Symmetric and non-symmetric Askey–Wilson functions and symmetries of the Askey–Wilson DAHA

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Work joint with Marta Mazzocco (Birmingham, UK).

Preprint:
Automorphisms of the DAHA of type $\check{C}_1 C_1$ and their action on Askey–Wilson polynomials and functions.
I. The flip $(a, b, c, d) \mapsto (a, b, qd^{-1}, qc^{-1})$,
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Plan

1. Operators with reflection terms and non-symmetric special functions
2. A special DAHA symmetry
3. Askey–Wilson functions
4. Non-symmetric Askey–Wilson functions

Part 1. Operators with reflection terms and non-symmetric special functions

Non-symmetric special functions

- Important multivariable special functions such as Heckman–Opdam polynomials and Macdonald polynomials are associated with a **root system** R.
- On R acts the **Weyl group** W, a finite group generated by reflections.
- A **multiplicity function** $\alpha \mapsto k_\alpha$ is a W-invariant function on R.
- The values k_α yield **parameters**.
- The usual multivariable special functions are W-invariant, i.e. **symmetric**, functions defined in terms of the root system data including the parameters.
- If the W-invariance is dropped then we get **non-symmetric** special functions.
- In the one-variable case $W = \{\mathrm{id}, s\}$, where $s\colon x \mapsto -x$.

Non-symmetric Bessel function and Dunkl operator

$J_\alpha(x)$ is the usual Bessel function.

$$\mathcal{J}_\alpha(\lambda;x) := \frac{2^\alpha\Gamma(\alpha+1)}{(\lambda x)^\alpha}\,J_\alpha(\lambda x) = {}_0F_1\left({-\atop \alpha+1};-\tfrac14(\lambda x)^2\right).$$

$$\mathcal{J}_{-1/2}(\lambda;x)=\cos(\lambda x),\quad \mathcal{J}_{1/2}(\lambda;x)=\frac{\sin(\lambda x)}{\lambda x}.$$

$$\left(\frac{d^2}{dx^2}+\frac{2\alpha+1}{x}\frac{d}{dx}\right)\mathcal{J}_\alpha(\lambda;x)=-\lambda^2\,\mathcal{J}_\alpha(\lambda;x).$$

Non-symmetric Bessel function:

$$\mathcal{E}_\alpha(\lambda;x) := \mathcal{J}_\alpha(\lambda;x)+\frac{i\lambda x}{2(\alpha+1)}\,\mathcal{J}_{\alpha+1}(\lambda;x);\quad \mathcal{E}_{-1/2}(\lambda;x)=e^{i\lambda x}.$$

Dunkl operator:

$$(Yf)(x) := \frac{df(x)}{dx}+(\alpha+\tfrac12)\,\frac{f(x)-f(-x)}{x}.$$

$$Y\mathcal{E}_\alpha(\lambda;\,.\,)=i\lambda\,\mathcal{E}_\alpha(\lambda;\,.\,);\qquad \text{for } \alpha=-\tfrac12:\quad \frac{d}{dx}e^{i\lambda x}=i\lambda\,e^{i\lambda x}.$$

Non-symmetric Jacobi polynomials

$P_n^{(\alpha,\beta)}(x)$ is the ordinary Jacobi polynomial.

$$R_n^{(\alpha,\beta)}(z) := \frac{n!}{(\alpha+1)_n}\, P_n^{(\alpha,\beta)}\left(\frac{z+z^{-1}}{2}\right).$$

$$R_n^{(-\frac{1}{2},-\frac{1}{2})}(e^{i\theta}) = \cos(n\theta), \quad R_n^{(\frac{1}{2},\frac{1}{2})}(e^{i\theta}) = \frac{\sin((n+1)\theta)}{(n+1)\sin\theta}.$$

Non-symmetric Jacobi polynomials:

$$E_n^{(\alpha,\beta)}(z) := R_n^{(\alpha,\beta)}(z) + \frac{n}{4(\alpha+1)}\,(z-z^{-1})R_{n-1}^{(\alpha+1,\beta+1)}(z) \qquad (n=0,1,2,\ldots),$$

$$E_{-n}^{(\alpha,\beta)}(z) := R_n^{(\alpha,\beta)}(z) - \frac{n+\alpha+\beta+1}{4(\alpha+1)}\,(z-z^{-1})R_{n-1}^{(\alpha+1,\beta+1)}(z) \qquad (n=1,2,\ldots).$$

$$E_n^{(-\frac{1}{2},-\frac{1}{2})}(e^{i\theta}) = e^{in\theta} \quad (n\in\mathbb{Z}).$$

Non-symmetric Jacobi polynomials (contd.)

Differential-reflection operator (generalized Dunkl operator):

$$(Yf)(z) := -z\,\frac{df(z)}{dz} + \frac{\alpha+\beta+1+(\alpha-\beta)z}{1-z^2}\,\big(f(z)-f(z^{-1})\big).$$

Non-symmetric Jacobi polynomials are eigenfunctions of Y:

$$\begin{aligned} YE_n^{(\alpha,\beta)} &= -nE_n^{(\alpha,\beta)} && (n=0,1,2,\ldots),\\ YE_{-n}^{(\alpha,\beta)} &= (n+\alpha+\beta+1)E_{-n}^{(\alpha,\beta)} && (n=1,2,\ldots).\end{aligned}$$

For $\alpha=\beta=-\frac{1}{2}$: $\quad (Yf)(e^{i\theta}) = i\,\dfrac{df(e^{i\theta})}{d\theta}$.

q-notation

Assume $q \in \mathbb{C}$, $q \neq 0$, $q^m \neq 1$ for $m \in \mathbb{Z}$.

q-shifted factorial:

$$(b;q)_k := (1-b)(1-qb)\ldots(1-q^{k-1}b),$$

$$(b;q)_\infty := \prod_{j=0}^{\infty}(1-q^j b) \quad (|q|<1),$$

$$(b_1,\ldots,b_s;q)_k := (b_1;q)_k \ldots (b_s;q)_k\,.$$

Terminating q-hypergeometric series ($n = 0, 1, 2, \ldots$):

$${}_r\phi_{r-1}\left(\begin{matrix} q^{-n}, a_2, \ldots, a_r \\ b_1, \ldots, b_{r-1}\end{matrix}; q, z\right) := \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} \frac{(a_2,\ldots,a_r;q)_k}{(b_1,\ldots,b_{r-1};q)_k} z^k.$$

Non-terminating q-hypergeometric series:

$${}_r\phi_{r-1}\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_{r-1}\end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_{r-1},q;q)_k} z^k \quad |z|<1.$$

Askey–Wilson polynomials

Let $p_n(x; a, b, c, d \,|\, q)$ be the usual Askey–Wilson polynomial (symmetric in a, b, c, d). Drop q in further notation.

$$R_n(z; a, b, c, d) := \frac{a^n}{(ab, ac, ad; q)_n} \, p_n\big(\tfrac{1}{2}(z + z^{-1}); a, b, c, d \,|\, q\big),$$

$$R_n(z) = R_n(z; a, b, c, d) = {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

$R_n(z)$ is eigenfunction of the (second order q-difference) **Askey–Wilson operator** $L = L^{a,b,c,d}$ (symmetric in a, b, c, d):

$$\begin{aligned}(Lf)(z) := {} & \big(1 + q^{-1}abcd\big) f(z) \\ & + \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)} \big(f(qz) - f(z)\big) \\ & + \frac{(a-z)(b-z)(c-z)(d-z)}{(1-z^2)(q-z^2)} \big(f(q^{-1}z) - f(z)\big).\end{aligned}$$

$$LR_n = (q^{-n} + abcdq^{n-1})R_n\,.$$

Non-symmetric Askey–Wilson polynomials

See Sahi (1999), Noumi & Stokman (2000, 2004), K (2007), K & Bouzeffour (2011), K & Mazzocco (2018).

$$E_n(z;a,b,c,d) := R_n(z;a,b,c,d) - \frac{aq^{1-n}(1-q^n)(1-q^{n-1}cd)}{(1-ab)(1-qab)(1-ac)(1-ad)} \times z^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d) \quad (n=0,1,2\ldots),$$

$$E_{-n}(z;a,b,c,d) := R_n(z;a,b,c,d) - \frac{q^{1-n}(1-q^{n-1}abcd)(1-q^n ab)}{b(1-ab)(1-qab)(1-ac)(1-ad)} \times z^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d) \quad (n=1,2,\ldots).$$

Symmetric in c, d and has a symmetry for $a \leftrightarrow b$.

Eigenfunctions of q-difference reflection operator

$$
\begin{aligned}
(Yf)(z) := {} & \frac{z\big(1+ab-(a+b)z\big)\big((c+d)q-(cd+q)z\big)}{q(1-z^2)(q-z^2)}\, f(z) \\
& + \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)} f(qz) \\
& + \frac{(1-az)(1-bz)\big((c+d)qz-(cd+q)\big)}{q(1-z^2)(1-qz^2)}\, f(z^{-1}) \\
& + \frac{(c-z)(d-z)\big(1+ab-(a+b)z\big)}{(1-z^2)(q-z^2)}\, f(qz^{-1}).
\end{aligned}
$$

Symmetric in a, b and in c, d.

$$
\begin{aligned}
YE_n &= q^{n-1} abcd\, E_n \qquad (n = 0, 1, 2, \ldots), \\
YE_{-n} &= q^{n-1} E_{-n} \qquad (n = 1, 2, \ldots).
\end{aligned}
$$

Symmetrization yields Askey–Wilson operator:
$(Y + q^{-1} abcd Y^{-1})f = Lf$ if $f(z) = f(z^{-1})$.

Double affine Hecke algebra (DAHA)

Assume $a, b, c, d, q \in \mathbb{C}^\star$, $q^m \neq 1$. The DAHA $\mathcal{H}(a,b,c,d)$ is the algebra generated by Z, Z^{-1}, T_0, T_1 with relations:

$$\begin{aligned}(T_1+ab)(T_1+1)&=0,\\ (T_0+q^{-1}cd)(T_0+1)&=0,\\ (T_1Z+a)(T_1Z+b)&=0,\\ (qT_0Z^{-1}+c)(qT_0Z^{-1}+d)&=0.\end{aligned}$$

Basic representation of the DAHA (on space of Laurent polynomials in z; later also on other function spaces):

$$(Zf)(z) := z\,f(z),$$

$$(T_1f)(z) := \frac{(a+b)z-(1+ab)}{1-z^2}\,f(z)+\frac{(1-az)(1-bz)}{1-z^2}\,f(z^{-1}),$$

$$(T_0f)(z) := \frac{q^{-1}z((cd+q)z-(c+d)q)}{q-z^2}\,f(z)-\frac{(c-z)(d-z)}{q-z^2}\,f(qz^{-1}).$$

Then $(Yf)(z)=(T_1T_0f)(z)$.

Part 2. A special DAHA symmetry

A special DAHA symmetry

Recall: $\mathcal{H}(a,b,c,d)$ is the algebra generated by Z, Z^{-1}, T_0, T_1 with relations:

$$\begin{aligned}(T_1+ab)(T_1+1)&=0,\\(T_0+q^{-1}cd)(T_0+1)&=0,\\(T_1Z+a)(T_1Z+b)&=0,\\(qT_0Z^{-1}+c)(qT_0Z^{-1}+d)&=0.\end{aligned}$$

Symmetric in a,b and in c,d.

A further symmetry is the algebra isomorphism
$T_0\to q^{-1}cdT_0\colon \mathcal{H}(a,b,c,d)\to\mathcal{H}(a,b,qd^{-1},qc^{-1})$.

Question: Can this symmetry also be found in special functions associated with the DAHA?

Symmetry realized in the basic representation

Recall the basic representation:

$$(Zf)(z) := z\,f(z),$$

$$(T_1 f)(z) := \frac{(a+b)z-(1+ab)}{1-z^2}\,f(z) + \frac{(1-az)(1-bz)}{1-z^2}\,f(z^{-1}),$$

$$\big(T_0(c,d)f\big)(z) := \frac{q^{-1}z((cd+q)z-(c+d)q)}{q-z^2}\,f(z) - \frac{(c-z)(d-z)}{q-z^2}\,f(qz^{-1}).$$

Two representations of $\mathcal{H}(a,b,c,d)$:
$(T_0, T_1, Z) \mapsto (T_0(c,d), T_1, Z)$,
$(T_0, T_1, Z) \mapsto (q^{-1}cdT_0(q^{-1}d, q^{-1}c), T_1, Z)$
Show that these reps are equivalent under conjugation by some invertible $w(Z)$:

$$w(Z)\,T_0(c,d)\,w(Z)^{-1} = q^{-1}cdT_0(q^{-1}d, q^{-1}c),$$

$$w(Z)\,T_1\,w(Z)^{-1} = T_1.$$

Computation of $w(z)$

$$w(Z)T_1w(Z)^{-1}=T_1 \quad \text{iff} \quad w(z)=w(z^{-1}).$$

$$w(Z)T_0(c,d)w(Z)^{-1}=q^{-1}cdT_0(q^{-1}d,q^{-1}c)$$

iff

$$\frac{w(qz^{-1})}{w(z)}=qz^{-2}\,\frac{(1-c^{-1}z)(1-d^{-1}z)}{(1-qc^{-1}z^{-1})(1-qd^{-1}z^{-1})}\,.$$

Possible solution:

$$w(z)=\frac{G_c(z)}{G_{qd^{-1}}(z)}=\frac{(qd^{-1}z,qd^{-1}z^{-1};q)_\infty}{(cz,cz^{-1};q)_\infty}\,,$$

where the **Gaussian** is $G_c(z)=\frac{1}{(cz,cz^{-1};q)_\infty}$.
The solution $w(z)$ is unique modulo multiplication by an invertible function $w_0(z)$ such that $w_0(z)=w_0(qz)=w_0(z^{-1})$.

Conjugation symmetry of the Askey–Wilson operator

$Y := T_1 T_0, \quad D := Y + q^{-1}abcdY^{-1}$,
also acting in the basic representation.

$$\begin{aligned} w(Z)T_1 w(Z)^{-1} &= T_1, \\ w(Z)T_0(c,d)w(Z)^{-1} &= q^{-1}cdT_0(q^{-1}d, q^{-1}c), \\ w(Z)Y(c,d)w(Z)^{-1} &= q^{-1}cdY(q^{-1}d, q^{-1}c)), \\ w(Z)D(c,d)w(Z)^{-1} &= q^{-1}cdD(q^{-1}d, q^{-1}c)). \end{aligned}$$

If $f(z) = f(z^{-1})$ then $(Df)(z) = (L^{a,b,c,d}f)(z) = \big(1 + q^{-1}abcd\big)f(z)$

$$+ \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\big(f(qz) - f(z)\big)$$

$$+ \frac{(a-z)(b-z)(c-z)(d-z)}{(1-z^2)(q-z^2)}\big(f(q^{-1}z) - f(z)\big),$$

$$w(Z)L^{a,b,c,d}w(Z)^{-1} = q^{-1}cdL^{a,b,q^{-1}d,q^{-1}c}.$$

Action of the symmetry on eigenfunctions

$$w(Z)L^{a,b,c,d}w(Z)^{-1} = q^{-1}cdL^{a,b,q^{-1}d,q^{-1}c}.$$

Proposition

Let ϕ be an eigenfunction of $L^{a,b,c,d}$ with eigenvalue λ, then $w\phi$ is an eigenfunction of $L^{a,b,qd^{-1},qc^{-1}}$ with eigenvalue $qc^{-1}d^{-1}\lambda$.

Recall: $LR_n = (q^{-n} + abcdq^{n-1})R_n$. So the Proposition is applicable to Askey–Wilson polynomials iff $cd \in q^{\mathbb{Z}}$.

Part 3. Askey–Wilson functions

Dual parameters

Assume $0 < q < 1$, $\mathrm{Re}\, a > 0$ and $q^{-1}abcd \in \mathbb{C}\backslash(-\infty, 0]$.
For $z \in \mathbb{C}\backslash(-\infty, 0]$ let $\sqrt{z}$ be such that $\sqrt{z} > 0$ if $z > 0$.
Then the *dual parameters* $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are well defined by

$$\tilde{a}^2 = q^{-1}abcd, \quad \tilde{a}\tilde{b} = ab, \quad \tilde{a}\tilde{c} = ac, \quad \tilde{a}\tilde{d} = ad.$$

Moreover, taking duals of dual parameters once more, we recover a, b, c, d.

Duality of Askey–Wilson polynomials:

$$R_n(q^m a; a, b, c, d) = R_m(q^n \tilde{a}; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}).$$

Askey–Wilson function

Very-well-poised-balanced series:

$$ {}_8W_7(a;b,c,d,e,f) = {}_8W_7\left(a;b,c,d,e,f;q,\frac{q^2a^2}{bcdef}\right) $$

$$:= {}_8\phi_7\left(\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f \\ a^{\frac12}, -a^{\frac12}, qa/b, qa/c, qa/d, qa/e, qa/f \end{matrix}; q, \frac{q^2a^2}{bcdef}\right), $$

$$ \left|\frac{q^2a^2}{bcdef}\right| < 1, \qquad \text{symmetric in } b,c,d,e,f. $$

Askey–Wilson function (Stokman, Selecta Math. (2003)):

$$ \mathfrak{E}^+(\gamma;z;a,b,c,d) := \frac{(q\gamma\tilde{a}d^{-1}z, q\gamma\tilde{a}d^{-1}z^{-1}, qad^{-1}, qa^{-1}d^{-1};q)_\infty}{(q\gamma\tilde{a}ad^{-1}, \gamma\tilde{a}^{-1}bc, qd^{-1}z, qd^{-1}z^{-1};q)_\infty} $$
$$ \times {}_8W_7(\gamma\tilde{a}ad^{-1}; az, az^{-1}, \gamma\tilde{a}, \gamma ab\tilde{a}^{-1}, \gamma ac\tilde{a}^{-1}) \quad (|\gamma\tilde{d}| > q). $$

Note the normalization $\mathfrak{E}^+(\gamma;a^{\pm1}) = 1$.

The Askey–Wilson function was earlier studied by Suslov (1987, 1997, 2001), Ismail & Rahman (1991), Rahman (1992, 1999), Koelink & Stokman (2001), Stokman (2002).

Askey–Wilson function extends A–W polynomial

$$\mathfrak{E}^+(\gamma;z;a,b,c,d) = {}_4\phi_3\left(\begin{matrix} az, az^{-1}, \tilde{a}\gamma, \tilde{a}\gamma^{-1} \\ ab, ac, ad \end{matrix}; q, q\right)$$
$$+ \frac{(az, az^{-1}, \tilde{a}\gamma, \tilde{a}\gamma^{-1}, qbd^{-1}, qcd^{-1}, qa^{-1}d^{-1}; q)_\infty}{(qd^{-1}z, qd^{-1}z^{-1}, q\gamma\tilde{a}a^{-1}d^{-1}, q\gamma^{-1}\tilde{a}a^{-1}d^{-1}, ab, ac, q^{-1}ad; q)_\infty}$$
$$\times {}_4\phi_3\left(\begin{matrix} qd^{-1}z, qd^{-1}z^{-1}, q\tilde{d}^{-1}\gamma, q\tilde{d}^{-1}\gamma^{-1} \\ qbd^{-1}, qcd^{-1}, q^2a^{-1}d^{-1} \end{matrix}; q, q\right).$$

Hence $\quad \mathfrak{E}^+(q^n\tilde{a};z;a,b,c,d) = R_n(z;a,b,c,d)$.

Also, for generic values of the parameters,

$$(qd^{-1}z, qd^{-1}z^{-1}, q\tilde{d}^{-1}\gamma, q\tilde{d}^{-1}\gamma^{-1}; q)_\infty\, \mathfrak{E}^+(\gamma;z;a,b,c,d)$$

extends to an analytic function in $z, \gamma \in \mathbb{C}\backslash\{0\}$. Accordingly the basic representation of the DAHA acts on the space of analytic functions on $\mathbb{C}\backslash\{0\}$ divided by $(qd^{-1}z, qd^{-1}z^{-1}; q)_\infty$.

Eigenfunction of Askey–Wilson operator:

$$\tilde{a}^{-1} L\, \mathfrak{E}^+(\gamma;\,.\,) = (\gamma + \gamma^{-1})\, \mathfrak{E}^+(\gamma;\,.\,).$$

This specializes for Askey–Wilson polynomials to

$$\tilde{a}^{-1} L\, \mathfrak{E}^+(q^n\tilde{a};\,.\,) = \big(q^n\tilde{a} + (q^n\tilde{a})^{-1}\big)\mathfrak{E}^+(q^n\tilde{a};\,.\,).$$

Symmetries of the Askey–Wilson function

$\mathfrak{E}^+(\gamma; z; a, b, c, d)$ is symmetric in b, c and **satisfies the** $c \leftrightarrow qd^{-1}$ **symmetry**

$$\mathfrak{E}^+(\gamma; z; a, b, c, d) = \frac{(qad^{-1}, qa^{-1}d^{-1}; q)_\infty}{(ac, a^{-1}c; q)_\infty} \\ \times \frac{G_c(z)}{G_{qd^{-1}}(z)} \, \mathfrak{E}^+(\gamma; z; a, b, qd^{-1}, qc^{-1})$$

(both symmetries are immediate from the ${}_8W_7$ expression).

Duality (derived from expression as sum of two ${}_4\phi_3$ terms):

$$\mathfrak{E}^+(\gamma; z; a, b, c, d) = \mathfrak{E}^+(z, \gamma; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}).$$

$a \leftrightarrow b$ **symmetry**:

$$\mathfrak{E}^+(\gamma; z; a, b, c, d) = \frac{(bc, qa^{-1}d^{-1}; q)_\infty}{(ac, qb^{-1}d^{-1}; q)_\infty} \frac{G_{\tilde{c}}(\gamma)}{G_{q\tilde{d}^{-1}}(\gamma} \, \mathfrak{E}^+(\gamma; z; b, a, c, d),$$

The kernel formula

Cherednik kernel (Stokman, JAT (2002)):

$$\mathfrak{E}^+(\gamma; z; a, b, c, d) = \frac{(bc, qad^{-1}, qbd^{-1}, qcd^{-1}, qa^{-1}d^{-1}; q)_\infty}{(qabcd^{-1}; q)_\infty (qd^{-1}z, qd^{-1}z^{-1}, q\tilde{d}^{-1}\gamma, q\tilde{d}^{-1}\gamma^{-1}; q)_\infty}$$
$$\times \sum_{m=0}^{\infty} (-1)^m (ad)^{-m} q^{\frac{1}{2}m(m+1)} \frac{1 - q^{2m}abcd^{-1}}{1 - abcd^{-1}} \frac{(ab, ac, abcd^{-1}; q)_m}{(qbd^{-1}, qcd^{-1}, q; q)_m}$$
$$\times R_m(z; a, b, c, qd^{-1})\, R_m(\gamma; \tilde{a}, \tilde{b}, \tilde{c}, q\tilde{d}^{-1}).$$

All symmetries of $\mathfrak{E}^+(\gamma; z; a, b, c, d)$ can be read off from this formula.

Part 4. Non-symmetric Askey–Wilson functions

Definition of non-symmetric Askey–Wilson function

$$\mathfrak{E}(\gamma;z;a,b,c,d) := \mathfrak{E}^+(\gamma;z;a,b,c,d) - \frac{\sqrt{qab^{-1}cd}}{(1-ab)(1-qab)(1-ac)(1-ad)}$$
$$\times\, \gamma^{-1}(1-\tilde{a}\gamma)(1-\tilde{b}\gamma)\, z^{-1}(1-az)(1-bz)\, \mathfrak{E}^+(\gamma;z;qa,qb,c,d).$$

This specializes for $\gamma = q^{-n}\tilde{a}^{-1}$ respectively $\gamma = q^n\tilde{a}$ to:

$$E_n(z;a,b,c,d) = R_n(z;a,b,c,d) - \frac{aq^{1-n}(1-q^n)(1-q^{n-1}cd)}{(1-ab)(1-qab)(1-ac)(1-ad)}$$
$$\times\, z^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d) \quad (n=0,1,2,\ldots),$$

$$E_{-n}(z;a,b,c,d) = R_n(z;a,b,c,d) - \frac{q^{1-n}(1-q^{n-1}abcd)(1-q^n ab)}{b(1-ab)(1-qab)(1-ac)(1-ad)}$$
$$\times\, z^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d) \quad (n=1,2,\ldots).$$

Properties

Eigenvalue equation:

$$\tilde{a}^{-1}\, Y_z\, \mathfrak{E}(\gamma; z; a, b, c, d) = \gamma^{-1}\mathfrak{E}(\gamma; z; a, b, c, d).$$

Normalization:

$$\mathfrak{E}(\gamma; a^{-1}; a, b, c, d) = 1.$$

Also,

$$(qd^{-1}z, qd^{-1}z^{-1}, q\tilde{d}^{-1}\gamma, q\tilde{d}^{-1}\gamma^{-1}; q)_\infty\, \mathfrak{E}(\gamma; z; a, b, c, d)$$

is an analytic function in z, γ for $z, \gamma \neq 0$.

Symmetries

Duality:

$$\mathfrak{E}(\gamma; z; a, b, c, d) = \mathfrak{E}(z, \gamma; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}),$$

$c \leftrightarrow qd^{-1}$ **symmetry**:

$$\mathfrak{E}(\gamma; z; a, b, c, d) = \frac{(qad^{-1}, qa^{-1}d^{-1}; q)_\infty}{(ac, a^{-1}c; q)_\infty} \times \frac{G_c(z)}{G_{qd^{-1}}(z)} \mathfrak{E}(\gamma; z; a, b, qd^{-1}, qc^{-1}).$$

$a \leftrightarrow b$ **symmetry**:

$$\mathfrak{E}(\gamma; z; a, b, c, d) = \frac{(bc, qa^{-1}d^{-1}; q)_\infty}{(ac, qb^{-1}d^{-1}; q)_\infty} \frac{G_{\tilde{c}}(\gamma)}{G_{q\tilde{d}^{-1}}(\gamma)} \mathfrak{E}(\gamma; z; b, a, c, d).$$

Derivation of kernel formula)

Substitute twice the kernel formula

$$\mathfrak{E}^+(\gamma;z;a,b,c,d)=\frac{(bc,qad^{-1},qbd^{-1},qcd^{-1},qa^{-1}d^{-1};q)_\infty}{(qabcd^{-1};q)_\infty(qd^{-1}z,qd^{-1}z^{-1},q\tilde{d}^{-1}\gamma,q\tilde{d}^{-1}\gamma^{-1};q)_\infty}$$
$$\times\sum_{m=0}^\infty(-1)^m(ad)^{-m}q^{\frac12 m(m+1)}\,\frac{1-q^{2m}abcd^{-1}}{1-abcd^{-1}}\,\frac{(ab,ac,abcd^{-1};q)_m}{(qbd^{-1},qcd^{-1},q;q)_m}$$
$$\times R_m(z;a,b,c,qd^{-1})\,R_m(\gamma;\tilde{a},\tilde{b},\tilde{c},q\tilde{d}^{-1})$$

in the right-hand side of

$$\mathfrak{E}(\gamma;z;a,b,c,d):=\mathfrak{E}^+(\gamma;z;a,b,c,d)-\frac{\sqrt{qab^{-1}cd}}{(1-ab)(1-qab)(1-ac)(1-ad)}$$
$$\times\gamma^{-1}(1-\tilde{a}\gamma)(1-\tilde{b}\gamma)\,z^{-1}(1-az)(1-bz)\,\mathfrak{E}^+(\gamma;z;qa,qb,c,d).$$

Then:

Kernel formula

$$\mathfrak{E}(\gamma; z; a,b,c,d) = \frac{(bc, qad^{-1}, qbd^{-1}, qcd^{-1}, qa^{-1}d^{-1}; q)_\infty}{(qabcd^{-1}; q)_\infty (qd^{-1}z, qd^{-1}z^{-1}, q\tilde{d}^{-1}\gamma, q\tilde{d}^{-1}\gamma^{-1}; q)_\infty}$$
$$\times \sum_{m=0}^{\infty} (-1)^m (ad)^{-m} q^{\frac{1}{2}m(m+1)} \frac{1-q^{2m}abcd^{-1}}{1-abcd^{-1}} \frac{(ab, ac, abcd^{-1}; q)_m}{(qbd^{-1}, qcd^{-1}, q; q)_m}$$
$$\times \Bigg(-\frac{ab(1-q^m)(1-q^m cd^{-1})}{(1-ab)(1-q^{2m}abcd^{-1})} \mathfrak{E}(q^m\sqrt{abcd^{-1}}; z; a,b,c,qd^{-1}) \times \mathfrak{E}(q^m\sqrt{abcd^{-1}}; \gamma; \tilde{a}, \tilde{b}, \tilde{c}, q\tilde{d}^{-1})$$
$$+ \frac{(1-q^m ab)(1-q^m abcd^{-1})}{(1-ab)(1-q^{2m}abcd^{-1})} \mathfrak{E}\left(\frac{1}{q^m\sqrt{abcd^{-1}}}; z; a,b,c,qd^{-1}\right) \times \mathfrak{E}\left(\frac{1}{q^m\sqrt{abcd^{-1}}}; \gamma; \tilde{a}, \tilde{b}, \tilde{c}, q\tilde{d}^{-1}\right)\Bigg).$$

This is an expansion in terms of products of two non-symmetric Askey–Wilson polynomials in z and γ, respectively.
It matches with the one-variable specialization of Stokman's (Selecta Math., 2003) multivariable kernel formula.
All symmetry properties of $\mathfrak{E}(\gamma; z; a,b,c,d)$ can be seen from the kernel formula.