The relationship between Zhedanov's algebra $AW(3)$ and DAHA

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Applications of Macdonald polynomials, BIRS, Banff, Alberta, Canada

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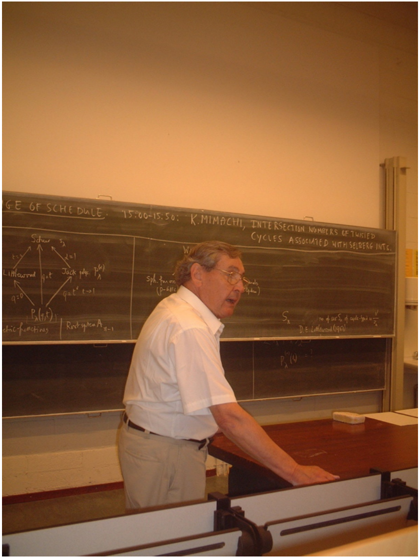
Picture of Macdonald in action



Picture of Macdonald relaxing

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Zhedanov's algebra $AW(3)$

Let $q \in \mathbb{C}$, $q \neq 0$, $q^m \neq 1$ $(m = 1, 2, \ldots)$.

q-commutator: $[X, Y]_q := q^{\frac{1}{2}} XY - q^{-\frac{1}{2}} YX$.

The algebra $AW(3)$ has:

- generators K_0, K_1, K_2,
- structure constants B, C_0, C_1, D_0, D_1,
- relations

$$
\begin{aligned}
[K_0, K_1]_q &= K_2, \\
[K_1, K_2]_q &= BK_1 + C_0 K_0 + D_0, \\
[K_2, K_0]_q &= BK_0 + C_1 K_1 + D_1.
\end{aligned}
$$

(Zhedanov, 1991)

Picture of Zhedanov

Choice of structure constants

Let a, b, c, d be complex parameters. Assume $a, b, c, d \neq 0$, $abcd \neq q^{-m}$ $(m = 0, 1, 2, \ldots)$.

Let e_1, e_2, e_3, e_4 be the elementary symmetric polynomials in a, b, c, d.

Put for the structure constants:

$$
\begin{aligned}
B &:= (1 - q^{-1})^2 (e_3 + q e_1), \\
C_0 &:= (q - q^{-1})^2, \\
C_1 &:= q^{-1} (q - q^{-1})^2 e_4, \\
D_0 &:= -q^{-3} (1 - q)^2 (1 + q)(e_4 + q e_2 + q^2), \\
D_1 &:= -q^{-3} (1 - q)^2 (1 + q)(e_1 e_4 + q e_3).
\end{aligned}
$$

Polynomial representation of $AW(3)$

Let $\mathcal{A}_{\text{sym}}$ be the space of symmetric Laurent polynomials $f[z] = f[z^{-1}]$.

Let the operator D_{sym} act on $\mathcal{A}_{\text{sym}}$ by

$$\begin{aligned}(D_{\text{sym}} f)[z] := A[z]\big(f[qz] - f[z]\big) \\ + A[z^{-1}]\big(f[q^{-1}z] - f[z]\big) + (1 + q^{-1}abcd)\,f[z],\end{aligned}$$

where

$$A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

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where

$$A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

The *polynomial representation* of $AW(3)$ on $\mathcal{A}_{\mathrm{sym}}$ is given by

$$\begin{aligned}(K_0 f)[z] &:= (D_{\mathrm{sym}}f)[z], \\ (K_1 f)[z] &:= (z + z^{-1})f[z].\end{aligned}$$

Askey-Wilson polynomials

Define and notate *Askey-Wilson polynomials* by

$$P_n[z] := \text{const. } {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right),$$

monic symmetric Laurent polynomial of degree n:

$$P_n[z] = P_n[z^{-1}] = z^n + \cdots + z^{-n}.$$

These are orthogonal polynomials (in variable $x := \frac{1}{2}(z + z^{-1})$) under certain conditions for q, a, b, c, d.

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Askey-Wilson polynomials satisfy

$$D_{\text{sym}} P_n = \lambda_n P_n, \quad \text{where} \quad \lambda_n := q^{-n} + q^{n-1}abcd.$$

Askey-Wilson polynomials as intertwining kernels

Askey-Wilson polynomials $P_n[z]$ are the kernel of an intertwining operator between the polynomial representation of $AW(3)$ on $\mathcal{A}_{\mathrm{sym}}$ (z-dependence) and a representation on $\mathrm{Fun}(\{0, 1, 2, \ldots\})$ (n-dependence):

$$(K_i)_z\, P_n[z] = (K_i)_n\, P_n[z].$$

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$$(K_i)_z\, P_n[z] = (K_i)_n\, P_n[z].$$

For K_0 2nd order q-difference equation:

$$A[z]P_n[qz] + B[z]P_n[z] + C[z]P_n[q^{-1}z] = \lambda_n P_n[z].$$

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For K_1 3-term recurrence relation:

$$(z + z^{-1})P_n[z] = a_n P_{n+1}[z] + b_n P_n[z] + c_n P_{n-1}[z].$$

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For K_1 3-term recurrence relation:

$$(z+z^{-1})P_n[z] = a_n P_{n+1}[z] + b_n P_n[z] + c_n P_{n-1}[z].$$

For K_2 q-structure relation:

$$\widetilde{A}[z]P_n[qz] + \widetilde{B}[z]P_n[z] + \widetilde{C}[z]P_n[q^{-1}z] = \widetilde{a}_n P_{n+1}[z] + \widetilde{b}_n P_n[z] + \widetilde{c}_n P_{n-1}[z].$$

Relations for $AW(3)$ in terms of K_0, K_1 only, and the Casimir operator

$AW(3)$ can also be considered as generated by K_0, K_1 with relations

$$(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = B\,K_1 + C_0\,K_0 + D_0,$$
$$(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = B\,K_0 + C_1\,K_1 + D_1.$$

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The Casimir operator

$$\begin{aligned}Q := {} & K_1K_0K_1K_0 - (q^2+1+q^{-2})K_0K_1K_0K_1 \\ & + (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\ & + B\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\ & + (q+1+q^{-1})\big(D_0K_0 + D_1K_1\big).\end{aligned}$$

commutes in $AW(3)$ with the generators K_0, K_1.

Value of the Casimir operator in the polynomial representation

In the polynomial representation (which is irreducible for generic values of a, b, c, d), Q becomes a constant scalar:

$$(Qf)[z] = Q_0 f[z],$$

where

$$Q_0 := q^{-4}(1-q)^2\Big(q^4(e_4-e_2)+q^3(e_1^2-e_1e_3-2e_2)$$
$$-q^2(e_2e_4+2e_4+e_2)+q(e_3^2-2e_2e_4-e_1e_3)+e_4(1-e_2)\Big).$$

A faithful representation on $\mathcal{A}_{\mathrm{sym}}$

Definition

$AW(3, Q_0)$ is the algebra $AW(3)$ with additional relation $Q = Q_0$.

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Theorem (THK, 2007)

$AW(3, Q_0)$ has the elements

$$K_0^n (K_1 K_0)^l K_1^m \quad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1)$$

as a linear basis.

The polynomial representation of $AW(3, Q_0)$ on $\mathcal{A}_{\mathrm{sym}}$ is faithful.

Proof of first part of theorem

$AW(3, Q_0)$ is spanned by elements $K_\alpha = K_{\alpha_1} \cdots K_{\alpha_k}$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i = 0$ or 1. Let $\rho(\alpha)$ the number of pairs (i, j) such that $i < j$, $\alpha_i = 1$, $\alpha_j = 0$. K_α has the form

$$K_0^n (K_1 K_0)^l K_1^m \qquad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1)$$

iff $\rho(\alpha) = 0$ or 1.

If $\rho(\alpha) > 1$ then K_α must have a substring

$$K_1 K_1 K_0 \quad \text{or} \quad K_1 K_0 K_0 \quad \text{or} \quad K_1 K_0 K_1 K_0.$$

By substitution of one of the three relations we see that each such string is a linear combination of elements K_β with $\rho(\beta) < \rho(\alpha)$.

Sketch of proof of second part of theorem

Note that

$$(D_{\mathrm{sym}})^n \, (Z + Z^{-1})^m \, P_j[z] = \lambda_{j+m}^n P_{j+m}[z] + \cdots ,$$

$$(D_{\mathrm{sym}})^{n-1} \, (Z + Z^{-1}) \, D_{\mathrm{sym}} \, (Z + Z^{-1})^{m-1} \, P_j[z]$$
$$= \lambda_{j+m}^{n-1} \lambda_{j+m-1} P_{j+m}[z] + \cdots .$$

Sketch of proof of second part of theorem

Note that

$$(D_{\rm sym})^n\,(Z+Z^{-1})^m\,P_j[z]=\lambda^n_{j+m}P_{j+m}[z]+\cdots\,,$$

$$(D_{\rm sym})^{n-1}\,(Z+Z^{-1})\,D_{\rm sym}\,(Z+Z^{-1})^{m-1}\,P_j[z]$$
$$=\lambda^{n-1}_{j+m}\lambda_{j+m-1}P_{j+m}[z]+\cdots\,.$$

If (with some $a_{m,l}$ or $b_{m,l}\neq 0$) we have:

$$\sum_{k=0}^m\sum_l a_{k,l}(D_{\rm sym})^l\,(Z+Z^{-1})^k$$
$$+\sum_{k=1}^m\sum_l b_{k,l}(D_{\rm sym})^{l-1}\,(Z+Z^{-1})\,D_{\rm sym}\,(Z+Z^{-1})^{k-1}=0$$

then we have for all j:

$$\sum_l(a_{m,l}\lambda^l_{j+m}+b_{m,l}\lambda^{l-1}_{j+m}\lambda_{j+m-1})=0.$$

Then consider maximal l for which $a_{m,l}$ or $b_{m,l}\neq 0$, and get a contradiction.

The center of $AW(3, Q_0)$

By a similar technique we can prove:

Theorem

The center of $AW(3, Q_0)$ consists of the scalars.

Double affine Hecke algebra of type $(C_1^\vee, C_1)$

The algebra $\tilde{\mathfrak{H}}$ has:

- q, a, b, c, d as before,
- generators $Z, Z^{-1}, T_1, T_0,$
- relations

$$(T_1 + ab)(T_1 + 1) = 0,$$
$$(T_0 + q^{-1}cd)(T_0 + 1) = 0,$$
$$(T_1 Z + a)(T_1 Z + b) = 0,$$
$$(qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) = 0.$$

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(Sahi, Noumi & Stokman, Macdonald's 2003 book; preceding work by Dunkl, Heckman, Cherednik.)

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(T_0 + q^{-1}cd)(T_0 + 1) &= 0, \\
(T_1 Z + a)(T_1 Z + b) &= 0, \\
(q T_0 Z^{-1} + c)(q T_0 Z^{-1} + d) &= 0.
\end{aligned}
$$

(Sahi, Noumi & Stokman, Macdonald's 2003 book; preceding work by Dunkl, Heckman, Cherednik.)

T_1 and T_0 are invertible.

$$Y := T_1 T_0, \qquad D := Y + q^{-1} abcd Y^{-1}.$$

Polynomial representation of $\tilde{\mathfrak{H}}$

Let $\mathcal{A}$ be the space of Laurent polynomials $f[z]$.

The *polynomial representation* of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$ is given by

$$
\begin{aligned}
(Zf)[z] &:= z\,f[z], \\
(T_1 f)[z] &:= -ab\,f[z] + \frac{(1-az)(1-bz)}{1-z^2}\left(f[z^{-1}] - f[z]\right), \\
(T_0 f)[z] &:= -q^{-1}cd\,f[z] + \frac{(c-z)(d-z)}{q-z^2}\left(f[z] - f[qz^{-1}]\right).
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$$(T_1 f)[z] := -ab\,f[z] + \frac{(1-az)(1-bz)}{1-z^2}\big(f[z^{-1}] - f[z]\big),$$

$$(T_0 f)[z] := -q^{-1}cd\,f[z] + \frac{(c-z)(d-z)}{q-z^2}\big(f[z] - f[qz^{-1}]\big).$$

Then

$$(T_1 f)[z] = -ab\,f[z] \quad \text{iff} \quad f[z] = f[z^{-1}],$$

and

$$(Df)[z] = (D_{\mathrm{sym}} f)[z] \quad \text{if} \quad f[z] = f[z^{-1}].$$

Eigenspaces of D

Let

$$\begin{aligned} Q_n[z] &:= a^{-1}b^{-1}z^{-1}(1-az)(1-bz)\,P_{n-1}[z; qa, qb, c, d \mid q] \\ &= z^n + \cdots + a^{-1}b^{-1}z^{-n}. \end{aligned}$$

Then

$$DQ_n = \lambda_n Q_n, \qquad T_1 Q_n = -Q_n.$$

Eigenspaces of D

Let

$$\begin{aligned} Q_n[z] &:= a^{-1}b^{-1}z^{-1}(1-az)(1-bz)\,P_{n-1}[z;qa,qb,c,d\mid q] \\ &= z^n + \cdots + a^{-1}b^{-1}z^{-n}. \end{aligned}$$

Then

$$DQ_n = \lambda_n Q_n, \qquad T_1 Q_n = -Q_n.$$

D has eigenvalues λ_n ($n = 0, 1, 2, \ldots$).

T_1 has eigenvalues $-1, -ab$.

D and T_1 commute.

The eigenspace of D for λ_n has basis P_n, Q_n ($n = 1, 2, \ldots$) or P_0 ($n = 0$).

Non-symmetric Askey-Wilson polynomials

Let

$$E_{-n} = \frac{ab}{ab-1}(P_n - Q_n) \qquad (n = 1, 2, \ldots),$$

$$E_n = \frac{(1-q^n ab)(1-q^{n-1}abcd)}{(1-ab)(1-q^{2n-1}abcd)} P_n - \frac{ab(1-q^n)(1-q^{n-1}cd)}{(1-ab)(1-q^{2n-1}abcd)} Q_n \qquad (n = 1, 2, \ldots).$$

Non-symmetric Askey-Wilson polynomials

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$$E_n = \frac{(1-q^n ab)(1-q^{n-1}abcd)}{(1-ab)(1-q^{2n-1}abcd)}\,P_n - \frac{ab(1-q^n)(1-q^{n-1}cd)}{(1-ab)(1-q^{2n-1}abcd)}\,Q_n \qquad (n = 1, 2, \ldots).$$

Then

$$YE_{-n} = q^{-n}\,E_{-n} \qquad (n = 1, 2, \ldots),$$

$$YE_n = q^{n-1}abcd\,E_n \qquad (n = 0, 1, 2, \ldots).$$

The $E_n[z]$ ($n \in \mathbb{Z}$) are the nonsymmetric Askey-Wilson polynomials. They form a biorthogonal system with respect to a suitable inner product given by a contour integral.

Relations for $\tilde{\mathfrak{H}}$ in terms of generators $Z^{\pm}, Y^{\pm}, T_1$

$$
\begin{aligned}
&T_1^2 = -(ab+1)T_1 - ab,\\
&T_1 Z = Z^{-1}T_1 + (ab+1)Z^{-1} - (a+b),\\
&T_1 Z^{-1} = ZT_1 - (ab+1)Z^{-1} + (a+b),\\
&T_1 Y = q^{-1}abcdY^{-1}T_1 - (ab+1)Y + ab(1+q^{-1}cd),\\
&T_1 Y^{-1} = q(abcd)^{-1}YT_1 + q(abcd)^{-1}(1+ab)Y - q(cd)^{-1}(1+q^{-1}cd),\\
&YZ = qZY + (1+ab)cd\,Z^{-1}Y^{-1}T_1 - (a+b)cd\,Y^{-1}T_1 - (1+q^{-1}cd)Z^{-1}T_1\\
&\qquad - (1-q)(1+ab)(1+q^{-1}cd)Z^{-1} + (c+d)T_1 + (1-q)(a+b)(1+q^{-1}cd),\\
&YZ^{-1} = q^{-1}Z^{-1}Y - q^{-2}(1+ab)cd\,Z^{-1}Y^{-1}T_1 + q^{-2}(a+b)cd\,Y^{-1}T_1\\
&\qquad + q^{-1}(1+q^{-1}cd)Z^{-1}T_1 - q^{-1}(c+d)T_1,\\
&Y^{-1}Z = q^{-1}ZY^{-1} - q(ab)^{-1}(1+ab)Z^{-1}Y^{-1}T_1 + (ab)^{-1}(a+b)Y^{-1}T_1\\
&\qquad + q(abcd)^{-1}(1+q^{-1}cd)Z^{-1}T_1 + q(abcd)^{-1}(1-q)(1+ab)(1+q^{-1}cd)Z^{-1}\\
&\qquad - (abcd)^{-1}(c+d)T_1 - (abcd)^{-1}(1-q)(1+ab)(c+d),\\
&Y^{-1}Z^{-1} = qZ^{-1}Y^{-1} + q(ab)^{-1}(1+ab)Z^{-1}Y^{-1}T_1 - (ab)^{-1}(a+b)Y^{-1}T_1\\
&\qquad - q^2(abcd)^{-1}(1+q^{-1}cd)Z^{-1}T_1 + q(abcd)^{-1}(c+d)T_1.
\end{aligned}
$$

Faithfulness of the polynomial representation of $\tilde{\mathfrak{H}}$

Theorem (Sahi)

The polynomial representation of $\tilde{\mathfrak{H}}$ is faithful.

The elements

$$Z^m Y^n T_1^i \qquad (m, n \in \mathbb{Z},\ i = 0, 1)$$

form a linear basis of $\tilde{\mathfrak{H}}$.

Central extension of $AW(3)$

Let the algebra $\widetilde{AW}(3)$ be generated by K_0, K_1, T_1 such that T_1 commutes with K_0, K_1 and with further relations

$$
\begin{aligned}
(T_1 + ab)(T_1 + 1) &= 0, \\
(q + q^{-1})K_1 K_0 K_1 - K_1^2 K_0 - K_0 K_1^2 &= B\, K_1 + C_0\, K_0 + D_0 \\
&\quad + E\, K_1(T_1 + ab) + F_0(T_1 + ab), \\
(q + q^{-1})K_0 K_1 K_0 - K_0^2 K_1 - K_1 K_0^2 &= B\, K_0 + C_1\, K_1 + D_1 \\
&\quad + E\, K_0(T_1 + ab) + F_1(T_1 + ab),
\end{aligned}
$$

where

$$
\begin{aligned}
E &:= -q^{-2}(1-q)^3(c+d), \\
F_0 &:= q^{-3}(1-q)^3(1+q)(cd+q), \\
F_1 &:= q^{-3}(1-q)^3(1+q)(a+b)cd.
\end{aligned}
$$

Polynomial representation of $\widetilde{AW}(3)$

The following element $\widetilde{Q}$ commutes with all elements of $\widetilde{AW}(3)$:

$$
\begin{aligned}
\widetilde{Q} := &(K_1K_0)^2 - (q^2+1+q^{-2})K_0(K_1K_0)K_1 \\
&+ (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\
&+ \big(B + E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\
&+ (q+1+q^{-1})\big(D_0 + F_0(T_1+ab)\big)K_0 \\
&+ (q+1+q^{-1})\big(D_1 + F_1(T_1+ab)\big)K_1 + G(T_1+ab),
\end{aligned}
$$

where G can be explicitly specified.

Polynomial representation of $\widetilde{AW}(3)$

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where G can be explicitly specified.

$\widetilde{AW}(3)$ acts on $\mathcal{A}$ such that K_0, K_1, T_1 act as D, $Z + Z^{-1}$, T_1, respectively, in the polynomial representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$.
This action is called the *polynomial representation* of $\widetilde{AW}(3)$ on $\mathcal{A}$.

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\widetilde{Q} :=& (K_1K_0)^2 - (q^2+1+q^{-2})K_0(K_1K_0)K_1 \\
&+ (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\
&+ \big(B + E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\
&+ (q+1+q^{-1})\big(D_0 + F_0(T_1+ab)\big)K_0 \\
&+ (q+1+q^{-1})\big(D_1 + F_1(T_1+ab)\big)K_1 + G(T_1+ab),
\end{aligned}
$$

where G can be explicitly specified.

$\widetilde{AW}(3)$ acts on $\mathcal{A}$ such that K_0, K_1, T_1 act as D, $Z+Z^{-1}$, T_1, respectively, in the polynomial representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$. This action is called the *polynomial representation* of $\widetilde{AW}(3)$ on $\mathcal{A}$.

Then $\widetilde{Q}$ acts as the constant Q_0.

A faithful representation on $\mathcal{A}$

Definition

$\widetilde{AW}(3, Q_0)$ is the algebra $\widetilde{AW}(3)$ with additional relation $\widetilde{Q} = Q_0$.

A faithful representation on $\mathcal{A}$

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Theorem (THK, 2007)

$\widetilde{AW}(3, Q_0)$ has the elements

$$K_0^n (K_1 K_0)^i K_1^m T_1^j \quad (m, n = 0, 1, 2, \ldots, \quad i, j = 0, 1)$$

as a linear basis.

The polynomial representation of $\widetilde{AW}(3, Q_0)$ on $\mathcal{A}$ is faithful.

$\widetilde{AW}(3, Q_0)$ has an injective embedding in $\tilde{\mathfrak{H}}$.

Definition of spherical subalgebra

From now on assume $ab \neq 1$. In $\tilde{\mathfrak{H}}$ we have:

$$\frac{T_1+1}{1-ab}\,\frac{T_1+1}{1-ab}=\frac{T_1+1}{1-ab}\,.$$

In the polynomial representation of $\tilde{\mathfrak{H}}$ we have:

$$(1-ab)^{-1}(T_1+1)f=\begin{cases}0 & \text{if } T_1 f=-f,\\ f & \text{if } T_1 f=-abf.\end{cases}$$

$P_{\mathrm{sym}} := (1-ab)^{-1}(T_1+1)$ projects $\mathcal{A}$ onto $\mathcal{A}_{\mathrm{sym}}$.

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$P_{\mathrm{sym}} := (1-ab)^{-1}(T_1+1)$ projects $\mathcal{A}$ onto $\mathcal{A}_{\mathrm{sym}}$.

Write

$$S(U) := P_{\mathrm{sym}} U P_{\mathrm{sym}} \qquad (U \in \tilde{\mathfrak{H}}).$$

Then

$$S(U)\,S(V) = S(U P_{\mathrm{sym}} V).$$

The image $S(\tilde{\mathfrak{H}})$ is a subalgebra of $\tilde{\mathfrak{H}}$, the spherical subalgebra.

Polynomial representation of spherical subalgebra

In the (faithful) polynomial representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$, the spherical subalgebra restricted to $\mathcal{A}_{\rm sym}$ yields a representation of $S(\tilde{\mathfrak{H}})$ on $\mathcal{A}_{\rm sym}$, which is also faithful. The following diagram is commutative.

$$\begin{array}{ccc} \tilde{\mathfrak{H}} & \longrightarrow & \mathrm{End}(\mathcal{A}) \\ \downarrow & & \downarrow \\ S(\tilde{\mathfrak{H}}) & \longrightarrow & \mathrm{End}(\mathcal{A}_{\rm sym}) \end{array}$$

$Z_{\tilde{\mathfrak{H}}}(T_1)$, the centralizer of T_1 in $\tilde{\mathfrak{H}}$, is a subalgebra on which S is an algebra homomorphism:

$$S(UV) = S(UP_{\rm sym}V) = S(U)\,S(V) \qquad (U, V \in Z_{\tilde{\mathfrak{H}}}(T_1)).$$

Note the embedding $\widetilde{AW}(3, Q_0) \hookrightarrow Z_{\tilde{\mathfrak{H}}}(T_1)$ with
$K_0 \mapsto Y + q^{-1}abcdY^{-1}, \quad K_1 \mapsto Z + Z^{-1}, \quad T_1 \mapsto T_1.$

$\widetilde{AW}(3,Q_0)$ and $AW(3,Q_0)$

$\widetilde{AW}(3,Q_0)$ generated by K_0, K_1, T_1 such that T_1 commutes with K_0, K_1 and $(T_1+ab)(T_1+1)=0$, and further relations

$$
\begin{aligned}
&(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 \\
&\quad = B\,K_1 + C_0\,K_0 + D_0 + E\,K_1(T_1+ab) + F_0(T_1+ab), \\
&(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 \\
&\quad = B\,K_0 + C_1\,K_1 + D_1 + E\,K_0(T_1+ab) + F_1(T_1+ab), \\
&Q_0 = (K_1K_0)^2 - (q^2+1+q^{-2})K_0(K_1K_0)K_1 \\
&\quad + (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\
&\quad + \big(B + E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\
&\quad + (q+1+q^{-1})\big(D_0 + F_0(T_1+ab)\big)K_0 \\
&\quad + (q+1+q^{-1})\big(D_1 + F_1(T_1+ab)\big)K_1 + G(T_1+ab).
\end{aligned}
$$

$AW(3,Q_0)$ is $\widetilde{AW}(3,Q_0)$ with additional relation $T_1=-ab$.

$AW(3, Q_0)$ mapped onto $S(\widetilde{AW}(3, Q_0))$

$\widetilde{AW}(3, Q_0)$ has basis $K_0^n (K_1 K_0)^i K_1^m T_1^j$

$$(m, n = 0, 1, 2, \ldots, \quad i, j = 0, 1).$$

$S(\widetilde{AW}(3, Q_0))$ has basis $(1 - ab)^{-1} K_0^n (K_1 K_0)^i K_1^m (T_1 + 1)$

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$$(m,n=0,1,2,\ldots\quad i=0,1).$$

$$U\mapsto(1-ab)^{-1}U(T_1+1)\colon AW(3,Q_0)\to S(\widetilde{AW}(3,Q_0))$$

is algebra isomorphism
because terms with factor T_1+ab in relations for $\widetilde{AW}(3,Q_0)$ are killed by factor T_1+1 in $S(\widetilde{AW}(3,Q_0))$.

Spherical subalgebra is isomorphic to $AW(3, Q_0)$

Theorem

$S(\tilde{\mathfrak{H}}) = S(\widetilde{AW}(3, Q_0))$, so the spherical subalgebra $S(\tilde{\mathfrak{H}})$ is isomorphic to the algebra $AW(3, Q_0)$.

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Sketch of Proof

$\tilde{\mathfrak{H}}$ has basis $Z^m Y^n T_1^i \quad (m, n \in \mathbb{Z},\ i = 0, 1)$.

$S(\tilde{\mathfrak{H}})$ is spanned by $(T_1 + 1) Z^m Y^n (T_1 + 1) \quad (m, n \in \mathbb{Z})$.

We say that

$$\sum_{k,l \in \mathbb{Z}} c_{k,l} Z^k Y^l = o(Z^m Y^n)$$

if $c_{k,l} \neq 0$ implies $|k| \leq |m|$, $|l| \leq |n|$, $(|k|, |l|) \neq (|m|, |n|)$.

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The result will follow from

$$(T_1+1)Z^m Y^n (T_1+1) \in (T_1+1)\Big(\widetilde{AW}(3, Q_0) + o(Z^m Y^n)\Big)(T_1+1).$$

Proof of theorem, continued

Write $(T_1 + 1)Z^m Y^n (T_1 + 1)$ as linear combination of

$$Z^{|m|}Y^{|n|}(T_1+1),\ Z^{-|m|}Y^{|n|}(T_1+1),\ Z^{-|m|}Y^{-|n|}(T_1+1),$$
$$Z^{-|m|}Y^{-|n|}(T_1+1) \quad \text{modulo } o(Z^{|m|}Y^{|n|})(T_1+1). \tag{1}$$

This is done by induction, starting with the $\tilde{\mathfrak{H}}$ relations for $T_1 Z$, $T_1 Z^{-1}$, $T_1 Y$, $T_1 Y^{-1}$.

Proof of theorem, continued

Write $(T_1+1)Z^mY^n(T_1+1)$ as linear combination of

$$Z^{|m|}Y^{|n|}(T_1+1),\ Z^{-|m|}Y^{|n|}(T_1+1),\ Z^{-|m|}Y^{-|n|}(T_1+1),$$
$$Z^{-|m|}Y^{-|n|}(T_1+1)\quad \text{modulo } o(Z^{|m|}Y^{|n|})(T_1+1). \tag{1}$$

This is done by induction, starting with the $\tilde{\mathfrak{H}}$ relations for T_1Z, T_1Z^{-1}, T_1Y, T_1Y^{-1}.

Also write $(T_1+1)K_1^mK_0^n(T_1+1)$ and $(T_1+1)K_1^{m-1}K_0K_1K_0^{n-1}(T_1+1)$ $(m,n=0,1,\ldots)$ as a linear combination of (1).

These latter linear combinations turn out to span the linear combinations obtained for $(T_1+1)Z^mY^n(T_1+1)$.

Proof of theorem, example

As an example see for $m, n > 0$:

$$(T_1 + 1)Z^m Y^n (T_1 + 1)$$
$$= \Big(Z^m Y^n - ab(q^{-1}abcd)^n Z^{-m} Y^{-n} + o(Z^m Y^n)\Big)(T_1 + 1)$$

and

$$(T_1+1)\Big(K_1^{m-1}(K_1K_0 - qK_0K_1)K_0^{n-1}\Big)(T_1+1) = (1-ab)(1-q^2)$$
$$\times \Big(Z^m Y^n - ab(q^{-1}abcd)^n Z^{-m} Y^{-n} + o(Z^m Y^n)\Big)(T_1 + 1).$$

Hence

$$(T_1 + 1)Z^m Y^n (T_1 + 1) = (1 - ab)^{-1}(1 - q^2)^{-1}$$
$$\times (T_1 + 1)\Big(K_1^{m-1}(K_1K_0 - qK_0K_1)K_0^{n-1} + o(Z^m Y^n)\Big)(T_1 + 1).$$

The subalgebra related to the -1 eigenspace of T_1

$P^-_{\mathrm{sym}} := (ab-1)^{-1}(T_1 + ab)$ projects $\mathcal{A}$ onto the -1 eigenspace $\mathcal{A}^-_{\mathrm{sym}}$ of T_1. Write

$$S^-(U) := P^-_{\mathrm{sym}} U P^-_{\mathrm{sym}} \qquad (U \in \tilde{\mathfrak{H}}).$$

The image $S^-(\tilde{\mathfrak{H}})$ is a subalgebra of $\tilde{\mathfrak{H}}$, and $S^-(\widetilde{AW}(3, Q_0))$ is a subalgebra of $S^-(\tilde{\mathfrak{H}})$.

Two isomorphic algebras

Theorem

Let $AW(3, Q_0; qa, qb, c, d)$ *be* $AW(3, Q_0)$ *with* a, b *replaced by* qa, qb*, respectively. Then* $K_0 \mapsto q(ab-1)^{-1}K_0(T_1 + ab)$ *and* $K_1 \mapsto (ab-1)^{-1}K_1(T_1 + ab)$ *extend to an algebra isomorphism* $AW(3, Q_0; qa, qb, c, d) \to S^-(\widetilde{AW}(3, Q_0))$.

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This is related to a result in Berest-Etingof-Ginzburg, Duke Math. J. (2003), Proposition 4.11 (see also Iain Gordon's lecture at this conference).

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Let $AW(3, Q_0; qa, qb, c, d)$ be $AW(3, Q_0)$ with a, b replaced by qa, qb, respectively. Then $K_0 \mapsto q(ab-1)^{-1}K_0(T_1+ab)$ and $K_1 \mapsto (ab-1)^{-1}K_1(T_1+ab)$ extend to an algebra isomorphism $AW(3, Q_0; qa, qb, c, d) \to S^-(\widetilde{AW}(3, Q_0))$.

This is related to a result in Berest-Etingof-Ginzburg, Duke Math. J. (2003), Proposition 4.11 (see also Iain Gordon's lecture at this conference).

Sketch of proof

Rewrite relations for $\widetilde{AW}(3, Q_0)$ while considering T_1+1 as a generator. Terms with factor T_1+1 in the relations for $\widetilde{AW}(3, Q_0)$ are killed by factor T_1+ab in $S^-(\widetilde{AW}(3, Q_0))$. In what remains, replace K_0 by $q^{-1}K_0$ and recognize the relations for $AW(3, Q_0; qa, qb, c, d)$.

The subalgebra $S^-(\tilde{\mathfrak{H}})$ is isomorphic to $AW(3, Q_0; qa, qb, c, d)$

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Theorem

$S^-(\tilde{\mathfrak{H}}) = S^-(\widetilde{AW}(3, Q_0))$, so the subalgebra $S^-(\tilde{\mathfrak{H}})$ is isomorphic to the algebra $AW(3, Q_0; qa, qb, c, d)$.

The proof is analogous to the proof that $S(\tilde{\mathfrak{H}}) = S(\widetilde{AW}(3, Q_0))$. One has to show that

$$(T_1+ab)Z^m Y^n (T_1+ab) \in (T_1+ab)\Big(\widetilde{AW}(3, Q_0)+o(Z^m Y^n)\Big)(T_1+ab).$$

The centralizer of T_1 in $\tilde{\mathfrak{H}}$

Corollary

The centralizer $Z_{\tilde{\mathfrak{H}}}(T_1)$ is equal to $\widetilde{AW}(3, Q_0)$.

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The centralizer $Z_{\tilde{\mathfrak{H}}}(T_1)$ is equal to $\widetilde{AW}(3, Q_0)$.

For the proof write $U \in \tilde{\mathfrak{H}}$ as

$$U = (1 - ab)^{-1}U(T_1 + 1) + (ab - 1)^{-1}U(T_1 + ab).$$

If $U \in Z_{\tilde{\mathfrak{H}}}(T_1)$ then so are $U(T_1 + 1)$ and $U(T_1 + ab)$. Hence

$$U(T_1 + 1) = (1 - ab)^{-1}(T_1 + 1)U(T_1 + 1),$$
$$U(T_1 + ab) = (ab - 1)^{-1}(T_1 + ab)U(T_1 + ab).$$

So $U(T_1 + 1) \in S(\tilde{\mathfrak{H}}) = S(\widetilde{AW}(3, Q_0)) \subset \widetilde{AW}(3, Q_0)$ and $U(T_1 + ab) \in S^-(\tilde{\mathfrak{H}}) = S^-(\widetilde{AW}(3, Q_0)) \subset \widetilde{AW}(3, Q_0)$.

The center of $\tilde{\mathfrak{H}}$

Corollary

The center of $\tilde{\mathfrak{H}}$ consists of the scalars.

Proof Let $U \in Z(\tilde{\mathfrak{H}})$. Then $U \in Z_{\tilde{\mathfrak{H}}}(T_1) = \widetilde{AW}(3, Q_0)$.
So $U \in Z(\widetilde{AW}(3, Q_0))$.
Then $U(T_1 + 1) \in Z(S(\tilde{\mathfrak{H}})) \sim Z(AW(3, Q_0)) \sim \mathbb{C}$ and
$U(T_1 + ab) \in Z(S^-(\tilde{\mathfrak{H}})) \sim Z(AW(3, Q_0; qa, qb, c, d)) \sim \mathbb{C}$.
So U is scalar. □

Further problems

1. In higher rank, any root system, describe in terms of generators and relations the algebra generated by polynomial multiplication and by the q-difference operators for which the Macdonald polynomials are eigenfunctions.
2. If thus the higher rank analogue of $AW(3, Q_0)$ is found, what is the analogue of $AW(3)$?
3. What about representations of $AW(3)$ for values of Q different from Q_0? Are there related special functions?
4. Higher rank analogues of my results in the nonsymmetric case.

Usage of Mathematica

I did computations in algebras defined by generators and relations in Mathematica with the aid of the package NCAlgebra, see `http://www.math.ucsd.edu/~ncalg/`

This was developed by J. W. Helton, R. L. Miller and M. Stankus, in particular for applications in systems engineering and control theory.

See my Mathematica notebooks on `http://www.science.uva.nl/~thk/art/`

References

Details of the first part of this lecture are in my paper

The relationship between Zhedanov's alsgebra $AW(3)$ *and the double affine Hecke algebra in the rank one case*,
SIGMA 3 (2007), 063;
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Details of the second part of this lecture are in my paper

Zhedanov's algebra AW(3) and the double affine Hecke algebra in the rank one case. II. The spherical subalgebra,
`arXiv:0711.2320v1 [math.QA]`.

Picture of Vadim Kuznetsov