Zhedanov’s Askey-Wilson algebra, Cherednik’s double affine Hecke algebras, and bispectrality.

Lecture 2: Zhedanov’s algebra

Tom Koornwinder

University of Amsterdam, T.H.Koornwinder@uva.nl,
http://www.science.uva.nl/~thk/

course of 3 lectures during the International Mathematical Meeting
IMM’09 on Harmonic Analysis & Partial Differential Equations,
Marrakech, Morocco, April 1 – 4, 2009

Version of 6 April 2009
Let \( \{p_n(x)\} \) be a system of OP’s which are eigenfunctions of some operator \( L \). Then (with \( (Xf)(x) := x f(x) \)):

\[
Lp_n = \lambda_n p_n, \quad Xp_n = A_n p_{n+1} + B_n p_n + C_n p_{n-1}.
\]

Then \( L \) and \( X \) will generate an associative algebra with identity consisting of linear operators acting on the space of polynomials. Certainly the structure operator \([L, X]\) will belong to this algebra. Are there further relations in the algebra?

Consider this for the Askey-Wilson polynomials

\[
P_n(x) = P_n[z] = P_n[z; a, b, c, d | q] \quad (x = \frac{1}{2}(z + z^{-1})).
\]

They satisfy \((LP_n)[z] = \lambda_n P_n[z]\)

with \( \lambda_n = q^{-n} + abcdq^{n-1} \) and

\[
\]

Zhedanov’s algebra $AW(3)$

Let $q \in \mathbb{C}$, $q \neq 0$, $q^m \neq 1$ ($m = 1, 2, \ldots$).

$q$-commutator: $[X, Y]_q := q^{\frac{1}{2}} XY - q^{-\frac{1}{2}} YX$.

Zhedanov (1991) introduced the algebra $AW(3)$:

- generators $K_0$, $K_1$, $K_2$,
- structure constants $B$, $C_0$, $C_1$, $D_0$, $D_1$,
- relations

$$[K_0, K_1]_q = K_2,$$
$$[K_1, K_2]_q = BK_1 + C_0 K_0 + D_0,$$
$$[K_2, K_0]_q = BK_0 + C_1 K_1 + D_1.$$

- The Casimir operator

$$Q := \left( q^{-\frac{1}{2}} - q^\frac{3}{2} \right) K_0 K_1 K_2 + q K_2^2 + B (K_0 K_1 + K_1 K_0) + q C_0 K_0^2$$
$$+ q^{-1} C_1 K_1^2 + (1 + q) D_0 K_0 + (1 + q^{-1}) D_1 K_1,$$

commutes in $AW(3)$ with the generators $K_0$, $K_1$, $K_2$. 

Tom Koornwinder lecture 2: Zhedanov’s algebra
Choice of structure constants

Let $a, b, c, d$ be complex parameters.

Let $e_1, e_2, e_3, e_4$ be the elementary symmetric polynomials in $a, b, c, d$.

Put for the structure constants:

\[
\begin{align*}
B & := (1 - q^{-1})^2(e_3 + qe_1), \\
C_0 & := (q - q^{-1})^2, \\
C_1 & := q^{-1}(q - q^{-1})^2 e_4, \\
D_0 & := -q^{-3}(1 - q)^2(1 + q)(e_4 + qe_2 + q^2), \\
D_1 & := -q^{-3}(1 - q)^2(1 + q)(e_1 e_4 + qe_3).
\end{align*}
\]
Basic representation of $AW(3)$

Let $A_{\text{sym}}$ be the space of symmetric Laurent polynomials $f[z] = f[z^{-1}]$.

Let $L$ be the operator acting on $A_{\text{sym}}$ for which the Askey-Wilson polynomials are eigenfunctions:

$$(Lf)[z] := A[z] (f[qz] - f[z]) + A[z^{-1}] (f[q^{-1}z] - f[z]) + (1 + q^{-1}abcd) f[z],$$


Then $K_0$ and $K_1$ satisfy the relations in $AW(3)$ with the given structure constants when they are realized as:

$$(K_0 f)[z] := (Lf)[z],$$

$$(K_1 f)[z] := (z + z^{-1}) f[z].$$

The representation of $AW(3)$ thus obtained is called the basic representation.
The basic representation realized on the space of terminating sequences

The Askey-Wilson polynomials can be interpreted as the kernel of an intertwining operator between the basic representation of $AW(3)$ and an equivalent representation on the space of terminating sequences $\{u_n\}_{n=0,1,2,\ldots}$.

Concretely, let $\langle \cdot, \cdot \rangle$ be the inner product for which the Askey-Wilson polynomials are orthogonal: $\langle P_n, P_m \rangle = h_n \delta_{n,m}$.

Define a map $f \mapsto \hat{f}$ from $A_{\text{sym}}$ onto the space of terminating sequences by $\hat{f}(n) := \langle f, P_n \rangle$. Then, corresponding to

$$K_0 P_n = \lambda_n P_n, \quad K_1 P_n = P_{n+1} + B_n P_n + C_n P_{n-1}$$

we have:

$$(K_0 f)(n) = \lambda_n \hat{f}(n),$$

$$(K_1 f)(n) = \hat{f}(n + 1) + B_n \hat{f}(n) + C_n \hat{f}(n - 1).$$
Clearly, $AW(3)$ can equivalently be described as an algebra with two generators $K_0, K_1$ and with two relations

$$(q + q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = BK_1 + C_0K_0 + D_0,$$

$$(q + q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = BK_0 + C_1K_1 + D_1.$$ 

Then the Casimir operator $Q$ can be written as

$$Q = K_1K_0K_1K_0 - (q^2 + 1 + q^{-2})K_0K_1K_0K_1 + (q + q^{-1})K_0^2K_1^2 + (q + q^{-1})(C_0K_0^2 + C_1K_1^2) + B((q + 1 + q^{-1})K_0K_1 + K_1K_0) + (q + 1 + q^{-1})(D_0K_0 + D_1K_1).$$
A duality for Askey-Wilson polynomials

From
\[
\frac{P_n[z; a, b, c, d \mid q]}{P_n[a; a, b, c, d \mid q]} = 4\phi_3\left(\begin{array}{c}
q^{-n}, q^{n-1}abcd, az, az^{-1} \\
ab, ac, ad
\end{array}; q, q\right)
\]

we have for \(m = 0, 1, 2, \ldots\) that
\[
\frac{P_n[q^m a; a, b, c, d \mid q]}{P_n[a; a, b, c, d \mid q]} = 4\phi_3\left(\begin{array}{c}
q^{-n}, q^{n-1}abcd, q^m a^2, q^{-m} \\\nab, ac, ad
\end{array}; q, q\right)
\]
\[
= 4\phi_3\left(\begin{array}{c}
q^{-n}, q^n a'^2, q^{m-1} a' b' c' d', q^{-m} \\\na' b', a' c', a' d'
\end{array}; q, q\right) = \frac{P_m[q^n a'; a', b', c', d' \mid q]}{P_m[a'; a', b', c', d' \mid q]},
\]

where
\[
a' = (q^{-1}abcd)^{\frac{1}{2}}, \quad b' = \frac{ab}{a'}, \quad c' = \frac{ac}{a'}, \quad d' = \frac{ad}{a'}.
\]
A duality for $AW(3)$

Not surprising, there is a similar duality for $AW(3)$.

Concretely, the relations

$$(q + q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = BK_1 + C_0K_0 + D_0,$$

$$(q + q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = BK_0 + C_1K_1 + D_1.$$  

with the structure parameters expressed in terms of $a, b, c, d$ as before, are preserved under the anti-automorphism generated by:

$$K_0 \rightarrow aK_1, \quad K_1 \rightarrow (q^{-1}abcd)^{-\frac{1}{2}}K_0, \quad a \rightarrow (q^{-1}abcd)^{\frac{1}{2}},$$

$$b \rightarrow \frac{ab}{(q^{-1}abcd)^{\frac{1}{2}}}, \quad c \rightarrow \frac{ac}{(q^{-1}abcd)^{\frac{1}{2}}}, \quad d \rightarrow \frac{ad}{(q^{-1}abcd)^{\frac{1}{2}}}.$$  

Also $a^{-2}Q$ ($Q$ the Casimir operator) is preserved under this transformation.
In the basic representation the Casimir operator $Q$ can be computed to become a constant scalar:

$$(Qf)[z] = Q_0 f[z],$$

where

$$Q_0 := q^{-4}(1 - q)^2 \left( q^4(e_4 - e_2) + q^3(e_1^2 - e_1e_3 - 2e_2) \\
- q^2(e_2e_4 + 2e_4 + e_2) + q(e_3^2 - 2e_2e_4 - e_1e_3) + e_4(e_1 - e_2) \right).$$

This fits nicely with the fact that the basic representation is irreducible for generic values of $a, b, c, d$.

The relation $Q = Q_0$ does not hold generally in $AW(3)$. We will pass to a quotient algebra of $AW(3)$ with this relation.
A faithful representation on $\mathcal{A}_{\text{sym}}$

Assumptions
$q \neq 0, \quad q^m \neq 1 \ (m = 1, 2, \ldots),$
$a, b, c, d \neq 0, \quad abcd \neq q^{-m} \ (m = 0, 1, 2, \ldots)$.

Definition
$AW(3, Q_0)$ is the algebra $AW(3)$ with additional relation
$Q = Q_0$.

Theorem (THK, 2007)
$AW(3, Q_0)$ has the elements

$$K_0^n (K_1 K_0)^l K_1^m \quad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1)$$

as a linear basis.

The basic representation of $AW(3, Q_0)$ on $\mathcal{A}_{\text{sym}}$ is faithful.
More representations of \( AW(3) \)

Assume \( abcd > 0 \). Recall:  
\[
[K_0, K_1]_q = K_2, \\
[K_1, K_2]_q = BK_1 + C_0 K_0 + D_0, \\
[K_2, K_0]_q = BK_0 + C_1 K_1 + D_1,
\]
realized on \( A_{\text{sym}} \) by

\[
(K_0 f)[z] := (q^{-1} abcd)^{-\frac{1}{2}} (Lf)[z], \\
(K_1 f)[z] := (z + z^{-1}) f[z],
\]
where \( C_0 = C_1 = (q - q^{-1})^2 \),

\[
B = q^{\frac{1}{2}} (1 - q^{-1})^2 e_4^{-\frac{1}{2}} (e_3 + q e_1),
\]

\[
D_0 = -q^{-5/2} (1 - q)^2 (1 + q) e_4^{-\frac{1}{2}} (e_4 + q e_2 + q^2),
\]

\[
D_1 = -q^{-2} (1 - q)^2 (1 + q) e_4^{-1} (e_1 e_4 + q e_3).
\]

Chosen values of \( B, D_0, D_1 \) impose three algebraic constraints on \( a, b, c, d \), but \( abcd \) can freely vary over the positive reals, leading to different representations of \( AW(3) \) (probably with different Casimir values).
More representations of $AW(3)$ (continued)

With $(K_0 f)[z] := (q^{-1} abcd)^{-\frac{1}{2}} (Lf)[z]$ we have

$$K_0 P_n[.; a, b, c, d | q] = ((abcdq^{2n-1})^{\frac{1}{2}} + (abcdq^{2n-1})^{-\frac{1}{2}}) \times P_n[.; a, b, c, d | q].$$

If $a', b', c', d'$ and $a, b, c, d$ give rise to the same structure constants $B, D_0, D_1$, while

$$a' b' c' d' = q^{2k} abcd \quad \text{for some } k \in \mathbb{Z},$$

then $P_n[.; a, b, c, d | q]$ and $P_{n-k}[.; a', b', c', d' | q]$ have the same eigenvalue for $K_0$. 
Yet another equivalent form for the generators and relations of $AW(3)$

Replace $K_0$ by $K_0 + \nu_0$ and $K_1$ by $K_1 + \nu_1$ ($\nu_0, \nu_1$ scalars). Also let $K_2 := [K_0, K_1]$ (the ordinary commutator). Write $R := 2 - q - q^{-1}$. Also use the notation for the anticommutator: $\{X, Y\} := XY + YX$. Then

\[
[K_1, K_2] = RK_1K_0K_1 + R\nu_1\{K_0, K_1\} + R\nu_0 K_1^2 + (2R\nu_0\nu_1 + B)K_1 \\
+ (R\nu_1^2 + C_0)K_0 + R\nu_0\nu_1^2 + B\nu_1 + C_0\nu_0 + D_0,
\]

\[
[K_2, K_0] = RK_0K_1K_0 + R\nu_1 K_0^2 + R\nu_0\{K_0, K_1\} + (2R\nu_0\nu_1 + B)K_0 \\
+ (R\nu_0^2 + C_1)K_1 + R\nu_0^2\nu_1 + B\nu_0 + C_1\nu_1 + D_1.
\]

After renaming the structure constants this becomes:

\[
[K_1, K_2] = RK_1K_0K_1 + S\{K_0, K_1\} + TK_1^2 + BK_1 + C_0K_0 + D_0,
\]

\[
[K_2, K_0] = RK_0K_1K_0 + SK_0^2 + T\{K_0, K_1\} + BK_0 + C_1K_1 + D_1.
\]
So we have the algebra generated by $K_0, K_1, K_2$ and with relations

\[
[K_0, K_1] = K_2, \\
[K_1, K_2] = R K_1 K_0 K_1 + S\{K_0, K_1\} + T K_1^2 + B K_1 + C_0 K_0 + D_0, \\
[K_2, K_0] = R K_0 K_1 K_0 + S K_0^2 + T\{K_0, K_1\} + B K_0 + C_1 K_1 + D_1.
\]

For suitable values of the structure constants $R = 2 - q - q^{-1}$, $S$, $T$, $B$, $C_0$, $C_1$, $D_0$, $D_1$ any system of OP’s in the ($q$-)Askey scheme can be associated with this algebra. For $R \neq 0$ we are in the $q$-Askey scheme, for $R = 0$ we are in the Askey scheme. For $R = S = T = 0$ we are in the case of a Lie algebra.
1. Restrict $a, b, c, d$ to the case of the continuous $q$-Jacobi polynomials:
\[ a = q^{1/2 \alpha + 1/4}, \ b = q^{1/2 \alpha + 3/4}, \ c = -q^{1/2 \beta + 1/4}, \ d = -q^{1/2 \beta + 3/4}. \]

2. Then, with $x = \frac{1}{2} (z + z^{-1})$,
\[
\frac{P_n[z]}{P_n[q^{1/2 \alpha + 1/4}]} = 4\phi_3 \left( q^{-n}, q^{n+\alpha+\beta+1}, q^{1/2 \alpha + 1/4} z, q^{1/2 \alpha + 1/4} z^{-1}; q, q \right) 
\]
\[
\frac{q^{1/2}}{2F_1} \left( -n, n + \alpha + \beta + 1; \frac{1}{2}(1 - x) \right) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}. 
\]

3. Consider $AW(3)$ with these $a, b, c, d$ and let
\[ K_0 \rightarrow -(1 - q)^2 K_0 + 1 + q^{\alpha + \beta + 1}, \quad K_1 \rightarrow 2K_1. \]

4. There appears the limit case of $AW(3)$ corresponding to the Jacobi polynomials.
In the Jacobi case $AW(3)$ is generated by $K_0, K_1, K_2$ with relations

\[
\begin{align*}
[K_0, K_1] &= K_2, \\
[K_1, K_2] &= 2K_1^2 - 2, \\
[K_2, K_0] &= 2\{K_0, K_1\} - (\alpha + \beta)(\alpha + \beta + 2)K_1 + \beta^2 - \alpha^2.
\end{align*}
\]

This is realized on the space of polynomials by:

\[
\begin{align*}
(K_0 f)(x) &= (1 - x^2)f''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)f'(x), \\
(K_1 f)(x) &= x f(x),
\end{align*}
\]

and

\[
K_0 P_n^{(\alpha,\beta)} = -n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}.
\]
Many representations of $AW(3)$ for Jacobi

Consider $AW(3)$ generated by $K_0, K_1, K_2$ for Jacobi parameters $(\gamma, 0)$, i.e.:

$$[K_0, K_1] = K_2,$$
$$[K_1, K_2] = 2K_1^2 - 2,$$
$$[K_2, K_0] = 2\{K_0, K_1\} - \gamma(\gamma + 2)K_1 - \gamma^2.$$

For any real $t$ this is realized by:

$$(K_0 f)(x) = (1 - x^2)f''(x) - (\gamma e^{-t} + (\gamma e^t + 2)x)f'(x)$$
$$- \left(\frac{1}{4}\gamma^2(e^{2t} - 1) + \frac{1}{2}\gamma(e^t - 1)\right)f(x),$$

$$(K_1 f)(x) = x f(x),$$

In this realization $P_n^{(\gamma \cosh t, \gamma \sinh t)}$ is an eigenfunction of $K_0$ with eigenvalue $-n(n + \gamma e^t + 1) - \left(\frac{1}{4}\gamma^2(e^{2t} - 1) + \frac{1}{2}\gamma(e^t - 1)\right)$. 

Tom Koornwinder lecture 2: Zhedanov’s algebra