Askey-Wilson polynomials and an embedding of Zhedanov’s algebra $AW(3)$ in a double affine Hecke algebra

Tom H. Koornwinder

University of Amsterdam, thk@science.uva.nl

July 2, 2007
9th OPSFA, Marseille, France
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Zhedanov’s algebra $AW(3)$

Let $q \in \mathbb{C}$, $q \neq 0$, $q^m \neq 1$ $(m = 1, 2, \ldots)$.

$q$-commutator: $[X, Y]_q := q^{1/2}XY - q^{-1/2}YX$.

The algebra $AW(3)$ has:

- generators $K_0, K_1, K_2$,
- structure constants $B, C_0, C_1, D_0, D_1$,
- relations

$$[K_0, K_1]_q = K_2,$$
$$[K_1, K_2]_q = BK_1 + C_0K_0 + D_0,$$
$$[K_2, K_0]_q = BK_0 + C_1K_1 + D_1.$$

(Zhedanov, 1991)
Picture of Zhedanov
Choice of structure constants

Let $a, b, c, d$ be complex parameters.

Let $e_1, e_2, e_3, e_4$ be the elementary symmetric polynomials in $a, b, c, d$.

Put for the structure constants:

$$B := (1 - q^{-1})^2(e_3 + qe_1),$$
$$C_0 := (q - q^{-1})^2,$$
$$C_1 := q^{-1}(q - q^{-1})^2 e_4,$$
$$D_0 := -q^{-3}(1 - q)^2(1 + q)(e_4 + qe_2 + q^2),$$
$$D_1 := -q^{-3}(1 - q)^2(1 + q)(e_1 e_4 + qe_3).$$
Basic representation of $\text{AW}(3)$

Let $\mathcal{A}_{\text{sym}}$ be the space of symmetric Laurent polynomials $f[z] = f[z^{-1}]$.

Let the operator $D_{\text{sym}}$ act on $\mathcal{A}_{\text{sym}}$ by

$$(D_{\text{sym}}f)[z] := A[z] \left( f[qz] - f[z] \right) + A[z^{-1}] \left( f[q^{-1}z] - f[z] \right) + (1 + q^{-1}abcd) f[z],$$

where

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where


The basic representation of $AW(3)$ on $\mathcal{A}_{\text{sym}}$ is given by

$$(K_0 f)[z] := (D_{\text{sym}} f)[z],$$

$$(K_1 f)[z] := (z + z^{-1}) f[z].$$
Define and notate *Askey-Wilson polynomials* by

\[ P_n[z] := \text{const. } 4\phi_3 \left( \begin{array}{c} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{array} ; q, q \right) , \]

monic symmetric Laurent polynomial of degree \( n \):

\[ P_n[z] = P_n[z^{-1}] = z^n + \cdots + z^{-n} . \]

These are OP’s (in variable \( x := \frac{1}{2}(z + z^{-1}) \)) under certain conditions for \( q, a, b, c, d \).
Askey-Wilson polynomials

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These are OP’s (in variable $x := \frac{1}{2}(z + z^{-1})$) under certain conditions for $q, a, b, c, d$. Askey-Wilson polynomials satisfy

$$D_{\text{sym}} P_n = \lambda_n P_n, \quad \text{where} \quad \lambda_n := q^{-n} + q^{n-1}abcd.$$
Askey-Wilson polynomials $P_n[z]$ are the kernel of an intertwining operator between the basic representation on $A_{\text{sym}}$ ($z$-dependence) and a representation on $\text{Fun}(\{0, 1, 2, \ldots\})$ ($n$-dependence):

$$(K_i)_z P_n[z] = (K_i)_n P_n[z].$$
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For $K_0$ 2nd order $q$-difference equation:

$$A[z]P_n[qz] + B[z]P_n[z] + C[z]P_n[q^{-1}z] = \lambda_n P_n[z].$$
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$$(z + z^{-1})P_n[z] = a_n P_{n+1}[z] + b_n P_n[z] + c_n P_{n-1}[z].$$
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$$(z + z^{-1})P_n[z] = a_n P_{n+1}[z] + b_n P_n[z] + c_n P_{n-1}[z].$$

For $K_2$ $q$-structure relation:

$$\tilde{A}[z]P_n[qz] + \tilde{B}[z]P_n[z] + \tilde{C}[z]P_n[q^{-1}z]$$

$$= \tilde{a}_n P_{n+1}[z] + \tilde{b}_n P_n[z] + \tilde{c}_n P_{n-1}[z].$$
The Casimir operator

\[ Q := (q^{-\frac{1}{2}} - q^{\frac{3}{2}})K_0 K_1 K_2 + qK_2^2 + B(K_0 K_1 + K_1 K_0) + qC_0 K_0^2 + q^{-1}C_1 K_1^2 + (1 + q)D_0 K_0 + (1 + q^{-1})D_1 K_1, \]

commutes in \( AW(3) \) with the generators \( K_0, K_1, K_2 \).
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commutes in \( AW(3) \) with the generators \( K_0, K_1, K_2 \).

In the basic representation (which is irreducible for generic values of \( a, b, c, d \)), \( Q \) becomes a constant scalar:

\[
(Qf)[z] = Q_0 f[z],
\]

where

\[
Q_0 := q^{-4}(1 - q)^2\left(q^4(e_4 - e_2) + q^3(e_1^2 - e_1e_3 - 2e_2)
- q^2(e_2e_4 + 2e_4 + e_2) + q(e_3^2 - 2e_2e_4 - e_1e_3) + e_4(e_1 - e_2)\right).
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**Assumption**

\( a, b, c, d \neq 0, \quad abcd \neq q^{-m} (m = 0, 1, 2, \ldots). \)
A faithful representation on $A_{\text{sym}}$

**Definition**

$AW(3, Q_0)$ is the algebra $AW(3)$ with additional relation $Q = Q_0$. 
A faithful representation on $A_{\text{sym}}$

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$AW(3, Q_0)$ is the algebra $AW(3)$ with additional relation $Q = Q_0$.

**Theorem (THK, 2007)**

$AW(3, Q_0)$ has the elements

$$K_0^n(K_1K_0)^lK_1^m \quad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1)$$

as a linear basis.

*The basic representation of $AW(3, Q_0)$ on $A_{\text{sym}}$ is faithful.*
Double affine Hecke algebra of type \((C_1^\vee, C_1)\)

The algebra \(\tilde{H}\) has:

- \(q, a, b, c, d\) as before,
- generators \(Z, Z^{-1}, T_1, T_0\),
- relations

\[
(T_1 + ab)(T_1 + 1) = 0, \\
(T_0 + q^{-1}cd)(T_0 + 1) = 0, \\
(T_1 Z + a)(T_1 Z + b) = 0, \\
(qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) = 0.
\]
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(Sahi, Noumi & Stokman, Macdonald’s 2003 book; preceding work by Dunkl, Heckman, Cherednik.)
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\(T_1\) and \(T_0\) are invertible.

\[
Y := T_1T_0, \quad D := Y + q^{-1}abcdY^{-1}.
\]
Basic representation of $\tilde{\mathcal{H}}$

Let $\mathcal{A}$ be the space of Laurent polynomials $f[z]$. The \textit{basic representation} of $\tilde{\mathcal{H}}$ on $\mathcal{A}$ is given by

\[(Zf)[z] := z f[z],\]
\[(T_1 f)[z] := -ab f[z] + \frac{(1 - az)(1 - bz)}{1 - z^2} (f[z^{-1}] - f[z]),\]
\[(T_0 f)[z] := -q^{-1} cd f[z] + \frac{(c - z)(d - z)}{q - z^2} (f[z] - f[qz^{-1}]).\]
Basic representation of $\tilde{\mathcal{H}}$

Let $\mathcal{A}$ be the space of Laurent polynomials $f[z]$. The *basic representation* of $\tilde{\mathcal{H}}$ on $\mathcal{A}$ is given by

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(Zf)[z] := z f[z],
\]
\[
(T_1 f)[z] := -ab f[z] + \frac{(1 - az)(1 - bz)}{1 - z^2} (f[z^{-1}] - f[z]),
\]
\[
(T_0 f)[z] := -q^{-1} cd f[z] + \frac{(c - z)(d - z)}{q - z^2} (f[z] - f[qz^{-1}]).
\]

Then

\[
(T_1 f)[z] = -ab f[z] \quad \text{iff} \quad f[z] = f[z^{-1}],
\]

and

\[
(Df)[z] = (D_{\text{sym}} f)[z] \quad \text{if} \quad f[z] = f[z^{-1}].
\]
Eigenspaces of $D$

Let

$$Q_n[z] := a^{-1}b^{-1}z^{-1}(1 - az)(1 - bz) P_{n-1}[z; qa, qb, c, d | q]$$

$$= z^n + \cdots + a^{-1}b^{-1}z^{-n}.$$ 

Then

$$DQ_n = \lambda_n Q_n, \quad T_1 Q_n = -Q_n.$$ 

$D$ has eigenvalues $\lambda_n$ ($n = 0, 1, 2, \ldots$). 

$T_1$ has eigenvalues $-1, -ab$. 

$D$ and $T_1$ commute. 

The eigenspace of $D$ for $\lambda_n$ is spanned by $P_n$ and $Q_n$ ($n = 1, 2, \ldots$).
Eigenspaces of $Y$

Let

$$E_{-n} = \frac{ab}{ab - 1} (P_n - Q_n) \quad (n = 1, 2, \ldots),$$

$$E_n = \frac{(1 - q^n ab)(1 - q^{n-1} abcd)}{(1 - ab)(1 - q^{2n-1} abcd)} P_n - \frac{ab(1 - q^n)(1 - q^{n-1} cd)}{(1 - ab)(1 - q^{2n-1} abcd)} Q_n \quad (n = 1, 2, \ldots).$$
Eigenspaces of $Y$

Let

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Then

$$YE_{-n} = q^{-n} E_{-n} \quad (n = 1, 2, \ldots),$$

$$YE_n = q^{n-1} abcd E_n \quad (n = 0, 1, 2, \ldots).$$
The basic representation of $\tilde{\mathfrak{H}}$ is faithful.

The elements

$$Z^m Y^n T_1^i$$ 

$(m, n \in \mathbb{Z}, \ i = 0, 1)$

form a linear basis of $\tilde{\mathfrak{H}}$. 

**Theorem (Sahi)**

*The basic representation of $\tilde{\mathfrak{H}}$ is faithful.*
Central extension of $AW(3)$

Let the algebra $\widetilde{AW}(3)$ be generated by $K_0$, $K_1$, $K_2$, $T_1$ such that $T_1$ commutes with $K_0$, $K_1$, $K_2$ and with further relations

$$(T_1 + ab)(T_1 + 1) = 0,$$

$$(q + q^{-1}) K_1 K_0 K_1 - K_1^2 K_0 - K_0 K_1^2 = B K_1 + C_0 K_0 + D_0 + E K_1 (T_1 + ab) + F_0 (T_1 + ab),$$

$$(q + q^{-1}) K_0 K_1 K_0 - K_0^2 K_1 - K_1 K_0^2 = B K_0 + C_1 K_1 + D_1 + E K_0 (T_1 + ab) + F_1 (T_1 + ab),$$

where

$$E := -q^{-2} (1 - q)^3 (c + d),$$

$$F_0 := q^{-3} (1 - q)^3 (1 + q) (cd + q),$$

$$F_1 := q^{-3} (1 - q)^3 (1 + q) (a + b) cd.$$
The following element $\tilde{Q}$ commutes with all elements of $\tilde{AW}(3)$:

$$
\tilde{Q} := (K_1 K_0)^2 - (q^2 + 1 + q^{-2}) K_0 (K_1 K_0) K_1 \\
+ (q + q^{-1}) K_0^2 K_1^2 + (q + q^{-1}) (C_0 K_0^2 + C_1 K_1^2) \\
+ (B + E(T_1 + ab)) ((q + 1 + q^{-1}) K_0 K_1 + K_1 K_0) \\
+ (q + 1 + q^{-1}) (D_0 + F_0(T_1 + ab)) K_0 \\
+ (q + 1 + q^{-1}) (D_1 + F_1(T_1 + ab)) K_1 + G(T_1 + ab),
$$

where $G$ can be explicitly specified.
The following element \( \tilde{Q} \) commutes with all elements of \( \tilde{AW}(3) \):

\[
\tilde{Q} := (K_1 K_0)^2 - (q^2 + 1 + q^{-2}) K_0(K_1 K_0) K_1
\]
\[
+ (q + q^{-1}) K_0^2 K_1^2 + (q + q^{-1})(C_0 K_0^2 + C_1 K_1^2)
\]
\[
+ (B + E(T_1 + ab))((q + 1 + q^{-1})K_0K_1 + K_1K_0)
\]
\[
+ (q + 1 + q^{-1})(D_0 + F_0(T_1 + ab))K_0
\]
\[
+ (q + 1 + q^{-1})(D_1 + F_1(T_1 + ab))K_1 + G(T_1 + ab),
\]

where \( G \) can be explicitly specified.

\( \tilde{AW}(3) \) acts on \( \mathcal{A} \) such that \( K_0, K_1, T_1 \) act as \( D_{\text{sym}}, Z + Z^{-1}, T_1 \), respectively, in the basic representation of \( \tilde{H} \) on \( \mathcal{A} \).

This action is called the \textit{basic representation} of \( \tilde{AW}(3) \) on \( \mathcal{A} \).
Basic representation of $\tilde{AW}(3)$

The following element $\tilde{Q}$ commutes with all elements of $\tilde{AW}(3)$:

$$
\tilde{Q} := (K_1 K_0)^2 - (q^2 + 1 + q^{-2})K_0(K_1 K_0)K_1
+ (q + q^{-1})K_0^2K_1^2 + (q + q^{-1})(C_0 K_0^2 + C_1 K_1^2)
+ (B + E(T_1 + ab))((q + 1 + q^{-1})K_0 K_1 + K_1 K_0)
+ (q + 1 + q^{-1})(D_0 + F_0(T_1 + ab))K_0
+ (q + 1 + q^{-1})(D_1 + F_1(T_1 + ab))K_1 + G(T_1 + ab),
$$

where $G$ can be explicitly specified.

$\tilde{AW}(3)$ acts on $\mathcal{A}$ such that $K_0, K_1, T_1$ act as $D_{\text{sym}}, Z + Z^{-1}, T_1$, respectively, in the basic representation of $\tilde{H}$ on $\mathcal{A}$. This action is called the \textit{basic representation} of $\tilde{AW}(3)$ on $\mathcal{A}$.

Then $\tilde{Q}$ acts as the constant $Q_0$. 
A faithful representation on $A$

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>$\tilde{AW}(3, Q_0)$ is the algebra $\tilde{AW}(3)$ with additional relation $\tilde{Q} = Q_0$.</td>
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</table>
A faithful representation on $\mathcal{A}$

**Definition**

$\tilde{\mathcal{A}}\mathcal{W}(3, Q_0)$ is the algebra $\tilde{\mathcal{A}}\mathcal{W}(3)$ with additional relation $\tilde{Q} = Q_0$.

**Theorem (THK, 2007)**

$\tilde{\mathcal{A}}\mathcal{W}(3, Q_0)$ has the elements

$$K_0^n(K_1K_0)^iK_1^mT_1^j \quad (m, n = 0, 1, 2, \ldots, \quad i, j = 0, 1)$$

as a linear basis.

The basic representation of $\tilde{\mathcal{A}}\mathcal{W}(3, Q_0)$ on $\mathcal{A}$ is faithful.

$\tilde{\mathcal{A}}\mathcal{W}(3, Q_0)$ has an injective embedding in $\tilde{\mathcal{N}}$. 
I did computations in algebras defined by generators and relations in Mathematica with the aid of the package NCAlgebra, see http://www.math.ucsd.edu/~ncalg/

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Details of this lecture in my paper

\textit{The relationship between Zhedanov’s algebras AW(3) and the double affine Hecke algebra in the rank one case}, \texttt{arXiv:math.QA/0612730v3}; \textbf{SIGMA 3 (2007), 063}. 
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*The relationship between Zhedanov’s algegra AW(3) and the double affine Hecke algebra in the rank one case*,


This is in the Vadim Kuznetsov memorial volume of SIGMA.
Picture of Vadim Kuznetsov