The hierarchy of hypergeometric functions and related algebras

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Hypergeometric series

*Pochhammer symbol:* \((a)_k := a(a+1)\ldots(a+k-1)\).

*Hypergeometric series:* \(_{r}F_{s}(a_1, \ldots, a_r; b_1, \ldots, b_s; z)\)

\[
= {_{r}F_{s}} \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_r)_k}{(b_1)_k \ldots (b_s)_k k!} z^k.
\]

Terminating if \(a_1 = -n\) (\(n\) nonnegative integer).
If nonterminating and \(s = r - 1\) then converges for \(|z| < 1\).

*Gauss hypergeometric series:* \(\,_{2}F_{1}(a, b; c; z)\).

*Jacobi polynomials:*

\(P_n^{(\alpha,\beta)}(x) := \text{const.} \,_{2}F_{1}(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1}{2}(1 - x))\).

*Orthogonality* \((\alpha, \beta > -1)\):

\[
\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) (1 - x)^\alpha (1 + x)^\beta \, dx = 0 \quad (n \neq m).
\]
The Gauss hypergeometric function / Jacobi polynomial case can be generalized in five different directions, which often can be combined, and ideally should always be combined.

1. Higher hypergeometric series; Askey scheme of hypergeometric orthogonal polynomials
2. $q$-hypergeometric series, elliptic and hyperbolic hypergeometric function
3. Non-symmetric functions (double affine Hecke algebras)
4. Four regular singularities (Heun equation)
5. Multivariable special functions associated with root systems (Heckman-Opdam, Macdonald, Macdonald-K, Cherednik, ...)

I will not discuss items 4 and 5 here. However, item 3 was inspired by the multi-variable case.
Plan of the lecture

**First part**
Higher hypergeometric series and $q$- and elliptic analogues

**Second part**
Double affine Hecke algebra in the Askey-Wilson case and relationship with Zhedanov algebra
Criteria for the \((q-)\)hypergeometric hierarchy

For hypergeometric and \(q\)-hypergeometric functions we will restrict to some cases which:

- have a rich set of transformations, which form a nice symmetry group;
- allow harmonic analysis: orthogonal polynomials or biorthogonal rational functions, or continuous analogues of these as kernels of integral transforms.

Then we mainly have:

- \(4F_3(1), 7F_6(1), 9F_8(1)\) hypergeometric functions, and \(q\)- and hyperbolic analogues, and only one elliptic analogue
- Moreover in these cases restrictions on parameters (balanced, very-well poised)
- Always distinction between terminating and non-terminating series
- In non-terminating cases alternative representations as hypergeometric (Mellin-Barnes type) integral; crucial role of gamma function (ordinary, \(q\)-, hyperbolic, elliptic)
Thomae's transformation formula rediscovered by Ramanujan:

\[
\begin{align*}
3F2\left( \begin{array}{c} a, b, c \\ d, e \end{array} ; 1 \right) &= \frac{\Gamma(d) \Gamma(e) \Gamma(d + e - a - b - c)}{\Gamma(a) \Gamma(d + e - a - c) \Gamma(d + e - a - b)} \\
&\quad \times 3F2\left( \begin{array}{c} d - a, e - a, d + e - a - b - c \\ d + e - a - c, d + e - a - b \end{array} ; 1 \right).
\end{align*}
\]

Hardy (Ramanujan, Twelve lectures on subject suggested by his life and work, 1940):

\[
\frac{1}{\Gamma(d) \Gamma(e) \Gamma(d + e - a - b - c)} 3F2\left( \begin{array}{c} a, b, c \\ d, e \end{array} ; 1 \right)
\]

is symmetric in \( d, e, d + e - b - c, d + e - c - a, d + e - a - b \).

Symmetry group \( S_5 = W(A_4) \) (Weyl group of root system \( A_4 \)).
Balanced $4\text{F}3(1)$

$r\text{F}_{r-1}(a_1, \ldots, a_r; b_1, \ldots, b_{r-1}; z)$ is called balanced if $b_1 + \ldots + b_{r-1} = a_1 + \ldots + a_r + 1$.

Beyer-Louck-Stein rediscovered Hardy’s $S_5$-symmetry for $3\text{F}2(1)$, and found symmetry group $S_6 = W(A_5)$ for terminating balanced $4\text{F}3(1)$:

$$4\text{F}3\left(\begin{array}{c}
-n, a, b, c \\
d, e, f
\end{array}; 1\right) \quad (d + e + f = -n + a + b + c + 1).$$

Related orthogonal polynomials: Wilson polynomials $W_n(x^2) :=$

$$\text{const.} \quad 4\text{F}3\left(\begin{array}{c}
-n, n + a + b + c + d - 1, a + ix, a - ix \\
a + b, a + c, a + d
\end{array}; 1\right),$$

and Racah polynomials. These form the top level of the Askey scheme of hypergeometric orthogonal polynomials.
Askey scheme

Wilson

cont. dual Hahn

Hahn

Meixner-Pollaczek

Meixner

Laguerre

Hermite

Jacobi

Krawtchouk

Racah

dual Hahn
Wilson functions

For *Wilson functions* (non-polynomial analogues of Wilson polynomials) one has to go to the $7\, F_6$ level.

Well-poised hypergeometric series:

$$r\, F_{r-1} \left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ 1 + a_1 - a_2, \ldots, 1 + a_1 - a_r \end{array} ; z \right).$$

This is very well-poised (VWP) if $a_2 = 1 + \frac{1}{2} a_1$.

Terminating VWP $7\, F_6(1) = \text{const.} \times \text{terminating balanced } 4\, F_3(1)$.

Non-terminating VWP $7\, F_6(1) = \text{linear combination of two balanced } 4\, F_3(1)\text{'s}$.

Wilson function transform (Groenevelt).
Terminating 2-balanced VWP $\, _9 F_8(1)$:
  Transformation formula (Bailey, Whipple).
  Biorthogonal rational functions (J. Wilson).

Non-terminating 2-balanced VWP $\, _9 F_8(1)$:
  Four-term transformation formula (Bailey).
Let $0 < q < 1$.

$q$-Pochhammer symbol:

\[(a; q)_k := (1 - a)(1 - qa) \ldots (1 - q^{k-1}a),\]
\[(a; q)_\infty := (1 - a)(1 - qa)(1 - q^2a) \ldots ,\]
\[(a_1, \ldots, a_r; q)_k := (a_1; q)_k \ldots (a_r; q)_k.\]

$q$-hypergeometric $r\phi_{r-1}$ series:

\[r\phi_{r-1} \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{array}; q, z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_{r-1}; q)_k (q; q)_k} z^k.\]

Terminating if $a_1 = q^{-n}$ ($n$ nonnegative integer).
If nonterminating then converges for $|z| < 1$.

Balanced if $b_1 \ldots b_{r-1} = qa_1 \ldots a_r$. 

Askey-Wilson polynomials and functions

Terminating balanced $4\phi_3$ of argument $q$:
- Symmetry group $S_6 = W(A_5)$ (Van der Jeugt & S. Rao).
- Askey-Wilson polynomials:

$$p_n\left(\frac{1}{2}(z+z^{-1})\right) := \text{const. } 4\phi_3\left(\begin{array}{c} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{array}; q, q \right).$$

- Askey-Wilson polynomials together with $q$-Racah polynomials form the top level of the $q$-Askey scheme.

**Very well-poised (VWP) $q$-hypergeometric series:**

$$r V_{r-1}(a_1; a_4, \ldots, a_r; q, z) := r\phi_{r-1}\left(\begin{array}{c} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \ldots, a_r \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \ldots, qa_1/a_r \end{array}; q, z \right).$$

Non-terminating very well-poised $8\phi_7$ of argument $\frac{q^2 a_1^2}{a_4 a_5 a_6 a_7 a_8}$:
- Sum of two non-terminating balanced $4\phi_3$’s of argument $q$.
- Symmetry group $W(D_5)$ (Van der Jeugt & S. Rao).
- Askey-Wilson functions (Stokman).
Bailey’s two-term $10\phi_9$ function

$$\Phi(a; b; c, d, e, f, g, h; q) :=$$

$$\left(\frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{aq}{g}, \frac{aq}{h}; q\right)_\infty$$

$$\times \left(\frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}, \frac{bg}{a}, \frac{bh}{a}; q\right)_\infty \times \left(\frac{b}{a}, \frac{aq}{q}; q\right)_\infty$$

$$\times 10\nabla_9(a; b, c, d, e, f, g, h; q, q)$$

$$+ \frac{\left(\frac{bq}{c}, \frac{bq}{d}, \frac{bq}{e}, \frac{bq}{f}, \frac{bq}{g}, \frac{bq}{h}, c, d, e, f, g, h; q\right)_\infty}{\left(\frac{a}{b}, \frac{b^2q}{a}; q\right)_\infty}$$

$$\times 10\nabla_9(b^2; a; b, bc/a, bd/a, be/a, bf/a, bg/a, bh/a; q, q),$$

where $a^3q^2 = bcdefgh$.

Bailey’s four-term transformation formula:

$$\Phi(a; b; c, d, e, f, g, h; q) = \Phi\left(\frac{a^2q}{cde}; b; \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, f, g, h; q\right).$$

Symmetry group $W(E_6)$ (Lievens & Van der Jeugt).
Terminating balanced very well-poised $\phi_9$’s of argument $q$:

- Bailey’s two-term transformation formula.
- Same symmetry group $W(E_6)$.
- Biorthogonal rational functions (Rahman, J. Wilson)

Dynkin diagram of $E_6$:
Let $p, q \in \mathbb{C}$ ($|p|, |q| < 1$).

**Elliptic gamma function** (Ruijsenaars):

$$\Gamma_e(z; p, q) := \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}.$$  

**Elliptic hypergeometric integral** (Spiridonov):

$$S_e(t; p, q) := \int_{C} \frac{\prod_{j=1}^{8} \Gamma_e(t_j z^\pm 1; p, q)}{\Gamma_e(z^\pm 2; p, q)} \frac{dz}{2\pi i z} \quad (\prod_{j=1}^{8} t_j = p^2 q^2),$$  

where $C$ is a deformation of the unit circle which separates the poles $t_j p^m q^n$ ($m, n = 0, 1, \ldots$) from the poles $t_j^{-1} p^{-m} q^{-n}$ ($m, n = 0, 1, \ldots$).

The transformations of $S_e(t; p, q)$ form a symmetry group which is isomorphic to $W(E_7)$ (Rains).
Put $t_6 = az$, $t_7 = a/z$, $f(z) = S_e(t; p, q)$. Then $f(z)$ satisfies the elliptic hypergeometric differential equation (Spiridonov):

$$A(z)(f(qz) - f(z)) + A(z^{-1})(f(q^{-1}z) - f(z)) + \nu f(z) = 0,$$

where $A(z)$ and $\nu$ are suitable products of theta functions

$$\theta(b; p) := (b, pb^{-1}; p)_\infty.$$

**Elliptic Pochhammer symbol:**

$$(a; q, p)_k := \theta(a; p)\theta(qa; p) \ldots \theta(q^{k-1}a; p).$$

**Elliptic hypergeometric series:**

$$rE_{r-1}\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{array} ; q, p; z \right) := \sum_{k=0}^{\infty} \frac{(a_1; q, p)_k \ldots (a_r; q, p)_k}{(b_1; q, p)_k \ldots (b_{r-1}; q, p)_k(q; q, p)_k} z^k,$$

where $a_1 \ldots a_r = b_1 \ldots b_{r-1} q$.

This is the elliptic balancing condition in order that $(k + 1)$-th term / $k$-th term is doubly periodic in $k$. 

**Elliptic hypergeometric differential equation and series**
Two-index biorthogonal rational elliptic hypergeometric functions

Very well-poised elliptic hypergeometric series:

\[ r V_{r-1}(a_1; a_6, \ldots, a_r; q, p) := \]
\[ r E_{r-1} \left( \begin{array}{c}
  a_1, qa_1^{1/2}, -qa_1^{1/2}, q(a_1/p)^{1/2}, -q(a_1p)^{1/2}, a_6, \ldots, a_r \\
  a_1^{1/2}, -a_1^{1/2}, (pa_1)^{1/2}, -(a_1/p)^{1/2}, qa_1/a_6, \ldots, qa_1/a_r \\
\end{array} \right); q, p; -1, \]

where \( a_6 \ldots a_r = q^{1/2} r^{-4} a_1^{1/2} r^{-3} \).

A certain terminating \(_{12} V_{11}\) satisfies the elliptic hypergeometric equation. It was first introduced by Frenkel & Turaev (elliptic 6j-symbol). They gave a transformation formula, and a \(_{10} V_{9}\) summation formula as a degenerate case.

Products \( R_n(z; q, p) R_m(z; p, q) \) of such rational functions satisfy a \textit{two-index biorthogonality} (Spiridonov).
In elliptic hypergeometric theory there are no transformation formulas below the $_{12}V_{11}$ level.

However, there is a limit case of the elliptic hypergeometric function, called *hyperbolic hypergeometric function*, started by Ruijsenaars, which is still above the $q$-case and with the following features:

- On top level again $W(E_7)$ symmetry.
- There is also a hyperbolic Askey-Wilson function.
- Has analytic continuation to $q$ on unit circle.
- Explicit expressions as products of two $q$-hypergeometric functions or a sum of two such products.

For details see the Thesis by Fokko van de Bult, *Hyperbolic Hypergeometric Functions*, 2007 (partly based on papers jointly with Rains and Stokman).
Second part
Double affine Hecke algebra in the Askey-Wilson case and relationship with Zhedanov algebra
Askey-Wilson operator acting on symmetric Laurent polynomials $f[z] = f[z^{-1}]$:

$$(D_{\text{sym}} f)[z] := A[z] (f[qz] - f[z]) + A[z^{-1}] (f[q^{-1}z] - f[z]) + (1 + q^{-1}abcd) f[z],$$

where


Askey-Wilson polynomials (monic symmetric Laurent polynomials $P_n[z] = P_n[z^{-1}] = z^n + \cdots + z^{-n}$):

$$P_n[z] := \text{const.} \ 4\phi_3 \left( \begin{array}{c} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \\ q, q \end{array} \right),$$

Eigenvalue equation:

$$D_{\text{sym}} P_n = \lambda_n P_n, \quad \text{where} \quad \lambda_n := q^{-n} + q^{n-1}abcd.$$
Let $0 < q < 1$, $a, b, c, d \in \mathbb{C}\setminus\{0\}$, $abcd \neq q^{-m}$ ($m = 0, 1, 2, \ldots$).

**Definition**

The *double affine Hecke algebra* $\tilde{H}$ of type $(C_1^\vee, C_1)$ is the algebra with generators $Z, Z^{-1}, T_1, T_0$ and with relations

\[
(T_1 + ab)(T_1 + 1) = 0,
\]

\[
(T_0 + q^{-1}cd)(T_0 + 1) = 0,
\]

\[
(T_1Z + a)(T_1Z + b) = 0,
\]

\[
(qT_0Z^{-1} + c)(qT_0Z^{-1} + d) = 0.
\]

(Sahi; Noumi & Stokman; Macdonald’s 2003 book)

$T_1$ and $T_0$ are invertible. Let

\[
Y := T_1 T_0, \quad D := Y + q^{-1}abcdY^{-1}.
\]
Let $\mathcal{A}$ be the space of Laurent polynomials $f[z]$. The polynomial representation of $\tilde{\mathcal{H}}$ on $\mathcal{A}$ is given by

$$(Zf)[z] := z f[z],$$

$$(T_1 f)[z] := -ab f[z] + \left(1 - az\right)\left(1 - bz\right) \frac{1}{1 - z^2} \left(f[z^{-1}] - f[z]\right),$$

$$(T_0 f)[z] := -q^{-1} cd f[z] + \frac{(c - z)(d - z)}{q - z^2} \left(f[z] - f[qz^{-1}]\right)$$

($q$-difference-reflection operators; $q$-analogues of the Dunkl operator). Then

$$(T_1 f)[z] = -ab f[z] \text{  iff  } f[z] = f[z^{-1}],$$

and

$$(Df)[z] = (D_{\text{sym}} f)[z] \text{  if  } f[z] = f[z^{-1}].$$
Eigenspaces of $D$

Let

$$Q_n[z] := a^{-1} b^{-1} z^{-1} (1 - az)(1 - bz) P_{n-1}[z; qa, qb, c, d | q].$$

Then

$$DQ_n = \lambda_n Q_n, \quad T_1 Q_n = -Q_n.$$  
$$DP_n = \lambda_n P_n, \quad T_1 P_n = -ab Q_n.$$

$D$ has eigenvalues $\lambda_n \ (n = 0, 1, 2, \ldots)$.  
$T_1$ has eigenvalues $-1, -ab$.  
$D$ and $T_1$ commute.  
The eigenspace of $D$ for $\lambda_n$ has basis $P_n, Q_n \ (n = 1, 2, \ldots)$ or $P_0 \ (n = 0)$.  

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Hypergeometric hierarchy
Non-symmetric Askey-Wilson polynomials

Let

\[ E_{-n} := \frac{ab}{ab - 1} (P_n - Q_n) \quad (n = 1, 2, \ldots), \]
\[ E_n := \frac{(1 - q^n ab)(1 - q^{n-1} abcd)}{(1 - ab)(1 - q^{2n-1} abcd)} P_n - \frac{ab(1 - q^n)(1 - q^{n-1} cd)}{(1 - ab)(1 - q^{2n-1} abcd)} Q_n \quad (n = 1, 2, \ldots). \]

Then

\[ YE_{-n} = q^{-n} E_{-n} \quad (n = 1, 2, \ldots), \]
\[ YE_n = q^{n-1} abcd E_n \quad (n = 0, 1, 2, \ldots). \]

The \( E_n[z] \ (n \in \mathbb{Z}) \) are the nonsymmetric Askey-Wilson polynomials. They form a biorthogonal system with respect to a suitable inner product given by a contour integral.
Zhedanov’s algebra $AW(3)$

**Definition**

Zhedanov’s algebra $AW(3)$ is the algebra generated by $K_0, K_1$ with relations

\[(q + q^{-1})K_1 K_0 K_1 - K_1^2 K_0 - K_0 K_1^2 = B K_1 + C_0 K_0 + D_0,\]

\[(q + q^{-1})K_0 K_1 K_0 - K_0^2 K_1 - K_1 K_0^2 = B K_0 + C_1 K_1 + D_1.\]

The *Casimir operator*

\[Q := K_1 K_0 K_1 K_0 - (q^2 + 1 + q^{-2})K_0 K_1 K_0 K_1 \]

\[+ (q + q^{-1})K_0^2 K_1^2 + (q + q^{-1})(C_0 K_0^2 + C_1 K_1^2) \]

\[+ B((q + 1 + q^{-1})K_0 K_1 + K_1 K_0) \]

\[+ (q + 1 + q^{-1})(D_0 K_0 + D_1 K_1).\]

commutes in $AW(3)$ with the generators $K_0, K_1$. 
The polynomial representation of $AW(3)$

Let $e_1, e_2, e_3, e_4$ be the elementary symmetric polynomials in $a, b, c, d$.

Put for the structure constants:

\[ B := (1 - q^{-1})^2(e_3 + qe_1), \]
\[ C_0 := (q - q^{-1})^2, \]
\[ C_1 := q^{-1}(q - q^{-1})^2 e_4, \]
\[ D_0 := -q^{-3}(1 - q)^2(1 + q)(e_4 + qe_2 + q^2), \]
\[ D_1 := -q^{-3}(1 - q)^2(1 + q)(e_1 e_4 + qe_3). \]

Then the polynomial representation of $AW(3)$ on the space $A_{\text{sym}}$ of symmetric Laurent polynomials in $z$ is given by

\[ (K_0 f)[z] := (D_{\text{sym}} f)[z], \]
\[ (K_1 f)[z] := (z + z^{-1}) f[z]. \]
The quotient algebra $AW(3, Q_0)$

In the polynomial representation (which is irreducible for generic values of $a, b, c, d$), $Q$ becomes a constant scalar:

$$(Qf)[z] = Q_0 f[z] \quad \text{where}$$

$$Q_0 := q^{-4}(1 - q)^2 \left( q^4(e_4 - e_2) + q^3(e_1^2 - e_1 e_3 - 2e_2) ight.$$  
$$- q^2(e_2 e_4 + 2e_4 + e_2) + q(e_3^2 - 2e_2 e_4 - e_1 e_3) + e_4(1 - e_2) \right).$$

Definition

$AW(3, Q_0)$ is the algebra $AW(3)$ with further relation $Q = Q_0$.

Theorem (K, 2007)

A basis of $AW(3, Q_0)$ is given by

$$K_0^n(K_1 K_0)^l K_1^m \quad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1).$$

The polynomial representation of $AW(3, Q_0)$ on $A_{\text{sym}}$ is faithful.
Central extension of $\tilde{\mathcal{A}W}(3)$

Let the algebra $\tilde{\mathcal{A}W}(3, Q_0)$ be generated by $K_0, K_1, T_1$ such that $T_1$ commutes with $K_0, K_1$ and with further relations

\[
(T_1 + ab)(T_1 + 1) = 0,
\]

\[
(q + q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 =BK_1 + C_0K_0 + D_0 + EK_1(T_1 + ab) + F_0(T_1 + ab),
\]

\[
(q + q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = BK_0 + C_1K_1 + D_1 + EK_0(T_1 + ab) + F_1(T_1 + ab),
\]

\[
\tilde{Q} := (K_1K_0)^2 - (q^2 + 1 + q^{-2})K_0(K_1K_0)K_1
\]

\[+ (q + q^{-1})K_0^2K_1^2 + (q + q^{-1})(C_0K_0^2 + C_1K_1^2)
\]

\[+ (B + E(T_1 + ab))((q + 1 + q^{-1})K_0K_1 + K_1K_0)
\]

\[+ (q + 1 + q^{-1})(D_0 + F_0(T_1 + ab))K_0
\]

\[+ (q + 1 + q^{-1})(D_1 + F_1(T_1 + ab))K_1 + G(T_1 + ab) = Q_0,
\]

where $E, F_0, F_1, G$ can be explicitly specified. Then $\tilde{Q}$ commutes with all elements of $\tilde{\mathcal{A}W}(3)$.
Connecting $\hat{A}W(3, Q_0)$ with $\tilde{\mathfrak{g}}$

**Theorem (K, 2007)**

$\hat{A}W(3, Q_0)$ acts on $A$ such that $K_0, K_1, T_1$ act as $D, Z + Z^{-1}, T_1$, respectively, in the polynomial representation of $\tilde{\mathfrak{g}}$ on $A$.

This representation is faithful.

$\hat{A}W(3, Q_0)$ has an injective embedding in $\tilde{\mathfrak{g}}$.

**Theorem (K, 2007)**

Let $ab \neq 1$.

$A\bar{W}(3, Q_0)$ is naturally isomorphic to the spherical subalgebra $(T_1 + 1)\tilde{\mathfrak{g}}(T_1 + 1)$.

$\hat{A}W(3, Q_0)$ is the centralizer of $T_1$ in $\tilde{\mathfrak{g}}$. 
On hypergeometric series:

On $q$-hypergeometric series:

On the Askey and the $q$-Askey scheme:
On elliptic hypergeometric functions:


On groups of transformations of hypergeometric functions:


On Zhedanov’s algebra and the double affine Hecke algebra:

See the following two papers and references given there.


T. H. Koornwinder, *Zhedanov’s algebra AW(3) and the double affine Hecke algebra in the rank one case. II. The spherical subalgebra*, arXiv/0711.2320v1.