Fractional integral and generalized Stieltjes transforms for hypergeometric functions as transmutation operators

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Oberwolfach workshop Combinatorics, 4–10 May 1980

École d'Été d'Analyse Harmonique, Tunis,
August–September 1984: excursion to Cap Bon

Tunisia, September 1984:
trip to Kairouan (with René Beerends)

About this talk see the preprint

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Fractional integrals

Riemann-Liouville fractional integral:

$$(I_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_a^x f(y)\,(x-y)^{\mu-1}\,dy \quad (\mathrm{Re}\,\mu > 0).$$

Weyl fractional integral:

$$(W_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_x^\infty f(y)\,(y-x)^{\mu-1}\,dy \quad (\mathrm{Re}\,\mu > 0).$$

$$\left(\frac{d}{dx}\right)^n (I_n f)(x) = f(x), \quad \left(\frac{d}{dx}\right)^n (W_n f)(x) = (-1)^n f(x).$$

Gauss hypergeometric function

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \qquad (|z| < 1). $$

It has a one-valued analytic continuation to $\mathbb{C} \backslash [1, \infty)$.

Hypergeometric equation:

$$ D_{a,b,c}\ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = 0, \quad \text{where} $$

$$ (D_{a,b,c} f)(z) := z(1-z)f''(z) + (c - (a+b+1)z)f'(z) - ab\, f(z). $$

Then $f(z) := {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$ is the unique solution of $D_{a,b,c} f = 0$ which is regular and equal to 1 at $z = 0$.

Fractional integrals and hypergeometric functions

Bateman's integral (1909):

$$\int_0^1 t^{c-1}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};tz\right)(1-t)^{\mu-1}\,dt=\frac{\Gamma(c)\Gamma(\mu)}{\Gamma(c+\mu)}\,{}_2F_1\left(\begin{matrix}a,b\\c+\mu\end{matrix};z\right)$$

$$(z\in\mathbb{C}\backslash[1,\infty),\ \mathrm{Re}\,c>0,\ \mathrm{Re}\,\mu>0).$$

As fractional integral:

$$\frac{1}{\Gamma(\mu)}\int_0^x y^{c-1}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};y\right)(x-y)^{\mu-1}\,dy$$

$$=\frac{\Gamma(c)}{\Gamma(c+\mu)}\,x^{c+\mu-1}\,{}_2F_1\left(\begin{matrix}a,b\\c+\mu\end{matrix};x\right)$$

$$(0<x<1,\ \mathrm{Re}\,c>0,\ \mathrm{Re}\,\mu>0).$$

Fractional generalization of:

$$\left(\frac{d}{dx}\right)^n\left(x^{c+n-1}\,{}_2F_1\left(\begin{matrix}a,b\\c+n\end{matrix};x\right)\right)=\frac{\Gamma(c+n)}{\Gamma(c)}\,x^{c-1}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right).$$

Askey-Fitch integral (1969):

$$\frac{1}{\Gamma(\mu)}\int_0^x y^{a-\mu-1}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};y\right)(x-y)^{\mu-1}\,dy$$
$$=\frac{\Gamma(a-\mu)}{\Gamma(a)}\,x^{a-1}\,{}_2F_1\left(\begin{matrix}a-\mu,b\\c\end{matrix};x\right)$$
$$(0<x<1,\ \mathrm{Re}\,a>\mathrm{Re}\,\mu>0).$$

Fractional generalization of:

$$\left(\frac{d}{dx}\right)^n\left(x^{a-1}\,{}_2F_1\left(\begin{matrix}a-n,b\\c\end{matrix};x\right)\right)=\frac{\Gamma(a)}{\Gamma(a-n)}\,x^{a-n-1}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right).$$

Fractional integrals and hypergeom. functions (cntd.)

Camporesi integral (2014, he gave a special case).
Rewrite the Askey-Fitch integral as

$$\frac{1}{\Gamma(\mu)} \int_{-\infty}^{x} (-y)^{-a} \, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; y^{-1}\right) (x-y)^{\mu-1} \, dy$$
$$= \frac{\Gamma(a-\mu)}{\Gamma(a)} (-x)^{-a+\mu} \, {}_2F_1\left(\begin{matrix} a-\mu, b \\ c \end{matrix}; x^{-1}\right) \qquad (x<0)$$

and twice apply

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-x)^{-a} \, {}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}; x^{-1}\right)$$
$$+ \left(a \longleftrightarrow b\right) \qquad (x<0).$$

Then:

Fractional integrals and hypergeom. functions (cntd.)

Camporesi integral:

$$\frac{1}{\Gamma(\mu)}\int_{-\infty}^{x} {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};y\right)(x-y)^{\mu-1}\,dy$$
$$=\frac{\Gamma(a-\mu)}{\Gamma(a)}\,\frac{\Gamma(b-\mu)}{\Gamma(b)}\,\frac{\Gamma(c)}{\Gamma(c-\mu)}\,{}_2F_1\left(\begin{matrix}a-\mu,b-\mu\\c-\mu\end{matrix};x\right)$$
$$(x<1,\ \mathrm{Re}\,a,\mathrm{Re}\,b>\mathrm{Re}\,\mu>0).$$

Fractional generalization of:

$$\left(\frac{d}{dx}\right)^{n}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)=\frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1\left(\begin{matrix}a+n,b+n\\c+n\end{matrix};x\right).$$

Fractional integrals and hypergeom. functions (cntd.)

The three fractional integrals of Bateman, Askey-Fitch and Camporesi have respectively two, one, and two variants by applying the following transformation formulas on both sides:

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) &= (1-z)^{-a}\,{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix};\frac{z}{z-1}\right) \qquad (z\notin[1,\infty)),\\
{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right) &= (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\c\end{matrix};z\right) \qquad (z\notin[1,\infty)).
\end{aligned}
$$

The parameter shift patterns of these eight formulas are:

I. Bateman: $c+;\quad a+,c+;\quad a+,b+,c+$
II. Askey-Fitch: $a-;\quad a+$
III. Camporesi: $a-,b-,c-;\quad a-,c-;\quad c-$

They are the fractional analogues of the eight parameter shifting n-th derivative formulas for hypergeometric functions.

Euler integral representation

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt $$
$$ (z \in \mathbb{C}\backslash[1,\infty),\ \mathrm{Re}\, c > \mathrm{Re}\, b > 0). $$

As fractional integral:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)\, x^{1-c}}{\Gamma(b)\Gamma(c-b)} \int_0^x y^{b-1}(1-y)^{-a}\,(x-y)^{c-b-1}\, dy $$
$$ (0 < x < 1,\ \mathrm{Re}\, c > \mathrm{Re}\, b > 0). $$

This is the case $c' = b$ of Bateman's integral:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)\, x^{1-c}}{\Gamma(c-c')\Gamma(c')} \int_0^x y^{c'-1}\, {}_2F_1\left(\begin{matrix} a, b \\ c' \end{matrix}; y\right)\,(x-y)^{c-c'-1}\, dy. $$

Proof by transmutation

Write $D_{a,b,c;x}$ for $D_{a,b,c}$ acting on a function of x.
We can prove the transmutation identity:

$$D_{a,b,c;x}\left(x^{1-c}\int_0^x y^{c'-1}f(y)\,(x-y)^{c-c'-1}\,dy\right) = x^{1-c}\int_0^x y^{c'-1}(D_{a,b,c'}f)(y)\,(x-y)^{c-c'-1}\,dy.$$

In particular for $c'=b$ (Euler integral representation).

In the identity, if $D_{a,b,c'}f=0$ then the integral on the left-hand side is annihilated by $D_{a,b,c;x}$.

Regularity and value at $x=0$ of the integral follows from regularity and value at 0 of f.

An operator T is a *transmutation operator* between differential operators D_1 and D_2 if

$$D_1 T = T D_2.$$

Pioneers are J. Delsarte (1938), Levitan (1951) and J. L. Lions (1956). A typical example is:

$D_2 = d^2/dx^2, \quad D_1 = d^2/dx^2 + (2\alpha+1)x^{-1}d/dx$ (Bessel).

Then:

$$(Tf)(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,x^{-2\alpha}\int_0^x f(y)\,(x^2-y^2)^{\alpha-\frac{1}{2}}\,dy \quad (\mathrm{Re}\,\alpha > -\tfrac{1}{2}).$$

If $f(x) = \cos(\lambda x)$ then

$$(Tf)(x) = \mathcal{J}_\alpha(\lambda x) := {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; -\frac{(\lambda x)^2}{4}\right).$$

Proof by transmutation (cntd.)

For the proof of the transmutation identity

$$D_{a,b,c;x}\left(x^{1-c}\int_0^x y^{c'-1}f(y)\,(x-y)^{c-c'-1}\,dy\right)$$
$$= x^{1-c}\int_0^x y^{c'-1}(D_{a,b,c'}f)(y)\,(x-y)^{c-c'-1}\,dy$$

we need two preliminary results:

1. The formal adjoint of $D_{a,b,c}$ is $D_{1-a,1-b,2-c}$.

2. A PDE for the fractional integral kernel:

$$y^{1-c'}D_{1-a,1-b,2-c';y}\left(y^{c'-1}(x-y)^{c-c'-1}\right)$$
$$= x^{c-1}D_{a,b,c;x}\left(x^{1-c}(x-y)^{c-c'-1}\right).$$

Proof by transmutation (cntd.)

Now we have formally:

$$
\begin{aligned}
&D_{a,b,c;x}\left(x^{1-c}\int_0^x f(y)\,y^{c'-1}\,(x-y)^{c-c'-1}\,dy\right)\\
&=\int_0^x f(y)\,y^{c'-1}\,D_{a,b,c;x}\Big(x^{1-c}(x-y)^{c-c'-1}\Big)dy\\
&=x^{1-c}\int_0^x f(y)\,D_{1-a,1-b,2-c';y}\Big(y^{c'-1}(x-y)^{c-c'-1}\Big)dy\\
&=x^{1-c}\int_0^x (D_{a,b,c'}f)(y)\,y^{c'-1}\,(x-y)^{c-c'-1}\,dy.
\end{aligned}
$$

All steps can be made rigorous for f regular and $\mathrm{Re}\,c'$ and $\mathrm{Re}\,(c-c')$ sufficiently large. Then relax the constraints by analytic continuation.

Key observation

Our proof of the transmutation identity works also for other integration boundaries:

$$D_{a,b,c;x}\left(|x|^{1-c}\int_m^M |y|^{c'-1} f(y)\,|x-y|^{c-c'-1}\,dy\right) = |x|^{1-c}\int_m^M |y|^{c'-1}(D_{a,b,c'}f)(y)\,|x-y|^{c-c'-1}\,dy.$$

Here m and M are two consecutive points where the integrand sufficiently vanishes. Such points may be x (but not necessarily), and also $-\infty$ or ∞.

So another solution of $D_{a,b,c'}f=0$ and/or other integration boundaries may yield another solution of $D_{a,b,c}g=0$.
This works also for the integrands of the other seven fractional integral transforms.

Key observation (cntd.)

In particular Euler type integral representations

$$g(x) = |x|^{1-c} \int_m^M |y|^{b-1} |1-y|^{-a} |x-y|^{c-b-1} \, dy$$

by

$$(D_{a,b,c} g)(x) = |x|^{1-c} \int_m^M |y|^{b-1} D_{a,b,b;y} \left(|1-y|^{-a} \right) |x-y|^{c-b-1} \, dy = 0.$$

This works for:

(m, M)	$(-\infty, x)$	$(-\infty, 0)$	$(x, 0)$	$(0, x)$	$(0, 1)$
$x \in$	$(-\infty, 0)$	$(0, \infty)$	$(-\infty, 0)$	$(0, 1)$	$(-\infty, 0)$

(m, M)	$(0, 1)$	$(x, 1)$	$(1, x)$	$(1, \infty)$	(x, ∞)
$x \in$	$(1, \infty)$	$(0, 1)$	$(1, \infty)$	$(-\infty, 1)$	$(1, \infty)$

Six solutions of the hypergeometric equation

For real x:

$$w_1(x;a,b,c) = {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};x\right) \quad (x\in(-\infty,1)),$$

$$w_2(x;a,b,c) = |x|^{1-c}\,{}_2F_1\left(\begin{matrix} a-c+1,b-c+1 \\ 2-c \end{matrix};x\right) \quad (x\in(-\infty,1)),$$

$$w_3(x;a,b,c) = |x|^{-a}\,{}_2F_1\left(\begin{matrix} a,a-c+1 \\ a-b+1 \end{matrix};x^{-1}\right) \quad (x\notin[0,1]),$$

$$w_4(x;a,b,c) = |x|^{-b}\,{}_2F_1\left(\begin{matrix} b,b-c+1 \\ b-a+1 \end{matrix};x^{-1}\right) \quad (x\notin[0,1]),$$

$$w_5(x;a,b,c) = {}_2F_1\left(\begin{matrix} a,b \\ a+b-c+1 \end{matrix};1-x\right) \quad (x\in(0,\infty)),$$

$$w_6(x;a,b,c) = |1-x|^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,c-b \\ c-a-b+1 \end{matrix};1-x\right) \quad (x\in(0,\infty)).$$

Euler type integral representations for the solutions

Integrand $|x|^{1-c}\,|y|^{b-1}\,|1-y|^{-a}\,|x-y|^{c-b-1}$:

$$w_1(x;a,b,c)=\frac{\Gamma(c)\,|x|^{1-c}}{\Gamma(b)\Gamma(c-b)}\int_{0<y/x<1}|y|^{b-1}(1-y)^{-a}\,|x-y|^{c-b-1}\,dy$$

$$(x<1,\ \mathrm{Re}\,c>\mathrm{Re}\,b>0),$$

$$w_3(x;a,b,c)=\frac{\Gamma(a-b+1)\,|x|^{1-c}}{\Gamma(a-c+1)\Gamma(c-b)}\int_{y/x>1}|y|^{b-1}|y-1|^{-a}\,|y-x|^{c-b-1}\,dy$$

$$(x\notin[0,1],\ \mathrm{Re}\,(a-c+1)>0,\ \mathrm{Re}\,(c-b)>0),$$

$$w_6(x;a,b,c)=\frac{\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(c-b)}$$

$$\times\,x^{1-c}\int_{0<\frac{1-y}{1-x}<1}y^{b-1}(1-y)^{-a}\,|x-y|^{c-b-1}\,dy$$

$$(x>0,\ \mathrm{Re}\,(c-b)>0,\ \mathrm{Re}\,a<1).$$

Euler type integral reps for the solutions (cntd.)

Integrand $|y|^{a-c}\,|1-y|^{c-b-1}\,|x-y|^{-a}$:

$$w_2(x;a,b,c)=\frac{\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-a)}\times\int_{0<y/x<1}|y|^{a-c}(1-y)^{c-b-1}\,|x-y|^{-a}\,dy$$

$$(x<1,\ \mathrm{Re}\,a,\mathrm{Re}\,(c-a)<1),$$

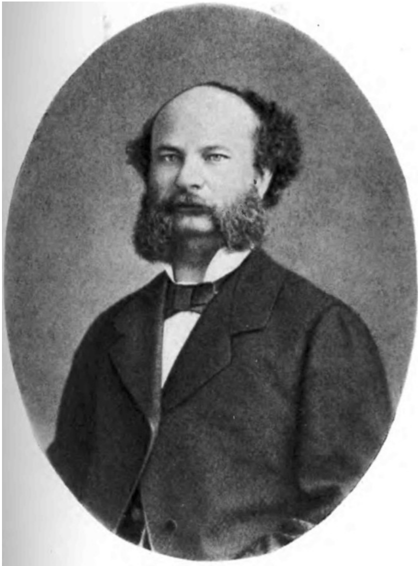
$$w_4(x;a,b,c)=\frac{\Gamma(b-a+1)}{\Gamma(-a)\Gamma(b+1)}\int_{y/x>1}|y|^{a-c}\,|y-1|^{c-b-1}\,|y-x|^{-a}\,dy$$

$$(x\notin[0,1],\ \mathrm{Re}\,a<1,\ \mathrm{Re}\,b>0),$$

$$w_6(x;a,b,c)=\frac{\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(c-b)}\int_{0<\frac{1-y}{1-x}<1}|y|^{a-c}|1-y|^{c-b-1}|x-y|^{-a}\,dy$$

$$(x>0,\ \mathrm{Re}\,(c-b)>0,\ \mathrm{Re}\,a<1).$$

Aleksei Letnikov (1837–1888)



Алексей Васильевич Летников
(1837—1888)

Aleksei Letnikov (cntd.)

A. V. Letnikov, *Research related to the theory of integrals of the form $\int_0^x (x-u)^{p-1} f(u)\,du$. Chapter III, Application to the integration of certain differential equations* (in Russian), Mat. Sbornik. 7 (1874), no. 2, 111–205.

This paper contains already the Euler type integral relations for the various solutions of the hypergeometric equation. Letnikov essentially used the transmutation method.

The paper remained unknown outside Russia. Mat. Sbornik was not reviewed in JFM before 1900. I thank Sergei Sitnik for bringing this work to my attention.

Letnikov is only mentioned in literature in connection with the Grünwald-Letnikov fractional derivative.

Fractional integrals for other solutions

Recall the eight parameter shift patterns for which we got fractional integral tranformation formulas for the solution w_1:

$c+; \quad a+, c+; \quad a+, b+, c+; \quad a-; \quad a+; \quad a-, b-, c-; \quad a-, c-; \quad c-$

Each of the other solutions w_j has also a fractional integral transformation formula for each of these eight patterns, and for each pattern the formula for each w_j has the same integrand, but the integration boundaries vary.

Furthermore, each Euler type integral representation for some w_j is a special case of some fractional integral transformation for that w_j.

Generalized Stieltjes transform

Widder (1938), for $z \in \mathbb{C}\backslash(-\infty, 0]$:

$$\alpha \mapsto f\colon\ f(z) = \int_0^\infty \frac{d\alpha(t)}{(z+t)^\rho} \quad \text{or} \quad \phi \mapsto f\colon\ f(z) = \int_0^\infty \frac{\phi(t)}{(z+t)^\rho}\, dt.$$

For $\rho = 1$ the classical Stieltjes transform.

Here we will denote by the generalized Stieltjes transform a map

$$f \mapsto g\colon\ g(x) = \int_m^M f(y)\, |y-x|^{\mu-1}\, dy \quad (x \in \mathbb{R}\backslash[m, M]),$$

where (m, M) is $(-\infty, 0)$ or $(0, 1)$ or $(1, \infty)$.

Generalized Stieltjes from Euler type integral rep.

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)=\frac{\Gamma(c)\,x^{1-c}}{\Gamma(b)\Gamma(c-b)}\int_0^x y^{b-1}(1-y)^{-a}\,(x-y)^{c-b-1}\,dy $$
$$ (0<x<1,\ \mathrm{Re}\,c>\mathrm{Re}\,b>0). $$

Replace x by x^{-1} and then substitute $y\mapsto y/x$:

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x^{-1}\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\,x^a\int_0^1 y^{b-1}(1-y)^{c-b-1}(x-y)^{-a}\,dy $$
$$ (x>1,\ \mathrm{Re}\,c>\mathrm{Re}\,b>0). $$

In this way obtain three Euler type integral representations with integrand $|y|^{a-c}\,|1-y|^{c-b-1}\,|x-y|^{-a}$ as generalized Stieltjes transforms, and three other with integrand $|x|^{1-c}\,|y|^{b-1}\,|1-y|^{-a}\,|x-y|^{c-b-1}$.

Generalized Stieltjes with Bateman type integrand

$$\int_{-\infty}^{0} (-y)^{c-1} w_1(y;a,b,c)\,(x-y)^{\mu-1}\,dy$$
$$= \frac{\Gamma(a-c-\mu+1)\Gamma(b-c-\mu+1)\Gamma(c)}{\Gamma(a+b-c-\mu+1)\Gamma(1-\mu)}\,x^{c-1+\mu} w_5(x;a,b,c+\mu)$$
$$(x>0,\ \mathrm{Re}\,(a-c-\mu+1), \mathrm{Re}\,(b-c-\mu+1), \mathrm{Re}\,(c)>0).$$

The left-hand side has the same integrand as Bateman's integral. That $x^{1-c-\mu}$ times the right-hand side is annihilated by $D_{a,b,c+\mu;x}$ can be expected by the transmutation argument.

Thus 24 generalized Stieltjes transforms mapping a w_i to a w_j can be obtained which all have a same integrand as one of the fractional integral transforms sending some w_i to itself.

They all follow from each other by applying standard tranformation formulas for the hypergeometric function to the left-hand side and/or right-hand side.

Gener. Stieltjes with Bateman type integrand (cntd.)

The box in row w_i and column w_j gives the parameter shift pattern of the generalized Stieltjes transform sending w_i to w_j:

	w_1	w_2	w_3	w_4	w_5	w_6
w_1			$b+, c+$	$a+, c+$	$c+$	$a+, b+, c+$
w_2			$a-, c-$	$b-, c-$	$a-, b-, c-$	$c-$
w_3	$b-, c-$	$a+, c+$			$b-$	$a+$
w_4	$a-, c-$	$b+, c+$			$a-$	$b+$
w_5	$c-$	$a+, b+, c+$	$b+$	$a+$		
w_6	$a-, b-, c-$	$c+$	$a-$	$b-$		

The six Euler type integral representations which are generalized Stieltjes transforms can be obtained by specialization of some of the above formulas.

Generalized Stieltjes transform sending ${}_2F_1$ to ${}_3F_2$

The following formula by Karp & Sitnik (2009) will imply all our 24 generalized Stieltjes transform formulas:

$$\int_0^1 t^{b-1}(1-t)^{d+e-b-c-1}\,{}_2F_1\left(\begin{matrix} d-c, e-c \\ d+e-b-c \end{matrix}; 1-t\right)(1-tz)^{-a}\,dt$$
$$= \frac{\Gamma(b)\Gamma(c)\Gamma(d+e-b-c)}{\Gamma(d)\Gamma(e)}\,{}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; z\right)$$
$$(z \in \mathbb{C}\backslash[1,\infty),\ \mathrm{Re}\,b, \mathrm{Re}\,c, \mathrm{Re}\,(d+e-b-c) > 0).$$

For the proof expand $(1-tz)^{-a}$ and evaluate

$$\int_0^1 t^{d+e-b-c-1}(1-t)^{b+k-1}\,{}_2F_1\left(\begin{matrix} d-c, e-c \\ d+e-b-c \end{matrix}; t\right)\,dt$$

by Bateman's integral and the Gauss summation formula.

For $d = a$ the ${}_3F_2$ becomes a ${}_2F_1$.

Stieltjes tranform sending $P_n^{(\alpha,\beta)}$ to $Q_n^{(\alpha,\beta)}$

A classical formula in Szegö's book gives for $x > 1$:

$$\int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \, \frac{P_n^{(\alpha,\beta)}(t)}{x-t} \, dt = \mathrm{const.}\, (x-1)^\alpha (x+1)^\beta \, Q_n^{(\alpha,\beta)}(x).$$

For n complex the right-hand side would be by the Karp-Sitnik formula a ${}_3F_2$. It specializes to a ${}_2F_1$ for n a nonnegative integer.

Appell hypergeometric F_2

$$F_2(a;b,b';c,c';x,y) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n\, m!\, n!}\, x^m y^n \quad (|x|+|y|<1).$$

Let $\Delta := \{(x,y) \in \mathbb{R}^2 \mid x,y \geq 0,\ x+y<1\}$.
For $(x,y) \in \Delta,\ \mathrm{Re}\, b, \mathrm{Re}\, b', \mathrm{Re}\,(c-b), \mathrm{Re}\,(c'-b') > 0$ we have an Euler type double integral representation:

$$F_2(a;b,b';c,c';x,y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}$$
$$\times \int_0^1 \int_0^1 u^{b-1} v^{b'-1} (1-u)^{c-b-1} (1-v)^{c'-b'-1} (1-ux-vy)^{-a}\, du\, dv$$
$$= \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}\, x^{1-c} y^{1-c'}$$
$$\times \int_{u=0}^{x} \int_{v=0}^{y} u^{b-1} v^{b'-1} (1-u-v)^{-a} (x-u)^{c-b-1} (y-v)^{c'-b'-1}\, du\, dv.$$

Double fractional integral transformation.

Appell hypergeometric F_2 (cntd.)

$$F_2(a; b, b'; b, b'; u, v) = (1 - u - v)^{-a}.$$

The Euler type double integral representation is a special case of a Bateman type double integral:

$$\int_{u=0}^{x} \int_{v=0}^{y} u^{c-1} v^{c'-1} F_2(a; b, b'; c, c'; u, v) \times \frac{(x-u)^{\mu-1}}{\Gamma(\mu)} \frac{(y-v)^{\mu'-1}}{\Gamma(\mu')} \, du \, dv$$

$$= \frac{\Gamma(c)}{\Gamma(c+\mu)} \frac{\Gamma(c')}{\Gamma(c'+\mu')} x^{c+\mu-1} y^{c'+\mu'-1} F_2(a; b, b'; c+\mu, c'+\mu'; x, y)$$

$$(x \in \Delta,\ \mathrm{Re}\, c, \mathrm{Re}\, c', \mathrm{Re}\, \mu, \mathrm{Re}\, \mu' > 0).$$

Appell hypergeometric F_2 (cntd.)

PDE's for F_2:

$$D_{a,b,c;x,y} := x(1-x)\partial_{xx} + \big(c - (a+b+1)x\big)\partial_x - ab \\ - (x\partial_x + b)\circ y\partial_y$$

$$D_{a,b,c;x,y}\Big(F_2(a;b,b';c,c';x,y)\Big) = 0,$$

$$D_{a,b',c';y,x}\Big(F_2(a;b,b';c,c';x,y)\Big) = 0.$$

Proof by transmutation of Bateman-type double integral works.
In particular in special case of Euler-type double integral representation.
Then changing the integration region should give double integral representations of other solutions of PDE's.

Indeed, for $x, y > 0$, $x + y < 1$ and for $\mathrm{Re}\, b, \mathrm{Re}\, b', \mathrm{Re}\,(c - b), \mathrm{Re}\,(c' - b') > 0$ we saw

$$F_2(a; b, b'; c, c'; x, y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}\, x^{1-c} y^{1-c'}$$
$$\times \int_{u=0}^{x} \int_{v=0}^{y} u^{b-1} v^{b'-1} (1-u-v)^{-a} (x-u)^{c-b-1} (y-v)^{c'-b'-1}\, du\, dv,$$

and for $x, y > 1$ and $\mathrm{Re}\, b, \mathrm{Re}\, b', \mathrm{Re}\,(1 - a) > 0$ we have

$$x^{-b} y^{-b'} F_3(b, b'; 1 + b - c, 1 + b' - c'; b + b' - a + 1; x^{-1}, y^{-1})$$
$$= \frac{\Gamma(b + b' - a + 1)}{\Gamma(b)\Gamma(b')\Gamma(1-a)}\, x^{1-c} y^{1-c'} \iint_{\Delta} u^{b-1} v^{b'-1} (1-u-v)^{-a}$$
$$\times (x-u)^{c-b-1} (y-v)^{c'-b'-1}\, du\, dv, \quad \text{where}$$

$$F_3(a, a'; b, b'; c; x, y) := \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}\, m!\, n!}\, x^m y^n \quad (|x|, |y| < 1).$$

Mourad, I wish you many years to come in discussing math with many people.