Zhedanov's Askey-Wilson algebra, Cherednik's double affine Hecke algebras, and bispectrality. Lecture 2: Zhedanov's algebra

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Algebra generated by L and X for bispectral OP's

Let $\{p_n(x)\}$ be a system of OP's which are eigenfunctions of some operator L. Then (with $(Xf)(x) := x\,f(x)$):

$$Lp_n = \lambda_n p_n, \quad Xp_n = A_n p_{n+1} + B_n p_n + C_n p_{n-1}.$$

Then L and X will generate an associative algebra with identity consisting of linear operators acting on the space of polynomials. Certainly the structure operator $[L, X]$ will belong to this algebra. Are there further relations in the algebra?

Consider this for the Askey-Wilson polynomials

$$P_n(x) = P_n[z] = P_n[z; a, b, c, d \mid q] \quad (x = \tfrac{1}{2}(z + z^{-1})).$$

They satisfy $\quad (LP_n)[z] = \lambda_n\, P_n[z]$

with $\quad \lambda_n = q^{-n} + abcdq^{n-1} \quad$ and

$$(Lf)[z] := A[z]\, f[qz] + A[z^{-1}]\, f[q^{-1}z] - (A[z] + A[z^{-1}])\, f[z]$$

$$+(1+q^{-1}abcd)f[z] \quad \text{with} \quad A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

Zhedanov's algebra $AW(3)$

Let $q \in \mathbb{C}$, $q \neq 0$, $q^m \neq 1$ $(m = 1, 2, \ldots)$.

q-commutator: $[X, Y]_q := q^{\frac{1}{2}} XY - q^{-\frac{1}{2}} YX$.

Zhedanov (1991) introduced the algebra $AW(3)$:

- generators K_0, K_1, K_2,
- structure constants B, C_0, C_1, D_0, D_1,
- relations

$$\begin{aligned} [K_0, K_1]_q &= K_2, \\ [K_1, K_2]_q &= BK_1 + C_0 K_0 + D_0, \\ [K_2, K_0]_q &= BK_0 + C_1 K_1 + D_1. \end{aligned}$$

- The Casimir operator

$$\begin{aligned} Q := (q^{-\frac{1}{2}} - q^{\frac{3}{2}}) K_0 K_1 K_2 + q K_2^2 + B(K_0 K_1 + K_1 K_0) + q C_0 K_0^2 \\ + q^{-1} C_1 K_1^2 + (1+q) D_0 K_0 + (1+q^{-1}) D_1 K_1, \end{aligned}$$

  commutes in $AW(3)$ with the generators K_0, K_1, K_2.

Picture of Zhedanov

Choice of structure constants

Let a, b, c, d be complex parameters.

Let e_1, e_2, e_3, e_4 be the elementary symmetric polynomials in a, b, c, d.

Put for the structure constants:

$$\begin{aligned}
B &:= (1-q^{-1})^2(e_3+qe_1),\\
C_0 &:= (q-q^{-1})^2,\\
C_1 &:= q^{-1}(q-q^{-1})^2 e_4,\\
D_0 &:= -q^{-3}(1-q)^2(1+q)(e_4+qe_2+q^2),\\
D_1 &:= -q^{-3}(1-q)^2(1+q)(e_1 e_4+qe_3).
\end{aligned}$$

Basic representation of $AW(3)$

Let $\mathcal{A}_{\rm sym}$ be the space of symmetric Laurent polynomials $f[z] = f[z^{-1}]$.

Let L be the operator acting on $\mathcal{A}_{\rm sym}$ for which the Askey-Wilson polynomials are eigenfunctions:

$$\begin{aligned}(Lf)[z] := {} & A[z]\,\big(f[qz]-f[z]\big)\\ & + A[z^{-1}]\,\big(f[q^{-1}z]-f[z]\big) + (1+q^{-1}abcd)\,f[z],\\ & \text{where}\quad A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,.\end{aligned}$$

Then K_0 and K_1 satisfy the relations in $AW(3)$ with the given structure constants when they are realized as:

$$\begin{aligned}(K_0 f)[z] &:= (Lf)[z],\\ (K_1 f)[z] &:= (z+z^{-1})f[z].\end{aligned}$$

The representation of $AW(3)$ thus obtained is called the *basic representation*.

The basic representation realized on the space of terminating sequences

The Askey-Wilson polynomials can be interpreted as the kernel of an intertwining operator between the basic representation of $AW(3)$ and an equivalent representation on the space of terminating sequences $\{u_n\}_{n=0,1,2,\ldots}$.
Concretely, let $\langle\,.\,,\,.\,\rangle$ be the inner product for which the Askey-Wilson polynomials are orthogonal: $\langle P_n, P_m\rangle = h_n\,\delta_{n,m}$.
Define a map $f \mapsto \widehat{f}$ from $\mathcal{A}_{\rm sym}$ onto the space of terminating sequences by $\widehat{f}(n) := \langle f, P_n\rangle$. Then, corresponding to

$$K_0 P_n = \lambda_n P_n, \quad K_1 P_n = P_{n+1} + B_n P_n + C_n P_{n-1}$$

we have:

$$\begin{aligned}(K_0 f)\widehat{\ }(n) &= \lambda_n \widehat{f}(n),\\ (K_1 f)\widehat{\ }(n) &= \widehat{f}(n+1) + B_n \widehat{f}(n) + C_n \widehat{f}(n-1).\end{aligned}$$

Equivalent form of relations for $AW(3)$

Clearly, $AW(3)$ can equivalently be described as an algebra with two generators K_0, K_1 and with two relations

$$(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = B\,K_1 + C_0\,K_0 + D_0,$$
$$(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = B\,K_0 + C_1\,K_1 + D_1.$$

Then the Casimir operator Q can be written as

$$\begin{aligned} Q = K_1K_0K_1K_0 - (q^2+1+q^{-2})K_0K_1K_0K_1 + (q+q^{-1})K_0^2K_1^2 \\ + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) + B\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\ + (q+1+q^{-1})(D_0K_0 + D_1K_1). \end{aligned}$$

A duality for Askey-Wilson polynomials

From

$$\frac{P_n[z;a,b,c,d\mid q]}{P_n[a;a,b,c,d\mid q]} = {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right)$$

we have for $m = 0, 1, 2, \ldots$ that

$$\frac{P_n[q^m a;a,b,c,d\mid q]}{P_n[a;a,b,c,d\mid q]} = {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, q^m a^2, q^{-m} \\ ab, ac, ad \end{matrix}; q, q\right)$$

$$= {}_4\phi_3\left(\begin{matrix} q^{-n}, q^n a'^2, q^{m-1}a'b'c'd', q^{-m} \\ a'b', a'c', a'd' \end{matrix}; q, q\right) = \frac{P_m[q^n a';a',b',c',d'\mid q]}{P_m[a';a',b',c',d'\mid q]},$$

where

$$a' = (q^{-1}abcd)^{\frac{1}{2}}, \quad b' = \frac{ab}{a'}, \quad c' = \frac{ac}{a'}, \quad d' = \frac{ad}{a'}.$$

A duality for $AW(3)$

Not surprising, there is a similar duality for $AW(3)$.

Concretely, the relations

$$(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = B\,K_1 + C_0\,K_0 + D_0,$$
$$(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = B\,K_0 + C_1\,K_1 + D_1.$$

with the structure parameters expressed in terms of a, b, c, d as before, are preserved under the anti-automorphism generated by:

$$K_0 \to aK_1, \quad K_1 \to (q^{-1}abcd)^{-\frac{1}{2}}K_0, \quad a \to (q^{-1}abcd)^{\frac{1}{2}},$$
$$b \to \frac{ab}{(q^{-1}abcd)^{\frac{1}{2}}}, \quad c \to \frac{ac}{(q^{-1}abcd)^{\frac{1}{2}}}, \quad d \to \frac{ad}{(q^{-1}abcd)^{\frac{1}{2}}}.$$

Also $a^{-2}Q$ (Q the Casimir operator) is preserved under this transformation.

The Casimir operator in the basic representation

In the basic representation the Casimir operator Q can be computed to become a constant scalar:

$$(Qf)[z] = Q_0\, f[z],$$

where

$$Q_0 := q^{-4}(1-q)^2\Big(q^4(e_4-e_2)+q^3(e_1^2-e_1e_3-2e_2)$$
$$-q^2(e_2e_4+2e_4+e_2)+q(e_3^2-2e_2e_4-e_1e_3)+e_4(e_1-e_2)\Big).$$

This fits nicely with the fact that the basic representation is irreducible for generic values of a, b, c, d.

The relation $Q = Q_0$ does not hold generally in $AW(3)$. We will pass to a quotient algebra of $AW(3)$ with this relation.

A faithful representation on $\mathcal{A}_{\rm sym}$

Assumptions $q \neq 0, \quad q^m \neq 1 \ (m = 1, 2, \ldots),$
$a, b, c, d \neq 0, \quad abcd \neq q^{-m} \ (m = 0, 1, 2, \ldots).$

Definition

$AW(3, Q_0)$ is the algebra $AW(3)$ with additional relation $Q = Q_0$.

Theorem (THK, 2007)

$AW(3, Q_0)$ has the elements

$$K_0^n (K_1 K_0)^l K_1^m \quad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1)$$

as a linear basis.

The basic representation of $AW(3, Q_0)$ on $\mathcal{A}_{\rm sym}$ is faithful.

More representations of $AW(3)$

Assume $abcd > 0$. Recall: $[K_0, K_1]_q = K_2$,

$$[K_1, K_2]_q = BK_1 + C_0 K_0 + D_0, \quad [K_2, K_0]_q = BK_0 + C_1 K_1 + D_1,$$

realized on $\mathcal{A}_{\mathrm{sym}}$ by

$$(K_0 f)[z] := (q^{-1}abcd)^{-\frac{1}{2}}(Lf)[z], \quad (K_1 f)[z] := (z + z^{-1})f[z],$$

where $\quad C_0 = C_1 = (q - q^{-1})^2,$

$$\begin{aligned} B &= q^{\frac{1}{2}}(1 - q^{-1})^2 e_4^{-\frac{1}{2}}(e_3 + qe_1), \\ D_0 &= -q^{-5/2}(1 - q)^2(1 + q)e_4^{-\frac{1}{2}}(e_4 + qe_2 + q^2), \\ D_1 &= -q^{-2}(1 - q)^2(1 + q)e_4^{-1}(e_1 e_4 + qe_3). \end{aligned}$$

Chosen values of B, D_0, D_1 impose three algebraic constraints on a, b, c, d, but $abcd$ can freely vary over the positive reals, leading to different representations of $AW(3)$ (probably with different Casimir values).

More representations of $AW(3)$ (continued)

With $(K_0 f)[z] := (q^{-1}abcd)^{-\frac{1}{2}}(Lf)[z]$ we have

$$K_0\, P_n[\,.\,; a,b,c,d \mid q] = \left((abcdq^{2n-1})^{\frac{1}{2}} + (abcdq^{2n-1})^{-\frac{1}{2}}\right) \times P_n[\,.\,; a,b,c,d \mid q].$$

If a',b',c',d' and a,b,c,d give rise to the same structure constants B, D_0, D_1, while

$$a'b'c'd' = q^{2k}abcd \quad \text{for some } k \in \mathbb{Z},$$

then $P_n[\,.\,; a,b,c,d \mid q]$ and $P_{n-k}[\,.\,; a',b',c',d' \mid q]$ have the same eigenvalue for K_0.

Yet another equivalent form for the generators and relations of $AW(3)$

Replace K_0 by $K_0+\nu_0$ and K_1 by $K_1+\nu_1$ (ν_0,ν_1 scalars). Also let $K_2:=[K_0,K_1]$ (the ordinary commutator). Write $R:=2-q-q^{-1}$. Also use the notation for the anticommutator: $\{X,Y\}:=XY+YX$. Then

$$\begin{aligned}
[K_1,K_2]&=R\,K_1K_0K_1+R\nu_1\{K_0,K_1\}+R\nu_0\,K_1^2+(2R\nu_0\nu_1+B)K_1\\
&\qquad+(R\nu_1^2+C_0)K_0+R\nu_0\nu_1^2+B\nu_1+C_0\nu_0+D_0,\\
[K_2,K_0]&=R\,K_0K_1K_0+R\nu_1\,K_0^2+R\nu_0\{K_0,K_1\}+(2R\nu_0\nu_1+B)K_0\\
&\qquad+(R\nu_0^2+C_1)K_1+R\nu_0^2\nu_1+B\nu_0+C_1\nu_1+D_1.
\end{aligned}$$

After renaming the structure constants this becomes:

$$\begin{aligned}
[K_1,K_2]&=R\,K_1K_0K_1+S\{K_0,K_1\}+T\,K_1^2+BK_1+C_0K_0+D_0,\\
[K_2,K_0]&=R\,K_0K_1K_0+S\,K_0^2+T\{K_0,K_1\}+BK_0+C_1K_1+D_1.
\end{aligned}$$

Equivalent form of $AW(3)$ (continued)

So we have the algebra generated by K_0, K_1, K_2 and with relations

$$[K_0, K_1] = K_2,$$
$$[K_1, K_2] = R\,K_1K_0K_1 + S\{K_0, K_1\} + T\,K_1^2 + BK_1 + C_0K_0 + D_0,$$
$$[K_2, K_0] = R\,K_0K_1K_0 + S\,K_0^2 + T\{K_0, K_1\} + BK_0 + C_1K_1 + D_1.$$

For suitable values of the structure constants $R = 2 - q - q^{-1}$, S, T, B, C_0, C_1, D_0, D_1 any system of OP's in the (q-)Askey scheme can be associated with this algebra. For $R \neq 0$ we are in the q-Askey scheme, for $R = 0$ we are in the Askey scheme. For $R = S = T = 0$ we are in the case of a Lie algebra.

$AW(3)$ from Askey-Wilson to Jacobi

1. Restrict a, b, c, d to the case of the *continuous q-Jacobi polynomials*:
$a = q^{\frac{1}{2}\alpha+\frac{1}{4}},\ b = q^{\frac{1}{2}\alpha+\frac{3}{4}},\ c = -q^{\frac{1}{2}\beta+\frac{1}{4}},\ d = -q^{\frac{1}{2}\beta+\frac{3}{4}}.$
2. Then, with $x = \frac{1}{2}(z+z^{-1})$,

$$\frac{P_n[z]}{P_n\left[q^{\frac{1}{2}\alpha+\frac{1}{4}}\right]} = {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}}z, q^{\frac{1}{2}\alpha+\frac{1}{4}}z^{-1} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)} \end{matrix}; q, q\right)$$

$$\xrightarrow{q\uparrow 1} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \tfrac{1}{2}(1-x)\right) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}.$$

3. Consider $AW(3)$ with these a, b, c, d and let
$K_0 \to -(1-q)^2 K_0 + 1 + q^{\alpha+\beta+1}, \quad K_1 \to 2K_1.$
4. There appears the limit case of $AW(3)$ corresponding to the Jacobi polynomials.

$AW(3)$ for Jacobi

In the Jacobi case $AW(3)$ is generated by K_0, K_1, K_2 with relations

$$
\begin{aligned}
&[K_0, K_1] = K_2, \\
&[K_1, K_2] = 2K_1^2 - 2, \\
&[K_2, K_0] = 2\{K_0, K_1\} - (\alpha+\beta)(\alpha+\beta+2)K_1 + \beta^2 - \alpha^2.
\end{aligned}
$$

This is realized on the space of polynomials by:

$$
\begin{aligned}
&(K_0 f)(x) = (1-x^2)f''(x) + (\beta - \alpha - (\alpha+\beta+2)x)f'(x), \\
&(K_1 f)(x) = x\, f(x),
\end{aligned}
$$

and $\quad K_0 P_n^{(\alpha,\beta)} = -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}$.

Many representations of $AW(3)$ for Jacobi

Consider $AW(3)$ generated by K_0, K_1, K_2 for Jacobi paramters $(\gamma, 0)$, i.e.:

$$\begin{aligned}
[K_0, K_1] &= K_2,\\
[K_1, K_2] &= 2K_1^2 - 2,\\
[K_2, K_0] &= 2\{K_0, K_1\} - \gamma(\gamma+2)K_1 - \gamma^2.
\end{aligned}$$

For any real t this is realized by:

$$\begin{aligned}
(K_0 f)(x) &= (1-x^2)f''(x) - (\gamma e^{-t} + (\gamma e^t + 2)x)f'(x)\\
&\qquad - \left(\tfrac{1}{4}\gamma^2(e^{2t}-1) + \tfrac{1}{2}\gamma(e^t-1)\right)f(x),\\
(K_1 f)(x) &= x\,f(x),
\end{aligned}$$

In this realization $P_n^{(\gamma\cosh t, \gamma\sinh t)}$ is an eigenfunction of K_0 with eigenvalue $\quad -n(n+\gamma e^t+1) - \left(\tfrac{1}{4}\gamma^2(e^{2t}-1) + \tfrac{1}{2}\gamma(e^t-1)\right)$.