Sklyanin algebra, part 2

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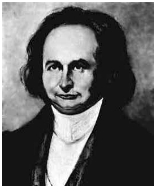
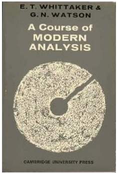
C. G. J. Jacobi (1829),
Fundamenta Nova Theoriae Functionum Ellipticarum





Jacobi Weierstrass Whittaker Watson [WW]

Jacobi theta functions (cntd.)

Let $q = e^{i\pi\tau} \quad (0 < |q| < 1, \mathrm{Im}\,\tau > 0)$.

Modified theta function (as in Gasper & Rahman):

$$\theta(w;q) := (w, q/w; q)_\infty = \frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} w^k.$$

$\theta(w^{-1};q) = -w^{-1}\theta(w;q) = \theta(qw;q)$.

Jacobi theta functions θ_a $(a = 1,2,3,4)$, or ϑ_a in [WW].

$\theta_a(z) = \theta_a(z,q) = \theta_a(z \mid \tau) = \vartheta_a(\pi z, q)$.

$$\theta_1(z) := i\, q^{1/4} (q^2;q^2)_\infty\, e^{-\pi i z}\, \theta(e^{2\pi i z}; q^2),$$

$$\theta_2(z) := q^{1/4} (q^2;q^2)_\infty\, e^{-\pi i z}\, \theta(-e^{2\pi i z}; q^2) = \theta_1(z + \tfrac{1}{2}),$$

$$\theta_3(z) := (q^2;q^2)_\infty\, \theta(-q\, e^{2\pi i z}; q^2) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2\pi i k z},$$

$$\theta_4(z) := (q^2;q^2)_\infty\, \theta(q\, e^{2\pi i z}; q^2) = \theta_3(z + \tfrac{1}{2}).$$

$\theta_1(z)$ is odd; $\theta_2(z), \theta_3(z), \theta_4(z)$ are even.

Fundamental theta identities: Weierstrass' formula

Weierstrass' fundamental theta identity is the three-term identity

$$\theta(xy, x/y, uv, u/v; q^2) - \theta(xv, x/v, uy, u/y; q^2) \\ = uy^{-1}\theta(yv, y/v, xu, x/u; q^2),$$

see Gasper & Rahman, (11.4.3). It was first obtained by Weierstrass in terms of the function $\sigma(z)$, see references to Weierstrass (1882) and Schwarz (1893) in arXiv:1401.5368, and [WW, p.451, Ex.5 and p.473, §21.43]. Some authors call it the Riemann identity, but it can't be found in Riemann's works.

For a quick proof divide the left-hand side by the right-hand side and consider the resulting expression as a meromorphic function $F(x)$ of x (the other variables generically fixed). Observe that the numerator vanishes at all (generically simple) zeros of the denominator. Thus F is entire analytic. It is also bounded (use that $F(q^2x) = F(x)$). By Liouville's theorem F is constant, which is 1 because $F(v) = 1$.

Fundamental theta identities: Jacobi's formulas

Jacobi's fundamental formulas [WW, §21.22] involve sums of products of four theta functions of the form

$$[a] := \theta_a(w)\theta_a(x)\theta_a(y)\theta_a(z), \quad [a]' := \theta_a(w')\theta_a(x')\theta_a(y')\theta_a(z'),$$

where

$$\begin{aligned} 2w' &= -w+x+y+z, \qquad & 2x' &= w-x+y+z,\\ 2y' &= w+x-y+z, \qquad & 2z' &= w+x+y-z. \end{aligned}$$

Then (the first one implies the others):

$$\begin{aligned} 2\,[1] &= [1]'+[2]'-[3]'+[4]', \quad & 2\,[2] &= [1]'+[2]'+[3]'-[4]',\\ 2\,[3] &= -[1]'+[2]'+[3]'+[4]', \quad & 2\,[4] &= [1]'-[2]'+[3]'+[4]'. \end{aligned}$$

These are easily seen to be equivalent with:

$$\begin{aligned} &[1]+[2]=[1]'+[2]', \quad [1]+[3]=[2]'+[4]', \quad [1]+[4]=[1]'+[4]',\\ &[1]-[2]=[4]'-[3]', \quad [1]-[3]=[1]'-[3]', \quad [1]-[4]=[2]'-[3]'. \end{aligned}$$

Fundamental theta identities: their equivalence

$$\begin{aligned}W(x,y,u,v;q)&:=\theta(xy,x/y,uv,u/v;q^2)-\theta(xv,x/v,uy,u/y;q^2)\\&\qquad-uy^{-1}\theta(yv,y/v,xu,x/u;q^2),\end{aligned}$$

$$\begin{aligned}&J(x,y,u,v;q):=2\theta(xy,x/y,uv,u/v;q^2)-\theta(xv,x/v,uy,u/y;q^2)\\&-\theta(-xv,-x/v,-uy,-u/y;q^2)-q^{-1}xu\,\theta(qxv,qx/v,quy,qu/y;q^2)\\&\qquad+q^{-1}xu\,\theta(-qxv,-qx/v,-quy,-qu/y;q^2).\end{aligned}$$

Then

$$\begin{aligned}&W(x,y,u,v;q)+W(-x,y,-u,v;q)-xyW(qx,qy,u,v;q)\\&\qquad-xyW(-qx,qy,-u,v;q)=J(x,y,u,v;q),\\&J(x,y,u,v;q)-uy^{-1}J(x,u,y,v;q)=2W(x,y,u,v;q).\end{aligned}$$

Hence the two identities $W=0$ and $J=0$ are equivalent.

See also K, `arXiv:1401.5368`.

Relations between squares of theta functions

$[1]-[4]=[2]'-[3]'$. Put $(x,y,u,v):=(y,y,z,z)$.
Then $(x',y',u',v')=(y,y,z,z)$. Hence

$$\begin{aligned}
&\theta_1^2(y)\theta_1^2(z)-\theta_2^2(y)\theta_2^2(z)+\theta_3^2(y)\theta_3^2(z)-\theta_4^2(y)\theta_4^2(z)=0,\\
&\theta_3^2(y)\theta_1^2(z)+\theta_4^2(y)\theta_2^2(z)-\theta_1^2(y)\theta_3^2(z)-\theta_2^2(y)\theta_4^2(z)=0,\\
&\big(\theta_1^4(y)+\theta_3^4(y)\big)\theta_1^2(z)+\big(\theta_3^2(y)\theta_4^2(y)-\theta_1^2(y)\theta_2^2(y)\big)\theta_2^2(z)\\
&\qquad-\big(\theta_1^2(y)\theta_4^2(y)+\theta_2^2(y)\theta_3^2(y)\big)\theta_4^2(z)=0.
\end{aligned}$$

By the first equation the functions $\theta_1^2,\theta_2^2,\theta_3^2,\theta_4^2$ span a linear space of dimension at most 2, hence equal to 2. In fact,

$$\begin{aligned}
\theta_1^2(\tfrac12)\theta_1^2(z)&=-\theta_3^2(\tfrac12)\theta_3^2(z)+\theta_4^2(\tfrac12)\theta_4^2(z),\\
\theta_2^2(0)\theta_2^2(z)&=\quad\theta_3^2(0)\theta_3^2(z)-\theta_4^2(0)\theta_4^2(z).
\end{aligned}$$

Some theta addition formulas

$$\theta(xy, x/y, uv, u/v; q^2) - \theta(xv, x/v, uy, u/y; q^2) = uy^{-1}\theta(yv, y/v, xu, x/u; q^2),$$

By the substitution $(x, u, v, y) \to (q^{\frac{1}{2}}y, q^{-\frac{1}{2}}z, q^{\frac{1}{2}}, -q^{\frac{1}{2}})$ we get

$$\theta(yz, qy/z, -1, -q; q^2) = \theta(y, qy, -z, -qz; q^2) + \theta(-y, -qy, z, qz; q^2).$$

Hence

$$\theta_1(y+z)\theta_4(y-z)\theta_2(0)\theta_3(0) = \theta_1(y)\theta_4(y)\theta_2(z)\theta_3(z) + \theta_2(y)\theta_3(y)\theta_1(z)\theta_4(z),$$

$$\theta_2(y+z)\theta_3(y-z)\theta_2(0)\theta_3(0) = \theta_2(y)\theta_3(y)\theta_2(z)\theta_3(z) - \theta_1(y)\theta_4(y)\theta_1(z)\theta_4(z).$$

Hence $\theta_2(z)\theta_3(z)\big(\theta_1(y+z)\theta_4(y-z) - \theta_1(y-z)\theta_4(y+z)\big) - \theta_1(z)\theta_4(z)\big(\theta_2(y+z)\theta_3(y-z) + \theta_2(y-z)\theta_3(y+z)\big) = 0.$

Variety associated with a set of relations

I follow the approach in S. P. Smith & J. T. Stafford, *Regularity in the four dimensional Sklyanin algebra*, Compositio Math. 83 (1992), 259–289, Section 2.

Let $X_0, \ldots, X_n$ be noncommuting variables.
Associate with a word $X_{i_1} \ldots X_{i_m}$ a monomial $x_{i_1,1} \ldots x_{i_m,m}$ in the commuting variables $x_{0,1}, \ldots, x_{n,1}, \ldots, x_{0,m}, \ldots, x_{n,m}$.

Associate with a set of homogeneous relations of degree m

$$\sum_{i_1,\ldots,i_m} c^{(j)}_{i_1,\ldots,i_m} X_{i_1} \ldots X_{i_m} = 0 \qquad (j = 1, \ldots, r).$$

a subset Γ of $(\mathbb{P}^n(\mathbb{C}))^m$ defined by the equations

$$\sum_{i_1,\ldots,i_m} c^{(j)}_{i_1,\ldots,i_m} x_{i_1,1} \ldots x_{i_m,m} = 0 \qquad (j = 1, \ldots, r).$$

Sklyanin algebra

α, β, γ means cyclic permutation of $1, 2, 3$.

Definition

Let J_{12}, J_{23}, J_{31} be complex constants, not equal to 0 or ± 1, such that

$$J_{12} + J_{23} + J_{31} + J_{12}\, J_{23}\, J_{31} = 0.$$

The Sklyanin algebra is the algebra $\mathcal{S}$ generated by S_0, S_1, S_2, S_3 with the six relations

$$\begin{aligned} S_0 S_\alpha - S_\alpha S_0 - i\, J_{\beta\gamma}(S_\beta S_\gamma + S_\gamma S_\beta) &= 0, \\ S_0 S_\alpha + S_\alpha S_0 + i(S_\beta S_\gamma - S_\gamma S_\beta) &= 0. \end{aligned}$$

The associated subset Γ of $\mathbb{P}^3 \times \mathbb{P}^3 := \mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^3(\mathbb{C})$ is defined by the six equations

$$\begin{aligned} x_0 y_\alpha - x_\alpha y_0 - i\, J_{\beta\gamma}(x_\beta y_\gamma + x_\gamma y_\beta) &= 0, \\ x_0 y_\alpha + x_\alpha y_0 + i(x_\beta y_\gamma - x_\gamma y_\beta) &= 0. \end{aligned}$$

Elliptic curve associated with Sklyanin algebra

Let π_1, π_2 be the projections of $\mathbb{P}^3 \times \mathbb{P}^3$ on the first respectively second factor of the direct product. Put $\Gamma_i := \pi_i(\Gamma) \subset \mathbb{P}^3$.

Theorem

1. $\pi_1 : \Gamma \to \Gamma_1$ *and* $\pi_2 : \Gamma \to \Gamma_2$ *are bijective maps.*
2. $\Gamma_1 = E \cup \{(1,0,0,0)\} \cup \{(0,1,0,0)\} \cup \{(0,0,1,0)\} \cup \{(0,0,0,1)\}$, *where*

$$
\begin{aligned}
E &= \{x \in \mathbb{P}^3 \mid g_1 = 0,\ g_2 = 0\},\\
g_1 &= -x_0^2 + x_1^2 + x_2^2 + x_3^2,\\
g_2 &= (1 + J_{12})x_1^2 + (1 + J_{12}J_{23})x_2^2 + (1 - J_{23})x_3^2.
\end{aligned}
$$

3. $\Gamma_1 = \Gamma_2$ *and thus* $\Gamma_1 \to \Gamma \to \Gamma_2$ *can be considered as a bijective map* $\sigma : \Gamma_1 \to \Gamma_2$. *It fixes the four points and leaves* E *invariant.*
4. E *is a smooth elliptic curve.*

Elliptic curve associated with Sklyanin algebra

Part of the proof of the above Theorem involves writing the six equations

$$x_0 y_\alpha - x_\alpha y_0 - i\, J_{\beta\gamma}(x_\beta y_\gamma + x_\gamma y_\beta) = 0,$$
$$x_0 y_\alpha + x_\alpha y_0 + i(x_\beta y_\gamma - x_\gamma y_\beta) = 0.$$

as $Ay = 0$, where A is a 6×4 matrix with entries wich are homogeneous of degree 1 in x_0, x_1, x_2, x_3. Then compute all 4×4 minors of A and observe that they are all equal to polynomials which are in the ideal generated by g_1 and g_2.

Of course, the computation can be done in Mathematica or Maple.

Parametrizing the elliptic curve

Fix $\eta \in \mathbb{C}$ such that η is not of order 4 in $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Write the structure constants as

$$J_{12} = \frac{\theta_4^2(\eta)\theta_1^2(\eta)}{\theta_2^2(\eta)\theta_3^2(\eta)}, \quad J_{23} = \frac{\theta_2^2(\eta)\theta_1^2(\eta)}{\theta_3^2(\eta)\theta_4^2(\eta)}, \quad J_{31} = -\frac{\theta_3^2(\eta)\theta_1^2(\eta)}{\theta_4^2(\eta)\theta_2^2(\eta)}.$$

Theorem

The map $z \mapsto (x_0, x_1, x_2, x_3)$ given by

$$x_0 = \theta_1(\eta)\theta_3(2z), \qquad x_1 = -i\,\theta_2(\eta)\theta_4(2z),$$
$$x_2 = \theta_3(\eta)\theta_1(2z), \qquad x_3 = \theta_4(\eta)\theta_2(2z),$$

sends $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ bijectively to $E \subset \mathbb{P}^3$.

Part of the proof is to verify: $g_1 = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ and

$$\theta_3^2(\eta)\,g_2 = \big(\theta_1^2(\eta)\theta_4^2(\eta)+\theta_2^2(\eta)\theta_3^2(\eta)\big)\frac{x_1^2}{\theta_2^2(\eta)}+\big(\theta_1^4(\eta)+\theta_3^4(\eta)\big)\frac{x_2^2}{\theta_3^2(\eta)}$$
$$+\big(\theta_3^2(\eta\theta_4^2(\eta)-\theta_1^2(\eta)\theta_2^2(\eta)\big)\frac{x_3^2}{\theta_4^2(\eta)} = 0 \quad \text{(use (1) and (2))}.$$

Parametrizing the elliptic curve

Theorem

The map $\sigma : E \to E$ is given by $\sigma(x(z)) := x(z+\eta)$.

Part of the proof consists of checking that the six equations

$$x_0 y_\alpha - x_\alpha y_0 - i\,J_{\beta\gamma}(x_\beta y_\gamma + x_\gamma y_\beta) = 0,$$
$$x_0 y_\alpha + x_\alpha y_0 + i(x_\beta y_\gamma - x_\gamma y_\beta) = 0.$$

hold for

$$\begin{aligned} x_0 &= \theta_1(\eta)\theta_3(2z), & x_1 &= -i\,\theta_2(\eta)\theta_4(2z),\\ x_2 &= \theta_3(\eta)\theta_1(2z), & x_3 &= \theta_4(\eta)\theta_2(2z),\\ y_0 &= \theta_1(\eta)\theta_3(2z+2\eta), & y_1 &= -i\,\theta_2(\eta)\theta_4(2z+2\eta),\\ y_2 &= \theta_3(\eta)\theta_1(2z+2\eta), & y_3 &= \theta_4(\eta)\theta_2(2z+2\eta), \end{aligned}$$

with

$$J_{12} = \frac{\theta_4^2(\eta)\theta_1^2(\eta)}{\theta_2^2(\eta)\theta_3^2(\eta)}, \quad J_{23} = \frac{\theta_2^2(\eta)\theta_1^2(\eta)}{\theta_3^2(\eta)\theta_4^2(\eta)}, \quad J_{31} = -\frac{\theta_3^2(\eta)\theta_1^2(\eta)}{\theta_4^2(\eta)\theta_2^2(\eta)}.$$

Parametrizing the elliptic curve

For instance,

$$x_0 y_3 + x_3 y_0 + i(x_1 y_2 - x_1 y_2) = 0$$

turns down to

$$\theta_2((\eta)\theta_3((\eta)\big(\theta_1(2z+2\eta)\theta_4(2z) - \theta_1(2z)\theta_4(2z+2\eta)\big) \\ - \theta_1((\eta)\theta_4((\eta)\big(\theta_2(2z+2\eta)\theta_3(2z) + \theta_2(2z)\theta_3(2z+2\eta)\big) = 0,$$

which is addition formula (3).

Connection with representations of Sklyanin algebra

The result in the above proof is equivalent to stating that $(S_i f)(z) := x_i(z) f(z+\eta)$ with the x_i as above gives a representation of $\mathcal{S}$ on the space of meromorphic functions. Indeed,

$$(S_i(S_j f))(z) = x_i(z) x_j(z+\eta) f(z+2\eta).$$

For a representation we need that for each relation

$$\sum_{i,j=0}^{3} c_{ij} S_i S_j = 0$$

we have

$$\sum_{i,j=0}^{3} c_{ij} x_i(z) x_j(z+\eta) f(z+2\eta) = 0.$$

In fact, in my previous notes (part 1), I already sketched the proof that we then have a representation. There, in formula (4), omit the term with $f(z-\eta)$ and take $\ell = 0$. We then still have a representation on the space of meromorphic functions.