Solutions of the system of pde's for Appell hypergeometric F_2, a tribute to Per O.M. Olsson

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Results presented here were
inspired by work in progress
by Enno Diekema,
and include some of his results.

Enno Diekema

Hypergeometric series

Pochhammer symbol:

$$(a)_n := a(a+1)(a+2)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (n=1,2,\ldots),$$

$$(a)_0 := 1.$$

A series $\displaystyle\sum_{n=0}^{\infty} t_n$ is called *hypergeometric* if $\displaystyle\frac{t_{n+1}}{t_n}$ is a rational function of n.

Example: $\displaystyle t_n = \frac{(a)_n(b)_n}{(c)_n\, n!}\, z^n, \quad \frac{t_{n+1}}{t_n} = \frac{(a+n)(b+n)}{(c+n)(n+1)}\, z.$

Gauss hypergeometric series:

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!}\, z^n \quad (|z|<1,\ c \neq 0,-1,-2,\ldots)$$

Standard formulas for ${}_2F_1(a, b; c; z)$

differential equation (Gauss):

$$\left(z(1-z)\frac{d^2}{dz^2} + (c-(a+b+1)z)\frac{d}{dz} - ab\right){}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = 0.$$

integral reprerepresentation (Euler):

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$$

$$(z \in \mathbb{C}\backslash[1,\infty),\ \operatorname{Re} c > \operatorname{Re} b > 0).$$

transformation formulas:

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) &= (1-z)^{-a}\,{}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{z}{z-1}\right) \\ &= (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right). \end{aligned}$$

evaluation formula:

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (\operatorname{Re}(c-a-b) > 0).$$

${}_2F_1(a, b; c; z)$ and spectral theory

There are many cases where Gauss hypergeometric functions occur as eigenfunctions for a spectral problem. Two examples:

1. Jacobi polynomials:

$$\Big(x(1-x)\frac{d^2}{dx^2} + \big(\alpha+1-(\alpha+\beta+2)x\big)\frac{d}{dx}\Big)p_n(x) =$$

$$-n(n+\alpha+\beta+1)p_n(x) \quad (x\in[0,1];\ n=0,1,2,\ldots;\ \alpha,\beta>-1),$$

$$p_n(x) = P_n^{(\alpha,\beta)}(1-2x) = \text{const.}\ {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};x\right).$$

2. Jacobi functions:

$$\Big(x(1-x)\frac{d^2}{dx^2} + \big(\alpha+1-(\alpha+\beta+2)x\big)\frac{d}{dx}\Big)\phi_\lambda(x) =$$

$$\begin{aligned}\tfrac14\big(\lambda^2+(\alpha+\beta+1)^2\big)\phi_\lambda(x) \quad &(x\in(-\infty,0];\ \lambda\in[0,\infty);\\ &\alpha>-1,\ \beta\in\mathbb{R},\ |\beta|\le\alpha+1),\end{aligned}$$

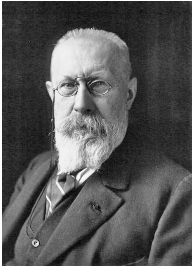
$$\phi_\lambda(x) = {}_2F_1\left(\begin{matrix}\frac12(\rho+i\lambda), \frac12(\rho-i\lambda)\\ \alpha+1\end{matrix};x\right), \qquad \rho=\alpha+\beta+1.$$

Bivariate hypergeometric series

A bivariate series $\sum_{m,n=0}^{\infty} t_{m,n}$ is called *hypergeometric* if $t_{m+1,n}/t_{m,n}$ and $t_{m,n+1}/t_{m,n}$ are rational functions of m, n.

Write these two rational functions each as a quotient of two polynomials in m, n without common factors. The maximum of the degrees of these four polynomials is called the *order* of the hypergeometric series.

Appell (1880, book 1926) introduced the series F_1, F_2, F_3, F_4 (of order two). Horn (1889, 1931) extended this to a classification of all series of order two (essentially 34 distinct convergent series).



Paul Appell

Appell hypergeometric series F_2

$$F_2(a; b_1, b_2; c_1, c_2; x, y) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n\, m!\, n!}\, x^m y^n,$$

convergent for $|x| + |y| < 1$.

Of order two: the summand $t_{m,n}$ satisfies

$$\frac{t_{m+1,n}}{t_{m,n}} = \frac{(a+m+n)(b_1+m)}{(c_1+m)(1+m)}\, x, \qquad \frac{t_{m,n+1}}{t_{m,n}} = \frac{(a+m+n)(b_2+n)}{(c_2+n)(1+n)}\, y.$$

$f = F_2(x, y)$ is the unique solution regular and equal to 1 at $(x, y) = (0, 0)$ of the system of pde's

$$x(1-x)f_{xx} - xyf_{xy} + \big(c_1 - (a+b_1+1)x\big)f_x - b_1 y f_y - ab_1 f = 0,$$
$$y(1-y)f_{yy} - xyf_{xy} + \big(c_2 - (a+b_2+1)y\big)f_y - b_2 x f_x - ab_2 f = 0.$$

Symmetry: $x \leftrightarrow y$, $b_1 \leftrightarrow b_2$, $c_1 \leftrightarrow c_2$.

Two eigenvalue equations, but pdo's do not commute.

Appell F_2 and spectral theory

Here connection with spectral theory is less obvious. However, put $a=\alpha+\beta+\gamma+\frac{1}{2}+m+n$, $b_1=-m$, $b_2=-n$, $c_1=\alpha+\frac{1}{2}$, $c_2=\beta+\frac{1}{2}$. Then a solution of the system

$$\Big(x(1-x)\partial_{xx}-xy\partial_{xy}+\big(\alpha+\tfrac{1}{2}-(\alpha+\beta+\gamma+n+\tfrac{3}{2})x\big)\partial_x \\ +my\partial_y+m(m+n+\alpha+\beta+\gamma+\tfrac{1}{2})\Big)f=0,$$

$$\Big(y(1-y)\partial_{yy}-xy\partial_{xy}+\big(\beta+\tfrac{1}{2}-(\alpha+\beta+\gamma+m+\tfrac{3}{2})y\big)\partial_y \\ +nx\partial_x+n(m+n+\alpha+\beta+\gamma+\tfrac{1}{2})\Big)f=0$$

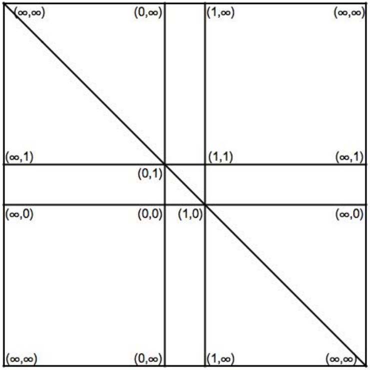
is $f=F_2(\alpha+\beta+\gamma+\frac{1}{2}+m+n;-m,-n;\alpha+\frac{1}{2},\beta+\frac{1}{2};x,y)$, while the solution space of the sum of the two pde's,

$$\Big(x(1-x)\partial_{xx}-2xy\partial_{xy}+y(1-y)\partial_{yy}+\big(\alpha+\tfrac{1}{2}-(\alpha+\beta+\gamma+\tfrac{3}{2})x\big)\partial_x \\ +\big(\beta+\tfrac{1}{2}-(\alpha+\beta+\gamma+\tfrac{3}{2})y\big)\partial_y+(m+n)(m+n+\alpha+\beta+\gamma+\tfrac{1}{2})\Big)f=0,$$

are all polynomials of degree $m+n$ which are orthogonal to all polynomials of lower degree with respect to the weight function $x^\alpha y^\beta(1-x-y)^\gamma$ on the triangle $x,y>0$, $x+y<1$.

Solving the F_2 system

In a neighbourhood of (x, y) the F_2-system of pde's has four linearly independent solutions except for points on the lines $x = 0$, $x = 1$, $y = 0$, $y = 1$, $x + y = 1$, the *fundamental loci*. Moreover, the lines $x = \infty$ and $y = \infty$ belong to the fundamental loci, as seen after the transformation $x \to x^{-1}$, $y \to y^{-1}$.



Compare with Gauss hypergeometric diff. equation. Two linearly independent solutions, except at singular points $0, 1, \infty$. S_3 acts on them by conformal maps.

Here: singular points $(0, 0), (1, 1), (\infty, \infty)$; $(1, 0), (\infty, 0), (\infty, 1)$; and by $x \leftrightarrow y$: $(0, 1), (0, \infty), (1, \infty)$.

More solutions generated by a 5-parameter solution

$F(a;b_1,b_2;c_1,c_2;x,y)$ means a solution of the F_2 system. Then the following transformed functions satisfy the same system:

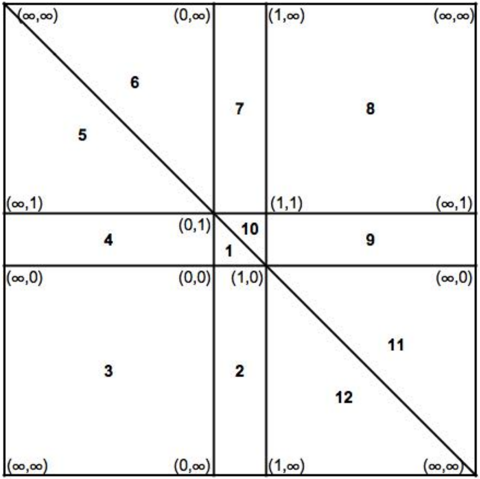
$$
\begin{aligned}
&(1)\quad F(a;b_1,b_2;c_1,c_2;x,y),\\
&(2)\quad F(a;b_2,b_1;c_2,c_1;y,x),\\
&(3)\quad x^{1-c_1}F(a-c_1+1;b_1-c_1+1,b_2;2-c_1,c_2;x,y),\\
&(4)\quad y^{1-c_2}F(a-c_2+1;b_1,b_2-c_2+1;c_1,2-c_2;x,y),\\
&(5)\quad x^{1-c_1}y^{1-c_2}F(a-c_1-c_2+2;b_1-c_1+1,b_2-c_2+1;2-c_1,2-c_2;x,y),\\
&(6)\quad (1-x)^{-a}F\left(a;c_1-b_1,b_2;c_1,c_2;\frac{x}{x-1},\frac{y}{1-x}\right),\\
&(7)\quad (1-y)^{-a}F\left(a;b_1,c_2-b_2;c_1,c_2;\frac{x}{1-y},\frac{y}{y-1}\right),\\
&(8)\quad (1-x-y)^{-a}F\left(a;c_1-b_1,c_2-b_2;c_1,c_2;\frac{x}{x+y-1},\frac{y}{x+y-1}\right).
\end{aligned}
$$

Altogether 32 expressions. For $F=F_2$: (1), (3), (4), (5) are linearly independent. But $(1)=(2)=(6)=(7)=(8)$.

A special birational map

$$(1-y)^{-a}F\left(a; b_1, c_2-b_2; c_1, c_2; \frac{x}{1-y}, \frac{y}{y-1}\right)$$

The *birational map* $(x,y) \to \left(\frac{x}{1-y}, \frac{y}{y-1}\right)$ maps critical lines and regions bounded by them as follows:



$$1 \longleftrightarrow 2$$
$$3 \longleftrightarrow 4$$
$$5 \longleftrightarrow 8$$
$$6 \longleftrightarrow 7$$
$$9 \longleftrightarrow 11$$
$$10 \longleftrightarrow 12$$

$$x=0 \longleftrightarrow x=0$$
$$y=0 \longleftrightarrow y=0$$
$$x=1 \longleftrightarrow x+y=1$$
$$y=1 \longleftrightarrow (\infty,\infty)$$
$$x=\infty \longleftrightarrow x=\infty$$
$$y=\infty \longleftrightarrow (0,1)$$

Per Olsson

Solutions of the F_2 system were studied, among others, by Borngässer (dissertation TH Darmstadt, 1933; Horn advisor); Erdélyi (1948, Proc. Roy. Soc. Edinburgh; 1950, Acta Math.),

and by Per O.M. Olsson. He worked at Dept. of Theoretical Physics of Stockholm University until 1963, next at Dept. of Theoretical Physics of KTH. He published several papers on Appell hypergeometric functions, starting 1963 and ending 1977, in the journals Arkiv för Fysik and J. Math. Phys., including coauthored papers with Nagel & Weissglas, and with Almström. His last published paper:

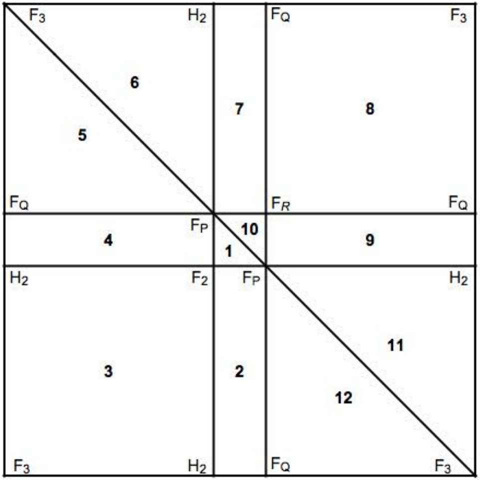
On the integration of the differential equations of five-parametric double-hypergeometric functions of second order, J. Math. Phys. 18 (1977), 1285–1294

is very important, but got few citations.

Who knows more about Per Olsson?

Solutions associated with a critical point

Solutions of the F_2 system given by Olsson, J. Math. Phys. (1977). There occur six essentially different functions associated with the various critical points.



F_2 at $(0,0)$ (4 sol's)

F_3 at (∞,∞) (1 sol.)

H_2 at $(\infty,0)$, $(0,\infty)$
(2 sol's at each pnt.)

F_P at $(1,0)$, $(0,1)$
(1 sol. at each pnt.)

F_Q at $(\infty,1)$, $(1,\infty)$
(1 sol. at each pnt.)

F_R at $(1,1)$ (2 sol's)

Solutions F_3 and H_2

Solution at (∞, ∞):

$$x^{-b_1} y^{-b_2} F_3(b_1, b_2; b_1 - c_1 + 1, b_2 - c_2 + 1; b_1 + b_2 - a + 1; x^{-1}, y^{-1}),$$

where $F_3(a_1, a_2; b_1, b_2; c; x, y) := \displaystyle\sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(c)_{m+n}\, m!\, n!} x^m y^n$

(Appell), of order 2, convergent if $|x|, |y| < 1$.

Solution at $(0, \infty)$: $\quad y^{-b_2} H_2(a - b_2, b_1, b_2, b_2 - c_1 + 1; c_1; x, -y^{-1}),$

where $H_2(a, b, c, d; e; x, y) := \displaystyle\sum_{m,n=0}^{\infty} \frac{(a)_{m-n} (b)_m (c)_n (d)_n}{(e)_m\, m!\, n!} x^m y^n$

(Horn), of order 2, convergent if $|x| < 1$, $|x| < |y|^{-1} - 1$.

Here $\quad (a)_k = \dfrac{\Gamma(a+k)}{\Gamma(a)} = \dfrac{1}{(a+k)_{-k}} = \dfrac{(-1)^k}{(-a+1)_{-k}} \quad$ if $k < 0$.

solutions F_P, F_Q

Solution at $(0,1)$: $\quad F_P(a;b_1,b_2;c_1,c_2;x,y)$

$$:= \sum_{m.n=0}^{\infty} \frac{(a)_{m+n}(a-c_2+1)_m(b_1)_m(b_2)_n}{(a+b_2-c_2+1)_{m+n}(c_1)_m\, m!\, n!}\, x^m(1-y)^n$$

(Diekema, original definition by Olsson more complicated), of order 3, convergent if $|x|<1$, $|y-1|<1$.

Solution at $(\infty,1)$: $F_Q(a;b_1,b_2;c_1,c_2;x,y) := x^{-b_1}y^{b_1-a}$

$$\times \sum_{m,n=0}^{\infty} \frac{(b_1+c_2-b_2-a)_{m-n}(b_1)_m(b_1-c_1+1)_m}{(b_1-a+1)_{m-n}(b_1+c_2-a)_{m-n}\, m!\, n!} \left(\frac{y}{x}\right)^m \left(\frac{1-y}{y}\right)^n$$

(Olsson), of order 3, convergent if $|x|>|y|+|y-1|>2|y-1|$.

Solution at $(1,1)$:

$$F_R(a;b_1,b_2;c_1,c_2;x,y) := \text{const.}\, x^{1-c_1} y^{1-c_2}$$
$$\times \sum_{m,n=0}^{\infty} \frac{(a-c_1+1)_m (a-c_2+1)_n (b_1-c_1+1)_m (b_2-c_2+1)_n}{(a+b_1-c_1+1)_m (a+b_2-c_2+1)_n\, m!\, n!}$$
$$\times\, {}_3F_2\left(\begin{matrix} a, b_1, b_2 \\ a+b_1-c_1+m+1, a+b_2-c_2+n+1 \end{matrix}; 1\right)$$
$$\times (1-x)^m (1-y)^n$$

(Olsson), not of hypergeometric type, convergent if $|x-1|, |y-1| < 1$ and $\mathrm{Re}(a-c_1-c_2+2) > 0$.

More solutions generated by a 5-parameter solution

Each of these solutions $F(a; b_1, b_2; c_1, c_2; x, y)$ gives more solutions

(2) $F(a; b_2, b_1; c_2, c_1; y, x)$,

(3) $x^{1-c_1} F(a - c_1 + 1; b_1 - c_1 + 1, b_2; 2 - c_1, c_2; x, y)$,

(4) $y^{1-c_2} F(a - c_2 + 1; b_1, b_2 - c_2 + 1; c_1, 2 - c_2; x, y)$,

(5) $x^{1-c_1} y^{1-c_2} F(a - c_1 - c_2 + 2; b_1 - c_1 + 1, b_2 - c_2 + 1; 2 - c_1, 2 - c_2; x, y)$,

(6) $(1 - x)^{-a} F\left(a; c_1 - b_1, b_2; c_1, c_2; \frac{x}{x - 1}, \frac{y}{1 - x}\right)$,

(7) $(1 - y)^{-a} F\left(a; b_1, c_2 - b_2; c_1, c_2; \frac{x}{1 - y}, \frac{y}{y - 1}\right)$,

(8) $(1 - x - y)^{-a} F\left(a; c_1 - b_1, c_2 b_2; c_1, c_2; \frac{x}{x + y - 1}, \frac{y}{x + y - 1}\right)$,

altogether 32 expressions. Olsson (1977) lists the equal ones (transformation formulas). Other solutions thus obtained may be associated to the same or another critical point, or only living at part of the neighbourhood of a critical point.

Olsson (1977), conclusion

Olsson gives also 3-term identitities involving three linearly independent solutions. Sufficiently many of them should give the monodromy of the system.

He also gives further expressions for the solutions as sums or integrals.

Olsson's classification of solutions seems complete and correct. However, no complete proofs are given, and not all reasoning is mathematically rigorous. But rewriting the proof of his classification in a rigorous way would not be difficult.

Less obvious would be to give all 3-term relations between solutions.

proof of Euler integral representation by transmutation

hypergeometric differential equation: $D_{a,b,c;z}\left({}_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix};z\right)\right)=0$,

$D_{a,b,c;z}(f(z)):=z(1-z)f''(z)+(c-(a+b+1)z)f'(z)-ab\,f(z)$.

transmutation formula:

$$D_{a,b,c+\mu;x}\left(x^{1-c-\mu}\int_0^x y^{c-1}f(y)\,(x-y)^{\mu-1}\,dy\right)$$
$$=x^{1-c-\mu}\int_0^x y^{c-1}D_{a,b,c;y}(f(y))\,(x-y)^{\mu-1}\,dy.$$

Bateman's integral:

$${}_2F_1\left(\begin{matrix}a,b\\c+\mu\end{matrix};x\right)=\frac{\Gamma(c+\mu)\,x^{1-c-\mu}}{\Gamma(c)\Gamma(\mu)}\int_0^x y^{c-1}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};y\right)\,(x-y)^{\mu-1}\,dy.$$

$c:=b$: **variant of Euler's integral representation:**

$${}_2F_1\left(\begin{matrix}a,b\\b+\mu\end{matrix};x\right)=\frac{\Gamma(b+\mu)\,x^{1-b-\mu}}{\Gamma(b)\Gamma(\mu)}\int_0^x y^{b-1}(1-y)^{-a}\,(x-y)^{\mu-1}\,dy.$$

Extending the transmutation method

The above proof is in

T. H. Koornwinder,
Fractional integral and generalized Stieltjes transforms for hypergeometric functions as transmutation operators, SIGMA 11 (2015), 074, 22 pp.; arXiv:1504.08144v3.

The same method can be used to prove similar formulas involving integration over other regions. The crucial observation is that the proof of the transmutation formula involves an integration by parts, where the boundary terms should vanish.

The same method can be used in the two-variable case.

Integral representations for F_2, F_3

These are already given by Appell, but they can also proved by the transmutation method.

Integral representation for F_2:

$$F_2(a;b_1,b_2;c_1,c_2;x,y) = \mathrm{const.}\, x^{1-c_1} y^{1-c_2} \int_{u=0}^{x} \int_{v=0}^{y} u^{b_1-1} v^{b_2-1} \times (1-u-v)^{-a} (x-u)^{c_1-b_1-1} (y-v)^{c_2-b_2-1}\, du\, dv,$$

$x,y>0$, $x+y<1$; $\mathrm{Re}\, b_1, \mathrm{Re}\, b_2, \mathrm{Re}(c_1-b_1), \mathrm{Re}(c_2-b_2) > 0$.

Integral representation for F_3:

$$x^{-b_1} y^{-b_2} F_3(b_1,b_2;1+b_1-c_1,1+b_2-c_2;b_1+b_2-a+1;x^{-1},y^{-1}) = \mathrm{const.}\, x^{1-c_1} y^{1-c_2} \iint_\Delta u^{b_1-1} v^{b_2-1} (1-u-v)^{-a} \times (x-u)^{c_1-b_1-1} (y-v)^{c_2-b_2-1}\, du\, dv,$$

$x,y>1$; $\mathrm{Re}\, b_1, \mathrm{Re}\, b_2, \mathrm{Re}(1-a) > 0$;
$\Delta := \{(u,v)\in\mathbb{R}^2 \mid u,v\ge 0,\ u+v<1\}$.

Integral representation for H_2

$$y^{-b_2} H_2(a-b_2, b_1, b_2-c_2+1, b_2; c; x, -y^{-1}) = \text{const.}\, x^{1-c_1} y^{1-c_2}$$
$$\times \int_{u=0}^{x} \int_{v=0}^{1-u} u^{b_1-1} v^{b_2-1} (1-u-v)^{-a} (x-u)^{c_1-b_1-1} (y-v)^{c_2-b_2-1}\, dv\, du$$
$$(0 < x < 1,\ y > 1; \quad \mathrm{Re}(1-a), \mathrm{Re}\, b_2, \mathrm{Re}\, b_1, \mathrm{Re}(c_1-b_1) > 0).$$

This is a rewritten form of an integral representation for H_2 which was first obtained by Tua & Kalla (1987) and recently independently by Diekema.

Integral representation for F_P

$$
\begin{aligned}
&|x|^{1-c_1} y^{1-c_2} F_P(a-c_1-c_2+2, b_1-c_1+1, b_2-c_2+1, 2-c_1, 2-c_2; x, y)\\
&= \text{const.}\, |x|^{1-c_1} y^{1-c_2} \int_{u=1}^{\infty} \int_{v=1-u}^{0} u^{b_1-1} (-v)^{b_2-1} (u+v-1)^{-a}\\
&\qquad \times (u-x)^{c_1-b_1-1} (y-v)^{c_2-b_2-1}\, dv\, du\\
&\big(x \in (-\infty, 0) \cup (0, 1), y > 0; \quad \operatorname{Re} a < 1,\ \operatorname{Re} b_2 > 0,\\
&\operatorname{Re}(a-c_1-c_2+2), \operatorname{Re}(a-c_1+1), \operatorname{Re} b_2, \operatorname{Re}(1-a) > 0\big),
\end{aligned}
$$

This is a rewritten form of an integral representation for F_P first given by Diekema. It was earlier incorrectly considered as an integral representation for H_2 by Yoshida (1980) and Kita (1992).