Zhedanov’s Askey-Wilson algebra, Cherednik’s double affine Hecke algebras, and bispectrality.

Lecture 1: The Askey and $q$-Askey scheme

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Plan of the course

1. The Askey and $q$-Askey scheme
2. Zhedanov’s algebra
3. Double affine Hecke algebra in the rank one case
Definition

Let \( \{p_n(x)\}_{n=0,1,\ldots} \) be a system of real-valued polynomials \( p_n(x) \) of degree \( n \) in \( x \). Let \( \mu \) be a positive Borel measure on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty \) for all \( n \). Then \( \{p_n(x)\} \) is called a system of \textit{orthogonal polynomials} (OP’s) if

\[
\int_{\mathbb{R}} p_n(x) x^k \, d\mu(x) = 0 \quad (k = 0, 1, \ldots, n-1). \tag{1}
\]

Theorem

Any system of orthogonal polynomials (with \( p_{-1}(x) := 0, \ p_0(x) := 1 \)) satisfies a recurrence relation of the form

\[
x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x). \tag{2}
\]

Conversely, if \( \{p_n(x)\} \) satisfies (2) with \( C_n A_{n-1} > 0 \) then there exists a positive Borel measure \( \mu \) on \( \mathbb{R} \) such that (1) holds.
General orthogonal polynomials (continued)

Notation

- Write $p_n(x) = k_n x^n + \cdots$.
- Write $h_n := \int_{\mathbb{R}} p_n(x)^2 \, d\mu(x)$. Then

  $$\int_{\mathbb{R}} p_n(x) p_m(x) \, d\mu(x) = h_n \delta_{n,m}.$$

Remarks

- The orthogonality measure $\mu$ is not necessarily uniquely determined (up to constant factor) by the recurrence relation (2). But if there exists an orthogonality measure $\mu$ with compact support then we have certainly uniqueness.
- Let $M$ be a linear operator acting on sequences $u = \{u_n\}_{n=0,1,\ldots}$ by $(M(u))_n := A_n u_{n+1} + B_n u_n + C_n u_{n-1}$. Then, if $\{p_n(x)\}$ satisfies the recurrence relation (2), then for each $x$ the sequence $\{p_n(x)\}$ is an eigenfunction of $M$ with eigenvalue $x$. 
Bispectrality

We speak about *bispectrality* if we have a linear operator $L_x$ acting on functions in the variable $x$ and a linear operator $M_\xi$ acting on functions in the variable $\xi$ such that there exists a function $\phi(x, \xi)$ in the two variables $x, \xi$ for which

$$L_x(\phi(x, \xi)) = \sigma(\xi) \phi(x, \xi), \quad (3)$$

$$M_\xi(\phi(x, \xi)) = \tau(x) \phi(x, \xi). \quad (4)$$

where $\sigma(\xi)$ and $\tau(x)$ are suitable eigenvalues.

In the case of OP’s the variable $\xi$ becomes the discrete variable $n$ and we have in general only equation (4). We are interested in OP’s which also satisfy (3).

*Structure equation* implied by (3) and (4):

$$[L_x, \tau(x)](\phi(x, \xi)) = [M_\xi, \sigma(\xi)](\phi(x, \xi)).$$

Classical orthogonal polynomials

These are essentially the only OP’s which are eigenfunctions of a second order differential operator (*Bochner’s theorem*).

- **Hermite polynomials** $H_n(x)$, \quad $H_n(x) = 2^n x^n + \cdots$,
  \[ d\mu(x) := e^{-x^2} \, dx, \quad \left( \frac{1}{2} \frac{d^2}{dx^2} - x \frac{d}{dx} \right) H_n(x) = -n H_n(x). \]

- **Laguerre polynomials** $L_\alpha^\alpha(x)$, \quad $L_\alpha^\alpha(0) = (\alpha + 1)_n/n!$, where
  \[ (a)_n := a(a+1) \cdots (a+n-1) \] (Pochhammer symbol).
  \[ d\mu(x) := \chi_{(0,\infty)}(x) \, x^\alpha \, e^{-x} \, dx \quad (\alpha > -1), \]
  \[ \left( x \frac{d^2}{dx^2} + (\alpha + 1 - x) \frac{d}{dx} \right) L_\alpha^\alpha(x) = -n L_\alpha^\alpha(x). \]

- **Jacobi polynomials** $P_n^{(\alpha,\beta)}(x)$, \quad $P_n^{(\alpha,\beta)}(1) = (\alpha + 1)_n/n!$,
  \[ d\mu(x) := \chi_{(-1,1)}(x) \, (1-x)^\alpha (1+x)^\beta \, dx \quad (\alpha, \beta > -1), \]
  \[ \left( (1-x^2) \frac{d^2}{dx^2} + (\beta - \alpha - (\alpha + \beta + 2)x) \frac{d}{dx} \right) P_n^{(\alpha,\beta)}(x) \]
  \[ = -n(n + \alpha + \beta + 1) P_n^{(\alpha,\beta)}(x). \]
Structure relation for OP’s satisfying an eigenvalue equation

Let \( \{p_n(x)\} \) be a system of OP’s such that there is a linear operator \( L \) acting on polynomials in \( x \) for which the \( p_n \) are eigenfunctions with eigenvalues \( \lambda_n \). Write \( (Xf)(x) := x f(x) \). Then, from

\[
Lp_n = \lambda_n p_n, \\
Xp_n = A_n p_{n+1} + B_n p_n + C_n p_{n-1},
\]

we have the structure relation

\[
[L, X] p_n = A_n(\lambda_{n+1} - \lambda_n) p_{n+1} - C_n(\lambda_n - \lambda_{n-1}) p_{n-1}.
\]

**Remark** Since \( L \) and \( X \) are symmetric operators with respect to the inner product \( \langle f, g \rangle := \int_{\mathbb{R}} f(x) g(x) \, d\mu(x) \), the structure operator \( [L, X] \) is anti-symmetric with respect to this inner product.
Structure relation for the classical OP’s

- Hermite polynomials:
  \[
  \left( \frac{d}{dx} - x \right) H_n(x) = -\frac{1}{2} H_{n+1}(x) + n H_{n-1}(x).
  \]

- Laguerre polynomials:
  \[
  \left( 2x \frac{d}{dx} + \alpha + 1 - x \right) L_\alpha^n(x) = (n+1)L_\alpha^{n+1}(x) - (n+\alpha)L_\alpha^{n-1}(x).
  \]

- Jacobi polynomials:
  \[
  \left( 2(1 - x^2) \frac{d}{dx} + \beta - \alpha - (\alpha + \beta + 2)x \right) P_{n}^{(\alpha,\beta)}(x) =
  - \frac{2(n+1)(n + \alpha + \beta + 1)}{2n + \alpha + \beta + 1} P_{n+1}^{(\alpha,\beta)}(x) + \frac{2(n+\alpha)(n + \beta)}{2n + \alpha + \beta + 1} P_{n-1}^{(\alpha,\beta)}(x).
  \]

Combine with 3-term recurrence relation. Then get the form
\[
\pi(x) p'_n(x) = a_n \pi p_{n+1}(x) + b_n \pi p_n(x) + c_n \pi p_{n-1}(x)
\]
for a polynomial \( \pi(x) \). Al-Salam & Chihara (1972) characterized the classical OP’s as OP’s with such a structure relation.
Let \( \{ p_n(x) \} \) be a system of classical OP’s and let \( L \) be the second order differential operator for which they are eigenfunctions. Then \( L \) and \( X \) will generate an associative algebra with identity of linear operators. Certainly the structure operator \( S := [L, X] \) will belong to this algebra. Are there further relations in the algebra? Let us try the commutators of \( S \) with \( L \) and \( X \).
Algebra generated by $L$ and $X$ for the classical OP’s (continued)

- **Hermite:**
  \[ [L, X] = S, \quad [X, S] = -1, \quad [S, L] = -X. \]

- **Laguerre:**
  \[ [L, X] = S, \quad [X, S] = -2X, \quad [S, L] = -2L - X + \alpha + 1. \]

- **Jacobi:**
  \[ [L, X] = S, \quad [X, S] = 2X^2 - 2, \]
  \[ [S, L] = 2(XL + LX) - (\alpha + \beta)(\alpha + \beta + 2)X + \beta^2 - \alpha^2. \]

Lie algebras and representations involved:
- **Hermite:** Heisenberg Lie algebra and its standard representation on a space of suitable functions on $\mathbb{R}$.
- **Laguerre:** the Lie algebra $sl(2, \mathbb{R})$ and its discrete series representation in a suitable model.
- **Jacobi:** quadratic terms; no (finite dimensional) Lie algebra.
The scheme of classical OP’s

\[
\begin{align*}
\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1 - 2\beta^{-1} x) &= L_n^{\alpha}(x). \\
\lim_{\alpha \to \infty} \alpha^{-\frac{1}{2} n} P_n^{(\alpha,\alpha)}(\alpha^{-\frac{1}{2}} x) &= H_n(x)/(2^n n!). \\
\lim_{\alpha \to \infty} \alpha^{-\frac{1}{2} n} L_n^{\alpha}((2\alpha)^{\frac{1}{2}} x + \alpha) &= (-1)^n H_n(x)/(2^{\frac{1}{2} n} n!).
\end{align*}
\]
A system \( \{ p_n(x) \}_{n=0}^{\infty} \) of OP’s is called *discrete* if the orthogonality measure \( \mu \) has discrete support \( \{ x_k \}_{k=0}^{\infty} \). Then

\[
\int_{\mathbb{R}} f(x) \, d\mu(x) = \sum_{k=0}^{\infty} f(x_k) \, w_k
\]

for certain positive *weights* \( w_k \).

We will also admit finite systems \( \{ p_n \}_{n=0,1,\ldots,N} \) of OP’s, where the orthogonality measure \( \mu \) has finite support \( \{ x_k \}_{k=0,1,\ldots,N} \). Then

\[
\int_{\mathbb{R}} f(x) \, d\mu(x) = \sum_{k=0}^{N} f(x_k) \, w_k
\]

for certain positive *weights* \( w_k \).
The Askey scheme

Extend the scheme of classical OP’s with the following classes:

- **OP’s of Hahn class** are OP’s which are eigenfunctions of a second order difference operator $L$ of one of the forms

  \[(Lf)(x) := a_n f(x - 1) + b_n f(x) + c_n f(x + 1) \quad \text{(discrete)},\]
  \[(Lf)(x) := a_n f(x - i) + b_n f(x) + c_n f(x + i) \quad \text{(continuous)}.\]

  These are the Hahn, continuous Hahn, Meixner-Pollaczek, Meixner, Krawtchouk and Charlier polynomials.

- **OP’s of quadratic lattice class** are OP’s which are eigenfunctions of a second order difference operator $L$ of one of the forms

  \[(Lf)(y^2) := a_n f((y - 1)^2) + b_n f(y^2) + c_n f((y + 1)^2) \quad \text{(discr.)},\]
  \[(Lf)(y^2) := a_n f((y - i)^2) + b_n f(y^2) + c_n f((y + i)^2) \quad \text{(cont.)}.\]

  These are the Wilson, Racah, dual Hahn and continuous dual Hahn polynomials.
Askey scheme

Wilson

Cont. dual Hahn

Cont. Hahn

Racah

Hahn

Dual Hahn

Meixner-Pollaczek

Jacobi

Meixner Krawtchouk

Laguerre

Charlier

Hermite

Dick Askey

discrete OP

quadratic lattice

Hahn class

classical OP
All OP’s in the Askey scheme are hypergeometric functions. The general *hypergeometric function* is defined by:

\[ \genfrac{}{}{0pt}{}{r}{s}F_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!} . \]

where \((a)_k := a(a + 1) \cdots (a + k - 1)\) (Pochhammer symbol).

If \(a_1 = -n\) \((n = 0, 1, 2, \ldots)\) then the series terminates after the term with \(k = n\). A hypergeometric function becomes undefined (singular) if one of the bottom parameters is a non-positive integer, say \(b_s = -N\), but the function remains well-defined if \(a_1 = -n\) with \(n = 0, 1, \ldots, N\), because the series then terminates before the term with \(k = N\).
Hahn polynomials are given by

\[ Q_n(x; \alpha, \beta, N) := \, _3F_2 \left( \begin{array}{c} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{array} ; 1 \right) \quad (n = 0, 1, \ldots, N). \]

They have a limit to Jacobi polynomials by

\[ Q_n(Nx; \alpha, \beta, N) = \, _3F_2 \left( \begin{array}{c} -n, n + \alpha + \beta + 1, -Nx \\ \alpha + 1, -N \end{array} ; 1 \right) \]

\[ \xrightarrow{N \to \infty} 2F_1 \left( \begin{array}{c} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{array} ; x \right) = \frac{P_n^{(\alpha, \beta)}(1 - 2x)}{P_n^{(\alpha, \beta)}(1)}. \]
Let $0 < q < 1$. Define the $q$-Pochhammer symbol by

$$(a; q)_k := (1 - a)(1 -aq) \ldots (1 - aq^{k-1}).$$

Also for $k = \infty$

$$(a; q)_\infty = (1 - a)(1 -aq)(1 -aq^2) \ldots \quad (\text{convergent}).$$

Put

$$(a_1, \ldots, a_r; q)_k := (a_1; q)_k \ldots (a_r; q)_k.$$

The $q$-Pochhammer symbol is a $q$-analogue of the Pochhammer symbol:

$$\frac{(q^a; q)_k}{(1 - q)_k^k} = \frac{1 - q^a}{1 - q} \frac{1 - q^{a+1}}{1 - q} \ldots \frac{1 - q^{a+k-1}}{1 - q} \xrightarrow{q \to 1} a(a + 1) \ldots (a + k - 1) = (a)_k.$$
Define the \( q \)-hypergeometric series by

\[
r\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; q, z \right) := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k ((-1)^k q^{1/2} k(k-1))^s-r+1}{(b_1; q)_k \cdots, (b_s; q)_k (q; q)_k} z^k.
\]

If \( a_1 = q^{-n} \) with \( n \) non-negative integer, then the series terminates after the term with \( k = n \).

The \( q \)-hypergeometric series is formally a \( q \)-analogue of ordinary hypergeometric series:

\[
\lim_{q \uparrow 1} r\phi_s \left( \begin{array}{c} q^{a_1}, \ldots, q^{a_r} \\ q^{b_1}, \ldots, q^{b_s} \end{array} ; q, (1 - q)^{s-r+1} z \right) = \, _rF_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; z \right).
\]
The $q$-Askey scheme

Parallel to the Askey scheme there is a $q$-Askey scheme in which the OP’s are expressed as terminating $q$-hypergeometric series. There are limit relations within the $q$-Askey scheme, and also from families in the $q$-Askey scheme to families in the Askey scheme. The $q$-Askey scheme consists of two classes:

- **OP’s of $q$-Hahn class** are OP’s which are eigenfunctions of a second order $q$-difference operator $L$ of the form
  \[
  (Lf)(x) := a_n f(q^{-1}x) + b_n f(x) + c_n f(qx).
  \]

- **OP’s of quadratic $q$-lattice class** are OP’s which are eigenfunctions of a second order $q$-difference operator $L$ of the form
  \[
  (Lf)(\frac{1}{2}(z + z^{-1})) := a_n f[q^{-1}z] + b_n f[z] + c_n f[qz],
  \]
  where
  \[
  f[z] := f(\frac{1}{2}(z + z^{-1})).
  \]
On the top level of the $q$-Askey scheme are the *Askey-Wilson polynomials*:

$$P_n[z] = P_n[z; a, b, c, d \mid q] = P_n\left(\frac{1}{2}(z + z^{-1}); a, b, c, d \mid q\right)$$

$$:= \frac{(ab, ac, ad; q)_n}{a^n(abcdq^{n-1}; q)_n} 4\phi_3\left(\begin{array}{c} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{array} ; q, q \right).$$

The right-hand side gives a *symmetric Laurent polynomial* in $z$:

$$P_n[z] = \sum_{k=-n}^{n} c_k z^k = P_n[z^{-1}] \quad (c_k = c_{-k}, \ c_n \neq 0).$$

Therefore it is an ordinary polynomial $P_n\left(\frac{1}{2}(z + z^{-1})\right)$ of degree $n$ in the variable $x := \frac{1}{2}(z + z^{-1})$. We have normalized $P_n[z]$ such that it is *monic* in $z$, i.e., $c_n = 1$. 
Askey-Wilson polynomials $P_n[z]$ satisfy the orthogonality relation

$$\frac{1}{4\pi i} \oint_C P_n[z] P_m[z] w[z] \frac{dz}{z} = h_n \delta_{n,m},$$

where

$$w(z) := \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty},$$

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty},$$

$$h_n = \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_{2n}(q^{n-1}abcd; q)_n}.$$

Here $C$ is the unit circle traversed in positive direction with deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to $\infty$.

For suitable $a, b, c, d$ this can be rewritten as an orthogonality relation for the $P_n(x)$ with respect to a positive measure $\mu$ supported on $[-1, 1]$ (or on its union with a finite discrete set).
Askey-Wilson polynomials are OP’s of quadratic $q$-lattice class. They are eigenfunctions of a second order $q$-difference operator $L$:


$$= (q^{-n} - 1)(1 - abcdq^{n-1})P_n[z],$$


With $(Xf)[z]) := (Z + Z^{-1}) f[z]$, we obtain for the structure operator:

$$([L, X]f)[z] := a[z] f[qz] - a[z^{-1}] f[q^{-1}z],$$

where $a[z] := \frac{(q^{-1} - 1)(1 - az)(1 - bz)(1 - cz)(1 - dz)}{z(1 - z^2)}$. 
There is a generalized Bochner theorem which characterizes
the Askey-Wilson polynomials and their limit cases as the only
polynomial solutions $p_n(x)$ of a second order difference
equation of the form

$$A(s)P_n(x(s+1)) + B(s)P_n(x(s)) + C(s)P_n(x(s-1)) = \lambda_n P_n(x(s)).$$

See Grünbaum & Haine (1996), Ismail (2003), Vinet &