# Askey-Wilson polynomials and an embedding of Zhedanov's algebra $AW(3)$ in a double affine Hecke algebra

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# Zhedanov's algebra $AW(3)$

Let $\quad q \in \mathbb{C}, \quad q \neq 0, \quad q^m \neq 1 \ (m = 1, 2, \ldots)$.

$q$-commutator: $\quad [X, Y]_q := q^{\frac{1}{2}} XY - q^{-\frac{1}{2}} YX$.

The algebra $AW(3)$ has:

- generators $K_0$, $K_1$, $K_2$,
- structure constants $B$, $C_0$, $C_1$, $D_0$, $D_1$,
- relations

$$
\begin{aligned}
[K_0, K_1]_q &= K_2, \\
[K_1, K_2]_q &= BK_1 + C_0 K_0 + D_0, \\
[K_2, K_0]_q &= BK_0 + C_1 K_1 + D_1.
\end{aligned}
$$

(Zhedanov, 1991)

# Picture of Zhedanov

# Choice of structure constants

Let $a, b, c, d$ be complex parameters.

Let $e_1, e_2, e_3, e_4$ be the elementary symmetric polynomials in $a, b, c, d$.

Put for the structure constants:

$$
\begin{aligned}
B &:= (1 - q^{-1})^2 (e_3 + q e_1), \\
C_0 &:= (q - q^{-1})^2, \\
C_1 &:= q^{-1} (q - q^{-1})^2 e_4, \\
D_0 &:= -q^{-3} (1 - q)^2 (1 + q)(e_4 + q e_2 + q^2), \\
D_1 &:= -q^{-3} (1 - q)^2 (1 + q)(e_1 e_4 + q e_3).
\end{aligned}
$$

# Basic representation of $AW(3)$

Let $\mathcal{A}_{\mathrm{sym}}$ be the space of symmetric Laurent polynomials $f[z] = f[z^{-1}]$.

Let the operator $D_{\mathrm{sym}}$ act on $\mathcal{A}_{\mathrm{sym}}$ by

$$\begin{aligned}(D_{\mathrm{sym}}f)[z] := A[z]\left(f[qz] - f[z]\right) \\ + A[z^{-1}]\left(f[q^{-1}z] - f[z]\right) + (1 + q^{-1}abcd)\, f[z],\end{aligned}$$

where

$$A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,.$$

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where

$$A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,.$$

The *basic representation* of $AW(3)$ on $\mathcal{A}_{\mathrm{sym}}$ is given by

$$\begin{aligned}(K_0 f)[z] &:= (D_{\mathrm{sym}} f)[z], \\ (K_1 f)[z] &:= (z + z^{-1}) f[z].\end{aligned}$$

# Askey-Wilson polynomials

Define and notate *Askey-Wilson polynomials* by

$$P_n[z] := \text{const.}\ {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right),$$

monic symmetric Laurent polynomial of degree $n$:

$$P_n[z] = P_n[z^{-1}] = z^n + \cdots + z^{-n}.$$

These are OP's (in variable $x := \frac{1}{2}(z + z^{-1})$ ) under certain conditions for $q, a, b, c, d$.

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Askey-Wilson polynomials satisfy

$$D_{\text{sym}} P_n = \lambda_n P_n, \quad \text{where} \quad \lambda_n := q^{-n} + q^{n-1}abcd.$$

# Askey-Wilson polynomials as intertwining kernels

Askey-Wilson polynomials $P_n[z]$ are the kernel of an intertwining operator between the basic representation on $\mathcal{A}_{\mathrm{sym}}$ ($z$-dependence) and a representation on $\mathrm{Fun}(\{0, 1, 2, \ldots\})$ ($n$-dependence):

$$(K_i)_z \, P_n[z] = (K_i)_n \, P_n[z].$$

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For $K_0$ 2nd order $q$-difference equation:

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For $K_1$ 3-term recurrence relation:

$$(z + z^{-1})P_n[z] = a_n P_{n+1}[z] + b_n P_n[z] + c_n P_{n-1}[z].$$

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For $K_2$ $q$-structure relation:

$$\widetilde{A}[z]P_n[qz] + \widetilde{B}[z]P_n[z] + \widetilde{C}[z]P_n[q^{-1}z] = \widetilde{a}_n P_{n+1}[z] + \widetilde{b}_n P_n[z] + \widetilde{c}_n P_{n-1}[z].$$

# Casimir operator for $AW(3)$

The Casimir operator

$$Q := (q^{-\frac{1}{2}} - q^{\frac{3}{2}})K_0K_1K_2 + qK_2^2 + B(K_0K_1 + K_1K_0) + qC_0K_0^2 \\ + q^{-1}C_1K_1^2 + (1+q)D_0K_0 + (1+q^{-1})D_1K_1,$$

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commutes in $AW(3)$ with the generators $K_0, K_1, K_2$.
In the basic representation (which is irreducible for generic values of $a, b, c, d$), $Q$ becomes a constant scalar:

$$(Qf)[z] = Q_0\, f[z],$$

where

$$Q_0 := q^{-4}(1-q)^2\Big(q^4(e_4 - e_2) + q^3(e_1^2 - e_1e_3 - 2e_2) \\ - q^2(e_2e_4 + 2e_4 + e_2) + q(e_3^2 - 2e_2e_4 - e_1e_3) + e_4(e_1 - e_2)\Big).$$

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**Assumption**
$a, b, c, d \neq 0, \quad abcd \neq q^{-m}\ (m = 0, 1, 2, \ldots).$

# A faithful representation on $\mathcal{A}_{\mathrm{sym}}$

**Definition**

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## Theorem (THK, 2007)

*$AW(3, Q_0)$ has the elements*

$$K_0^n (K_1 K_0)^l K_1^m \quad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1)$$

*as a linear basis.*

*The basic representation of $AW(3, Q_0)$ on $\mathcal{A}_{\mathrm{sym}}$ is faithful.*

# Double affine Hecke algebra of type $(C_1^\vee, C_1)$

The algebra $\tilde{\mathfrak{H}}$ has:

- $q, a, b, c, d$ as before,
- generators $Z, Z^{-1}, T_1, T_0$,
- relations

$$\begin{aligned}(T_1 + ab)(T_1 + 1) &= 0,\\ (T_0 + q^{-1}cd)(T_0 + 1) &= 0,\\ (T_1 Z + a)(T_1 Z + b) &= 0,\\ (qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) &= 0.\end{aligned}$$

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$T_1$ and $T_0$ are invertible.

$$Y := T_1 T_0, \qquad D := Y + q^{-1}abcdY^{-1}.$$

# Basic representation of $\tilde{\mathfrak{H}}$

Let $\mathcal{A}$ be the space of Laurent polynomials $f[z]$.

The *basic representation* of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$ is given by

$$
\begin{aligned}
(Zf)[z] &:= z\, f[z], \\
(T_1 f)[z] &:= -ab\, f[z] + \frac{(1-az)(1-bz)}{1-z^2}\left(f[z^{-1}] - f[z]\right), \\
(T_0 f)[z] &:= -q^{-1}cd\, f[z] + \frac{(c-z)(d-z)}{q-z^2}\left(f[z] - f[qz^{-1}]\right).
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\end{aligned}
$$

Then

$$(T_1 f)[z] = -ab\,f[z] \quad \text{iff} \quad f[z] = f[z^{-1}],$$

and

$$(Df)[z] = (D_{\mathrm{sym}} f)[z] \quad \text{if} \quad f[z] = f[z^{-1}].$$

# Eigenspaces of $D$

Let

$$\begin{aligned} Q_n[z] &:= a^{-1}b^{-1}z^{-1}(1-az)(1-bz)\,P_{n-1}[z;qa,qb,c,d\mid q] \\ &= z^n + \cdots + a^{-1}b^{-1}z^{-n}. \end{aligned}$$

Then

$$DQ_n = \lambda_n Q_n, \qquad T_1 Q_n = -Q_n.$$

$D$ has eigenvalues $\lambda_n$ $(n = 0, 1, 2, \ldots)$.

$T_1$ has eigenvalues $-1, -ab$.

$D$ and $T_1$ commute.

The eigenspace of $D$ for $\lambda_n$ is spanned by $P_n$ and $Q_n$ $(n = 1, 2, \ldots)$.

# Eigenspaces of $Y$

Let

$$E_{-n} = \frac{ab}{ab-1}\,(P_n - Q_n) \qquad (n = 1, 2, \ldots),$$

$$E_n = \frac{(1-q^n ab)(1-q^{n-1}abcd)}{(1-ab)(1-q^{2n-1}abcd)}\,P_n - \frac{ab(1-q^n)(1-q^{n-1}cd)}{(1-ab)(1-q^{2n-1}abcd)}\,Q_n \qquad (n = 1, 2, \ldots).$$

# Eigenspaces of $Y$

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Then

$$YE_{-n} = q^{-n}\,E_{-n} \qquad (n = 1, 2, \ldots),$$

$$YE_n = q^{n-1}abcd\,E_n \qquad (n = 0, 1, 2, \ldots).$$

# Faithfulness of the basic representation of $\tilde{\mathfrak{H}}$

**Theorem (Sahi)**

*The basic representation of $\tilde{\mathfrak{H}}$ is faithful.*

*The elements*

$$Z^m Y^n T_1^i \qquad (m, n \in \mathbb{Z},\ i = 0, 1)$$

*form a linear basis of $\tilde{\mathfrak{H}}$.*

# Central extension of $AW(3)$

Let the algebra $\widetilde{AW}(3)$ be generated by $K_0$, $K_1$, $K_2$, $T_1$ such that $T_1$ commutes with $K_0, K_1, K_2$ and with further relations

$$
\begin{aligned}
(T_1 + ab)(T_1 + 1) &= 0, \\
(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 &= B\,K_1 + C_0\,K_0 + D_0 \\
&\quad + E\,K_1(T_1+ab) + F_0(T_1+ab), \\
(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 &= B\,K_0 + C_1\,K_1 + D_1 \\
&\quad + E\,K_0(T_1+ab) + F_1(T_1+ab),
\end{aligned}
$$

where

$$
\begin{aligned}
E &:= -q^{-2}(1-q)^3(c+d), \\
F_0 &:= q^{-3}(1-q)^3(1+q)(cd+q), \\
F_1 &:= q^{-3}(1-q)^3(1+q)(a+b)cd.
\end{aligned}
$$

# Basic representation of $\widetilde{AW}(3)$

The following element $\widetilde{Q}$ commutes with all elements of $\widetilde{AW}(3)$:

$$\begin{aligned}\widetilde{Q} :=& (K_1K_0)^2 - (q^2+1+q^{-2})K_0(K_1K_0)K_1 \\ &+ (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\ &+ \big(B + E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\ &+ (q+1+q^{-1})\big(D_0 + F_0(T_1+ab)\big)K_0 \\ &+ (q+1+q^{-1})\big(D_1 + F_1(T_1+ab)\big)K_1 + G(T_1+ab),\end{aligned}$$

where $G$ can be explicitly specified.

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&+ (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\
&+ \big(B + E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\
&+ (q+1+q^{-1})\big(D_0 + F_0(T_1+ab)\big)K_0 \\
&+ (q+1+q^{-1})\big(D_1 + F_1(T_1+ab)\big)K_1 + G(T_1+ab),
\end{aligned}
$$

where $G$ can be explicitly specified.

$\widetilde{AW}(3)$ acts on $\mathcal{A}$ such that $K_0, K_1, T_1$ act as $D_{\rm sym}$, $Z+Z^{-1}$, $T_1$, respectively, in the basic representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$.
This action is called the *basic representation* of $\widetilde{AW}(3)$ on $\mathcal{A}$.

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$\widetilde{AW}(3)$ acts on $\mathcal{A}$ such that $K_0, K_1, T_1$ act as $D_{\mathrm{sym}}$, $Z+Z^{-1}$, $T_1$, respectively, in the basic representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$.
This action is called the *basic representation* of $\widetilde{AW}(3)$ on $\mathcal{A}$.

Then $\widetilde{Q}$ acts as the constant $Q_0$.

# A faithful representation on $\mathcal{A}$

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*as a linear basis.*

*The basic representation of $\widetilde{AW}(3, Q_0)$ on $\mathcal{A}$ is faithful.*

*$\widetilde{AW}(3, Q_0)$ has an injective embedding in $\tilde{\mathfrak{H}}$.*

# References

I did computations in algebras defined by generators and relations in Mathematica with the aid of the package NCAlgebra, see `http://www.math.ucsd.edu/~ncalg/`

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Details of this lecture in my paper

*The relationship between Zhedanov's alsgebra* $AW(3)$ *and the double affine Hecke algebra in the rank one case*, `arXiv:math.QA/0612730v3`; SIGMA 3 (2007), 063.

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This is in the Vadim Kuznetsov memorial volume of SIGMA.

# Picture of Vadim Kuznetsov