Zhedanov's Askey-Wilson algebra, Cherednik's double affine Hecke algebras, and bispectrality. Lecture 1: The Askey and q-Askey scheme

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Plan of the course

1. The Askey and q-Askey scheme
2. Zhedanov's algebra
3. Double affine Hecke algebra in the rank one case

General orthogonal polynomials

Definition

Let $\{p_n(x)\}_{n=0,1,\ldots}$ be a system of real-valued polynomials $p_n(x)$ of degree n in x. Let μ be a positive Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty$ for all n. Then $\{p_n(x)\}$ is called a system of *orthogonal polynomials* (OP's) if

$$\int_{\mathbb{R}} p_n(x)\, x^k \, d\mu(x) = 0 \quad (k = 0, 1, \ldots, n-1). \tag{1}$$

Theorem

Any system of orthogonal polynomials (with $p_{-1}(x) := 0$, $p_0(x) := 1$) satisfies a recurrence relation of the form

$$x\, p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x). \tag{2}$$

Conversely, if $\{p_n(x)\}$ satisfies (2) with $C_n A_{n-1} > 0$ then there exists a positive Borel measure μ on $\mathbb{R}$ such that (1) holds.

General orthogonal polynomials (continued)

Notation

- Write $p_n(x) = k_n x^n + \cdots$.
- Write $h_n := \int_{\mathbb{R}} p_n(x)^2 \, d\mu(x)$. Then

$$\int_{\mathbb{R}} p_n(x) \, p_m(x) \, d\mu(x) = h_n \, \delta_{n,m}.$$

Remarks

- The orthogonality measure μ is not necessarily uniquely determined (up to constant factor) by the recurrence relation (2). But if there exists an orthogonality measure μ with compact support then we have certainly uniqueness.
- Let M be a linear operator acting on sequences $u = \{u_n\}_{n=0,1,\ldots}$ by $(M(u))_n := A_n u_{n+1} + B_n u_n + C_n u_{n-1}$. Then, if $\{p_n(x)\}$ satisfies the recurrence relation (2), then for each x the sequence $\{p_n(x)\}$ is an eigenfunction of M with eigenvalue x.

Bispectrality

We speak about *bispectrality* if we have a linear operator L_x acting on functions in the variable x and a linear operator M_ξ acting on functions in the variable ξ such that there exists a function $\phi(x, \xi)$ in the two variables x, ξ for which

$$L_x(\phi(x,\xi)) = \sigma(\xi)\,\phi(x,\xi), \tag{3}$$

$$M_\xi(\phi(x,\xi)) = \tau(x)\,\phi(x,\xi). \tag{4}$$

where $\sigma(\xi)$ and $\tau(x)$ are suitable eigenvalues.

In the case of OP's the variable ξ becomes the discrete variable n and we have in general only equation (4). We are interested in OP's which also satisfy (3).

Structure equation implied by (3) and (4):

$$[L_x, \tau(x)]\,(\phi(x,\xi)) = [M_\xi, \sigma(\xi)]\,(\phi(x,\xi)).$$

Here $[A, B] := AB - BA$ (commutator).

Classical orthogonal polynomials

These are essentially the only OP's which are eigenfunctions of a second order differential operator (*Bochner's theorem*).

- Hermite polynomials $H_n(x)$, $\quad H_n(x) = 2^n x^n + \cdots,$
  $d\mu(x) := e^{-x^2}\,dx, \quad \left(\frac{1}{2}\frac{d^2}{dx^2} - x\frac{d}{dx}\right)H_n(x) = -nH_n(x).$
- Laguerre polynomials $L_n^\alpha(x)$, $\quad L_n^\alpha(0) = (\alpha+1)_n/n!$, where $(a)_n := a(a+1)\ldots(a+n-1)$ (Pochhammer symbol).
  $d\mu(x) := \chi_{(0,\infty)}(x)\,x^\alpha\,e^{-x}\,dx \quad (\alpha > -1),$
  $$\left(x\frac{d^2}{dx^2} + (\alpha+1-x)\frac{d}{dx}\right)L_n^\alpha(x) = -n\,L_n^\alpha(x).$$
- Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\quad P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!$,
  $d\mu(x) := \chi_{(-1,1)}(x)\,(1-x)^\alpha(1+x)^\beta\,dx \quad (\alpha,\beta > -1),$
  $$\left((1-x^2)\frac{d^2}{dx^2} + (\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}\right)P_n^{(\alpha,\beta)}(x) = -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x).$$

Structure relation for OP's satisfying an eigenvalue equation

Let $\{p_n(x)\}$ be a system of OP's such that there is a linear operator L acting on polynomials in x for which the p_n are eigenfunctions with eigenvalues λ_n. Write $(Xf)(x) := x\,f(x)$. Then, from

$$\begin{aligned} Lp_n &= \lambda_n p_n, \\ Xp_n &= A_n p_{n+1} + B_n p_n + C_n p_{n-1}, \end{aligned}$$

we have the *structure relation*

$$[L, X]\,p_n = A_n(\lambda_{n+1} - \lambda_n)\,p_{n+1} - C_n(\lambda_n - \lambda_{n-1})p_{n-1}.$$

Remark Since L and X are symmetric operators with respect to the inner product $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\,g(x)\,d\mu(x)$, the *structure operator* $[L, X]$ is anti-symmetric with respect to this inner product.

Structure relation for the classical OP's

- Hermite polynomials:

$$\left(\frac{d}{dx}-x\right)H_n(x)=-\tfrac{1}{2}H_{n+1}(x)+nH_{n-1}(x).$$

- Laguerre polynomials:

$$\left(2x\frac{d}{dx}+\alpha+1-x\right)L_n^\alpha(x)=(n+1)L_{n+1}^\alpha(x)-(n+\alpha)L_{n-1}^\alpha(x).$$

- Jacobi polynomials:

$$\left(2(1-x^2)\frac{d}{dx}+\beta-\alpha-(\alpha+\beta+2)x\right)P_n^{(\alpha,\beta)}(x)=$$
$$-\frac{2(n+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+1}\,P_{n+1}^{(\alpha,\beta)}(x)+\frac{2(n+\alpha)(n+\beta)}{2n+\alpha+\beta+1}\,P_{n-1}^{(\alpha,\beta)}(x).$$

Combine with 3-term recurrence relation. Then get the form $\pi(x)\,p_n'(x)=a_np_{n+1}(x)+b_np_n(x)+c_np_{n-1}(x)$ for a polynomial $\pi(x)$. Al-Salam & Chihara (1972) characterized the classical OP's as OP's with such a structure relation.

Algebra generated by L and X for the classical OP's

Let $\{p_n(x)\}$ be a system of classical OP's and let L be the second order differential operator for which they are eigenfunctions. Then L and X will generate an associative algebra with identity of linear operators. Certainly the structure operator $S := [L, X]$ will belong to this algebra. Are there further relations in the algebra? Let us try the commutators of S with L and X.

Algebra generated by L and X for the classical OP's (continued)

- Hermite:

$$[L, X] = S, \quad [X, S] = -1, \quad [S, L] = -X.$$

- Laguerre:

$$[L, X] = S, \quad [X, S] = -2X, \quad [S, L] = -2L - X + \alpha + 1.$$

- Jacobi:

$$[L, X] = S, \quad [X, S] = 2X^2 - 2,$$

$$[S, L] = 2(XL + LX) - (\alpha + \beta)(\alpha + \beta + 2)X + \beta^2 - \alpha^2.$$

Lie algebras and representations involved:

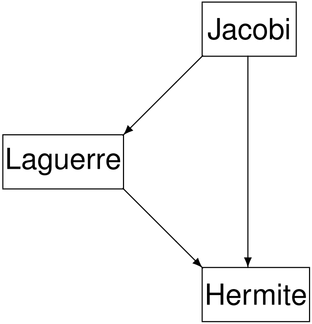
- Hermite: Heisenberg Lie algebra and its standard representation on a space of suitable functions on $\mathbb{R}$.
- Laguerre: the Lie algebra $sl(2, \mathbb{R})$ and its discrete series representation in a suitable model.
- Jacobi: quadratic terms; no (finite dimensional) Lie algebra.

The scheme of classical OP's

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x) = L_n^\alpha(x).$$

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} P_n^{(\alpha,\alpha)}(\alpha^{-\frac{1}{2}}x) = H_n(x)/(2^n n!).$$

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} L_n^\alpha\big((2\alpha)^{\frac{1}{2}}x+\alpha\big) = (-1)^n H_n(x)/(2^{\frac{1}{2}n} n!).$$



Discrete OP's

A system $\{p_n(x)\}_{n=0}^{\infty}$ of OP's is called *discrete* if the orthogonality measure μ has discrete support $\{x_k\}_{k=0}^{\infty}$. Then

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \sum_{k=0}^{\infty} f(x_k)\, w_k$$

for certain positive *weights* w_k.

We will also admit finite systems $\{p_n\}_{n=0,1,\ldots,N}$ of OP's, where the orthogonality measure μ has finite support $\{x_k\}_{k=0,1,\ldots,N}$. Then

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \sum_{k=0}^{N} f(x_k)\, w_k$$

for certain positive *weights* w_k.

The Askey scheme

Extend the scheme of classical OP's with the following classes:

- OP's of Hahn class are OP's which are eigenfunctions of a second order difference operator L of one of the forms

$$(Lf)(x) := a_n f(x-1) + b_n f(x) + c_n f(x+1) \qquad \text{(discrete)},$$
$$(Lf)(x) := a_n f(x-i) + b_n f(x) + c_n f(x+i) \qquad \text{(continuous)}.$$

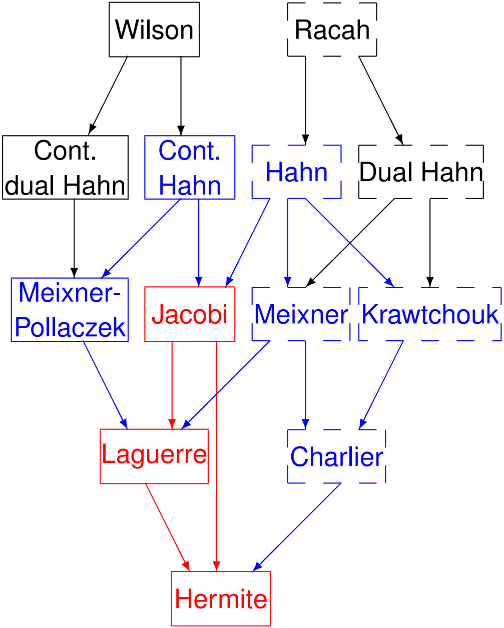
  These are the Hahn, continuous Hahn, Meixner-Pollaczek, Meixner, Krawtchouk and Charlier polynomials.

- OP's of quadratic lattice class are OP's which are eigenfunctions of a second order difference operator L of one of the forms

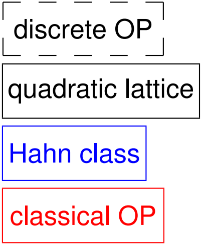
$$(Lf)(y^2) := a_n f\big((y-1)^2\big) + b_n f(y^2) + c_n f\big((y+1)^2\big) \quad \text{(discr.)},$$
$$(Lf)(y^2) := a_n f\big((y-i)^2\big) + b_n f(y^2) + c_n f\big((y+i)^2\big) \qquad \text{(cont.)}.$$

  These are the Wilson, Racah, dual Hahn and continuous dual Hahn polynomials.

Askey scheme



Dick Askey



Hypergeometric functions

All OP's in the Askey scheme are hypergeometric functions. The general *hypergeometric function* is defined by:

$$ {}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k\ldots(a_r)_k}{(b_1)_k\ldots(b_s)_k}\,\frac{z^k}{k!}\,. $$

where $(a)_k := a(a+1)\ldots(a+k-1)$ (Pochhammer symbol).

If $a_1 = -n$ ($n = 0,1,2,\ldots$) then the series terminates after the term with $k = n$. A hypergeometric function becomes undefined (singular) if one of the bottom parameters is a non-positive integer, say $b_s = -N$, but the function remains well-defined if $a_1 = -n$ with $n = 0,1,\ldots,N$, because the series then terminates before the term with $k = N$.

Example: Hahn polynomials

Hahn polynomials are given by

$$Q_n(x;\alpha,\beta,N) := {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x\\ \alpha+1, -N\end{matrix}; 1\right) \quad (n=0,1,\ldots,N).$$

They have a limit to Jacobi polynomials by

$$Q_n(Nx;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -Nx\\ \alpha+1, -N\end{matrix}; 1\right)$$
$$\xrightarrow{N\to\infty} {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; x\right) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}.$$

q-Pochhammer symbol

Let $0 < q < 1$. Define the *q-Pochhammer symbol* by

$$(a;q)_k := (1-a)(1-aq)\ldots(1-aq^{k-1}).$$

Also for $k = \infty$:

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ldots \qquad \text{(convergent)}.$$

Put

$$(a_1,\ldots,a_r;q)_k := (a_1;q)_k\ldots(a_r;q)_k.$$

The q-Pochhammer symbol is a q-analogue of the Pochhammer symbol:

$$\frac{(q^a;q)_k}{(1-q)^k} = \frac{1-q^a}{1-q}\,\frac{1-q^{a+1}}{1-q}\ldots\frac{1-q^{a+k-1}}{1-q}$$
$$\xrightarrow{q\to1} a(a+1)\ldots(a+k-1) = (a)_k.$$

q-Hypergeometric series

Define the *q-hypergeometric series* by

$$ {}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_r;q)_k \left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{s-r+1} z^k}{(b_1;q)_k \ldots, (b_s;q)_k \, (q;q)_k} \,. $$

If $a_1 = q^{-n}$ with n non-negative integer, then the series terminates after the term with $k = n$.

The q-hypergeometric series is formally a q-analogue of ordinary hypergeometric series:

$$ \lim_{q\uparrow 1} {}_r\phi_s\left(\begin{matrix} q^{a_1},\ldots,q^{a_r} \\ q^{b_1},\ldots,q^{b_s} \end{matrix}; q, (1-q)^{s-r+1} z\right) \\ = {}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; z\right). $$

The q-Askey scheme

Parallel to the Askey scheme there is a q-Askey scheme in which the OP's are expressed as terminating q-hypergeometric series. There are limit relations within the q-Askey scheme, and also from families in the q-Askey scheme to families in the Askey scheme. The q-Askey scheme consists of two classes:

- OP's of q-Hahn class are OP's which are eigenfunctions of a second order q-difference operator L of the form

$$(Lf)(x) := a_n f(q^{-1}x) + b_n f(x) + c_n f(qx).$$

- OP's of quadratic q-lattice class are OP's which are eigenfunctions of a second order q-difference operator L of the form

$$(Lf)\big(\tfrac{1}{2}(z+z^{-1})\big) := a_n f[q^{-1}z] + b_n f[z] + c_n f[qz], \quad \text{where} \quad f[z] := f(\tfrac{1}{2}(z+z^{-1})).$$

Askey-Wilson polynomials

On the top level of the q-Askey scheme are the *Askey-Wilson polynomials*:

$$P_n[z] = P_n[z; a, b, c, d \mid q] = P_n\big(\tfrac{1}{2}(z + z^{-1}); a, b, c, d \mid q\big)$$
$$:= \frac{(ab, ac, ad; q)_n}{a^n (abcdq^{n-1}; q)_n} \, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

The right-hand side gives a *symmetric Laurent polynomial* in z:

$$P_n[z] = \textstyle\sum_{k=-n}^{n} c_k z^k = P_n[z^{-1}] \qquad (c_k = c_{-k},\ c_n \neq 0).$$

Therefore it is an ordinary polynomial $P_n\big(\frac{1}{2}(z + z^{-1})\big)$ of degree n in the variable $x := \frac{1}{2}(z + z^{-1})$. We have normalized $P_n[z]$ such that it is *monic* in z, i.e., $c_n = 1$.

Askey-Wilson polynomials: orthogonality

Askey-Wilson polynomials $P_n[z]$ satisfy the orthogonality relation

$$\frac{1}{4\pi i}\oint_C P_n[z]\,P_m[z]\,w[z]\,\frac{dz}{z} = h_n\,\delta_{n,m}, \quad \text{where}$$

$$w(z) := \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty},$$

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty},$$

$$\frac{h_n}{h_0} = \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_{2n}(q^{n-1}abcd; q)_n}.$$

Here C is the unit circle traversed in positive direction with deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to ∞.

For suitable a, b, c, d this can be rewritten as an orthogonality relation for the $P_n(x)$ with respect to a positive measure μ supported on $[-1, 1]$ (or on its union with a finite discrete set).

Askey-Wilson polynomials as eigenfunctions of L

Askey-Wilson polynomials are OP's of quadratic q-lattice class. They are eigenfunctions of a second order q-difference operator L:

$$(LP_n)[z] := A[z]\,P_n[qz] + A[z^{-1}]\,P_n[q^{-1}z] - (A[z] + A[z^{-1}])\,P_n[z] \\ = (q^{-n}-1)(1-abcdq^{n-1})P_n[z],$$

$$\text{where} \quad A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,.$$

With $(Xf)[z]) := (Z+Z^{-1})\,f[z]$, we obtain for the structure operator:

$$([L,X]f)[z] := a[z]\,f[qz] - a[z^{-1}]\,f[q^{-1}z],$$

$$\text{where} \quad a[z] := \frac{(q^{-1}-1)(1-az)(1-bz)(1-cz)(1-dz)}{z(1-z^2)}\,.$$

Generalized Bochner theorem

There is a generalized Bochner theorem which characterizes the Askey-Wilson polynomials and their limit cases as the only polynomial solutions $p_n(x)$ of a second order difference equation of the form

$$A(s)P_n\big(x(s+1)\big)+B(s)P_n\big(x(s)\big)+C(s)P_n\big(x(s-1)\big)=\lambda_n P_n\big(x(s)\big).$$

See Grünbaum & Haine (1996), Ismail (2003), Vinet & Zhedanov (2008).