1$. *Polynomials $f(x)$ of degree $\leq d$ are uniquely determined by values $f(\overline{\alpha})$ on $\overline{\alpha} := q^\alpha w_\alpha \tau$ ($\alpha \in \Lambda_n$, $|\alpha| \leq d$).*

The case BC_n (Disveld, Stokman, K)
Simple roots $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n$. $\alpha \in \mathbb{Z}^n$, $|\alpha| := \sum_{i=1}^d |\alpha_i|$.
Let w_α be shortest element in W_n such that $w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$.
Write $w_\alpha = \sigma_\alpha \pi_\alpha$ ($\sigma_\alpha \in \{\pm 1\}^n$, $\pi_\alpha \in S_n$). Then, for $i < j$,

$$\pi_\alpha^{-1}(i) < \pi_\alpha^{-1}(j) \iff |\alpha_i| > |\alpha_j| \text{ or } 0 \leq \alpha_i = \pm\alpha_j.$$

Let $0 < |q|, |a| < 1$. *Laurent polynomials $f(x)$ of degree $\leq d$ are uniquely determined by values $f(\overline{\alpha})$ on $\overline{\alpha} := q^\alpha w_\alpha \tau$ ($\alpha \in \mathbb{Z}^n$, $|\alpha| \leq d$).* Here $(w_\alpha \tau)_i := (\tau_{\pi_\alpha^{-1}(i)})^{\mathrm{sgn}(\alpha_i)}$, where $\mathrm{sgn}(0) := 1$.

Idea of the BC_n existence proof (constructive by recurrence)

$I \subseteq [1,n] := \{1,2,\ldots,n\}$.

$$\begin{aligned} \Lambda_{n,d} &:= \{\mu \in \mathbb{Z}^n \mid |\mu| \le d\}, \\ R(n,d,I) &:= \{\alpha \in \Lambda_{n,d} \mid \alpha_j \ne 0 \text{ for all } j \text{ and } \alpha_i \ne -1 \text{ if } i \in I^c\}, \\ T(n,d,I) &:= \{\alpha \in \Lambda_{n,d} \mid \alpha_i \ne 0 \text{ if } i \in I^c\}. \end{aligned}$$

Note that

$$R\big(n,d,[1,n]\big) = T(n,d,\emptyset), \qquad T\big(n,d,[1,n]\big) = \Lambda_{n,d}.$$

By complete induction with respect to $n+d$:

1. For every map $\overline{f}\colon R(n,d,I) \to \mathbb{C}$ there is a Laurent polynomial $f \in \mathcal{P}_n$ such that $f(\overline{\alpha}(q,\tau)) = \overline{f}(\alpha)$ for all $\alpha \in R(n,d,I)$ and $\deg\big(x_J f(x)\big) \le d-n+k \quad \forall J \subseteq I$.
2. For every map $\overline{f}\colon T(n,d,I) \to \mathbb{C}$ there is a Laurent polynomial $f \in \mathcal{P}_n$ such that $f(\overline{\alpha}(q,\tau)) = \overline{f}(\alpha)$ for all $\alpha \in T(n,d,I)$ and $\deg\big(x_J f(x)\big) \le d \qquad \forall J \subseteq I^c$.

Both items are successively proved by complete induction with respect to $|I|$.

The case $n=1$: $\overline{\alpha}=q^\alpha\tau^{\mathrm{sgn}(\alpha)}$ $(\alpha\in\mathbb{Z},\, 0<|q|,|\tau|<1)$.
For given $\overline{f}\colon[-d,d]\to\mathbb{C}$ find Laurent polynomial $f(x)$ of degree $\le d$ such that $f(\overline{\alpha})=\overline{f}(\alpha)$ $(|\alpha|\le d)$.
First deal with interpolation points $\overline{0}=\tau$ and $\overline{-1}=q^{-1}\tau^{-1}$.
Then use $\overline{\alpha}=q^{\alpha-\mathrm{sgn}(\alpha)}(q\tau)^{\mathrm{sgn}(\alpha)}$ and reduce to interpolation problem with d,τ replaced by $d-1,q\tau$. In practice:

Ansatz $f(x)=c_1+(x-\tau)f_1(x)$. Then $f(\tau)=c_1$,
$f(\overline{\alpha})=c_1+(\overline{\alpha}-\tau)f_1(\overline{\alpha})$ for $|\alpha|\le d$, $\alpha\ne0$, then $\overline{\alpha}-\tau\ne0$.

Ansatz $f_1(x)=c_2+(x^{-1}-q\tau)f_2(x)$. Then $f_1(q^{-1}\tau^{-1})=c_2$,
$f_1(\overline{\alpha})=c_2+(\overline{\alpha}^{-1}-q\tau)f_2(\overline{\alpha})$ for $|\alpha|\le d$, $\alpha\ne0,-1$,
then $\overline{\alpha}^{-1}-q\tau\ne0$.

With $\overline{\alpha}=q^\alpha(q\tau)^{\mathrm{sgn}(\alpha)}$ and given $\overline{f_2}\colon[-d+1,d-1]\to\mathbb{C}$
find Laurent polynomial $f_2(x)$ of degree $\le d-1$ such that
$f_2(\overline{\alpha})=\overline{f_2}(\alpha)$ $(|\alpha|\le d-1)$.

Note: If $f_2(x)$ of degree $\le d-1$ then $(x^{-1}-q\tau)f_2(x)$ of degree $\le d$ and $(x-\tau)f_1(x)=(x-\tau)(x^{-1}-q\tau)f_2(x)$ of degree $\le d$.

Symmetric BC-type interpolation polynomials in different normalization:
For $\lambda \in \Lambda_n^+$ there is a unique W_n-invariant Laurent polynomial $R_\lambda(x;q,\tau)$ of degree $|\lambda|$ such that

$$R_\lambda(\overline{\mu};q,\tau) = \delta_{\lambda,\mu} \quad \text{if } \mu \in \Lambda_n^+,\ |\mu| \le |\lambda|,\ \overline{\mu} := q^\mu \tau.$$

Nonsymmetric BC-type interpolation polynomials:
For $\alpha \in \mathbb{Z}^n$ there is a unique Laurent polynomial $G_\alpha(x;q,\tau)$ of degree $|\alpha|$ such that

$$G_\alpha(\overline{\beta};q,\tau) = \delta_{\alpha,\beta} \quad \text{if } \beta \in \mathbb{Z}^n,\ |\beta| \le |\alpha|,\ \overline{\beta} := q^\beta w_\beta \tau.$$

Expansion of R_λ in terms of the G_α:

$$R_\lambda(x) = \sum_{\alpha \in W_n \lambda} G_\alpha(x) \quad (\lambda \in \Lambda_n^+).$$

The case $n=1$:

$$G_m(x;q,s)=\frac{\left(qsx,sx^{-1};q\right)_m}{\left(q^{1+m}s^2,q^{-m};q\right)_m},\qquad m\in\mathbb{Z}_{\geq 0},$$

$$G_{-m}(x;q,s)=\frac{q^m sx\left(qsx;q\right)_{m-1}\left(sx^{-1};q\right)_{m+1}}{\left(q^m s^2;q\right)_{m+1}\left(q^{1-m};q\right)_{m-1}},\quad m\in\mathbb{Z}_{>0}.$$

Then, with $R_m(x;q,s)=\frac{(sx,sx^{-1};q)_m}{(q^m s^2,q^{-m};q)_m}\quad (m\in\mathbb{Z}_{\geq 0})$,

$$R_0(x)=G_0(x),\qquad R_m(x)=G_m(x)+G_{-m}(x)\quad (m\in\mathbb{Z}_{>0}).$$

However, a **binomial formula** $\quad E_n(z^{-1};a,b,c,d;q)$

$$=\sum_{k=0}^{|n|}\frac{p_k^+\left(z_{\tilde a,q}(n);\tilde a;q\right)p_k^+(z;a;q)}{q^{-k}(ab,ac,ad,q;q)_k}-\sum_{k=1}^{|n|}\frac{p_k^-\left(z_{\tilde a,q}(n);\tilde a,\tilde b\;q\right)p_k^-(z;a,b;q)}{q^{-k}ab(ab;q)_{k+1}(ac,ad;q)_k(q;q)_{k-1}},$$

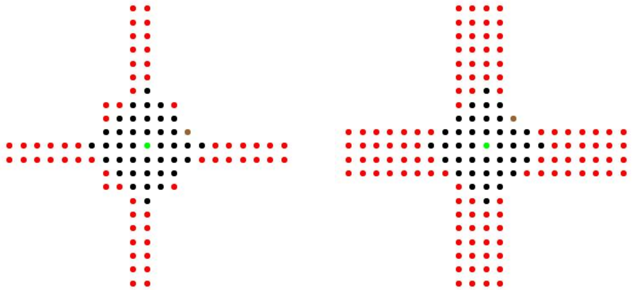
where $\quad p_k^+(z;a;q):=(az,az^{-1};q)_k\quad (k\geq 0)$

$$p_k^-(z;a,b;q):=az^{-1}(z-a)(z-b)(qaz,qaz^{-1};q)_{k-1}\quad (k\geq 1),$$

$$z_{a,q}(n):=aq^n\quad (n\geq 0),\qquad z_{a,q}(-n):=a^{-1}q^{-n}\quad (n>0).$$

Extra-vanishing:

By computer algebra experiments there is indication of extra-vanishing for $G_\alpha(x;q,\tau)$ in the principal specialization $\tau := t^\delta a \quad (|a|,|t|<1)$, i.e.,
$G_\alpha(q^\beta t^\delta a; q, t^\delta a) = 0$ not only if $\beta \in \mathbb{Z}^n$, $|\beta| \le |\alpha|$, $\beta \ne \alpha$, but also for certain other $\beta \in \mathbb{Z}^n$, depending on α, but not on q, t, a.



$\alpha = (3,1)$ $\qquad$ $\alpha = (2,2)$

green dot $= (0,0)$, brown dot $= \alpha$, black dots = other β with $|\beta| \le |\alpha|$,
red dots = points γ with $\overline{\gamma}$ extravanishing

Comparison of interpolation polynomials for A-type and BC-type (usually in principal specialization $\tau = t^\delta$ or $t^\delta a$):

- *Duality*. For A-type (non)symmetric: yes; for BC-type (non)-symmetric: no.
- *Limit from BC-type to A-type*. For symmetric polynomials: yes; for nonsymmetric polynomials: quite probably.
- *Highest degree part*. For (non)-symmetric A-type: (non)-symmetric Macdonald polynomials; for symmetric BC-type: Macdonald polynomials, not Koornwinder polynomials; for non-symmetric BC-type probably nonsymmetric Macdonald polynomials.
- *Extra-vanishing*. For (non)symmetric A-type and for symmetric BC-type yes. Probably also for nonsymmetric BC-type.
- *Combinatorial formula*. Yes for symmetric A-type and BC-type. Unknown for nonsymmetric A-type and BC-type.

Comparison (continued)

- *Binomial formula*. For (non)symmetric Macdonald and A-type interpolation polynomials: yes; for Koornwinder polynomials: yes; for (non)symmetric BC-type interpolation polynomials: unknown; for nonsymmetric Koornwinder polynomials: probably not.
- *Eigenvalue equation*. For (non)symmetric A-type: yes; for symmetric BC-type: with a $q^{\frac{1}{2}}$-shift in the a-parameter (Rains, 2005); for non-symmetric BC-type: unknown.