Extended abstract of the lecture

Orthogonal polynomials in several variables potentially useful in pde

by Tom H. Koornwinder, T.H.Koornwinder@uva.nl

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Tom H. Koornwinder

A system of orthogonal polynomials (OP’s) \( \{p_n\}_{n=0}^\infty \) on \( \mathbb{R} \) with respect to a positive measure \( \mu \) on \( \mathbb{R} \) is called classical if there is a second order differential operator \( L \) such that \( Lp_n = \lambda_n p_n \) \( (n = 0, 1, 2, \ldots) \) for certain eigenvalues \( \lambda_n \). By a theorem of Bochner [1] there are three families of classical OP’s (up to an affine transformation of the argument of the OP):

1. Hermite: \( p_n = H_n \), \( d\mu(x) = e^{-x^2} dx \) on \( \mathbb{R} \),
   \[(Lf)(x) = \frac{1}{2}f''(x) - xf'(x), \quad \lambda_n = -n.\]
2. Laguerre: \( p_n = L_n^\alpha \), \( d\mu(x) = x^\alpha e^{-x} dx \) on \([0, \infty)\), \( \alpha > -1 \),
   \[(Lf)(x) = xf''(x) + (\alpha + 1 - x)f'(x), \quad \lambda_n = -n.\]
3. Jacobi: \( p_n = P_n^{(\alpha, \beta)} \), \( d\mu(x) = (1-x)^\alpha(1+x)^\beta dx \) on \([-1, 1]\), \( \alpha, \beta > -1 \),
   \[(Lf)(x) = (1-x^2)f''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)f'(x), \quad \lambda_n = -n(n+\alpha+\beta+1).\]

Let \( \mu \) be a positive measure on \( \mathbb{R}^d \) such that \( \int_{\mathbb{R}} |x|^\alpha d\mu(x) < \infty \) \( (\alpha \in (\mathbb{Z}_{\geq 0})^d) \) and the support of \( \mu \) has nonempty interior. Let \( \mathcal{P}_n \) consist of all polynomials \( p \) of degree \( \leq n \) such that \( \int_{\mathbb{R}^d} pqd\mu = 0 \) for all polynomials \( q \) of degree \( < n \). Then \( \mathcal{P}_n \) has the same dimension \( \binom{n+d-1}{n} \) as the space of homogeneous polynomials of degree \( n \) in \( d \) variables. Furthermore, the spaces \( \mathcal{P}_n \) \( (n = 0, 1, 2, \ldots) \) are mutually orthogonal in \( L^2(\mu) \). We call \( \{\mathcal{P}_n\}_{n=0}^\infty \) a system of orthogonal polynomials with respect to the measure \( \mu \).

As a refinement of this notion we may choose an orthogonal basis \( \{p_n\}_{\alpha_1+\cdots+\alpha_d=n} \) for each space \( \mathcal{P}_n \), and call the polynomials \( p_n \) orthogonal polynomials. Of course, there are many ways to choose such orthogonal bases.

A system \( \{\mathcal{P}_n\} \) of orthogonal polynomials in \( d \) variables is called classical if there is a second order pdo \( L \) acting on the space of polynomials such that \( \mathcal{P}_n \) is an eigenspace of \( L \) for a certain eigenvalue \( \lambda_n \) \( (n = 0, 1, 2, \ldots) \). As a refinement there may be, apart from \( L = L_1, d-1 \) further pdo’s \( L_2, \ldots, L_d \) such that \( L_1, L_2, \ldots, L_d \) commute, are self-adjoint with respect to \( \mu \), and have one-dimensional joint eigenspaces. Then we have OP’s \( p_n \) with \( L_ip_n = \lambda_n^{(i)} p_n \).

It was shown by Krall & Sheffer [8] and Kwon, Lee & Littlejohn [9] that there are five families of classical orthogonal polynomials in 2 variables, as follows:

1. \( dp(x, y) = e^{-x^2-y^2} dx dy \) on \( \mathbb{R}^2 \), \( L = \frac{1}{2}(\partial_{xx} + \partial_{yy}) - x\partial_x - y\partial_y, \quad \lambda_n = -n.\)
2. \( dp(x, y) = x^\alpha y^\beta e^{-x-y} dx dy \) on \([0, \infty) \times [0, \infty)\), \( \alpha, \beta > -1 \),
   \( L = x\partial_{xx} + y\partial_{yy} + (1 + \alpha - x)\partial_x + (1 + \beta - y)\partial_y, \quad \lambda_n = -n.\)
3. \( dp(x, y) = y^\beta e^{-x^2-y^2} dx dy \) on \( \mathbb{R} \times [0, \infty) \), \( \beta > -1 \),
   \( L = \frac{1}{2}x\partial_{xx} + y\partial_{yy} - x\partial_x + (1 + \beta - y)\partial_y, \quad \lambda_n = -n.\)
4. \( dp(x, y) = x^\alpha y^\beta(1-x-y)^\gamma dx dy \) on \( \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, x + y \leq 1\} \), \( \alpha, \beta, \gamma > -1 \), \( L = x(1-x)\partial_{xx} + y(1-y)\partial_{yy} - 2xy\partial_{xy} + (\alpha + 1 - (\alpha + \beta + \gamma + 3)x)\partial_x + (\beta + 1 - (\alpha + \beta + \gamma + 3)y)\partial_y, \quad \lambda_n = -n(n + \alpha + \beta + \gamma + 2).\)
5. \( d\mu(x, y) = (1 - x^2 - y^2)^\alpha dx dy \) on \( \{(x, y) \in \mathbb{R}^2 \mid (x^2 + y^2) \leq 1\} \), \( \alpha > -1 \), 
\[ L = (1 - x^2)\partial_{xx} + (1 - y^2)\partial_{yy} - 2xy\partial_{xy} - (2\alpha + 3)(x\partial_x + y\partial_y), \]
\[ \lambda_n = -n(n + 2\alpha + 2). \]

Orthogonal bases \( \{p_{n,k}\}_{k=0,1,...,n} \) for \( \mathcal{P}_n \) \((n = 0, 1, 2, \ldots) \) in these five cases can be obtained by Gram-Schmidt orthogonalization of the monomials \( 1, x, y, x^2, xy, y^2, \ldots, x^n, x^{n-1}y, \ldots, x^{n-k}y^k, \ldots \). The resulting polynomials are as follows.

1. \( p_{n,k}(x, y) = H_{n-k}(x)H_k(y). \)
2. \( p_{n,k}(x, y) = L_{n-k}^\alpha(x)L_n^\beta(y). \)
3. \( p_{n,k}(x, y) = H_{n-k}(x)L_n^\beta(y). \)
4. \( p_{n,k}(x, y) = P_n(\alpha,\beta+\gamma+2k+1)(1 - 2x)(1 - x)^kP_k^{(\beta,\gamma)}(1 - 2y/(1 - x)). \)
5. \( p_{n,k}(x, y) = P_n(\alpha+k+\frac{1}{2},\alpha+k+\frac{1}{2})(1 - x^2)^{k/2}P_k^{(\alpha,\alpha)}(y/\sqrt{1 - x^2}). \)

The expansions in monomials of these polynomials \( p_{n,k} \) do not involve all monomials \( x^{m-j}y^j \) with \( (m, j) \) equal or less than \( (n, k) \) in the lexicographic ordering. For classes 1, 2 and 3 \( p_{n,k}(x, y) \) only contains monomials \( x^{m-j}y^j \) with \( m - j \leq n - k \) and \( j \leq k \). For classes 4 and 5 \( p_{n,k}(x, y) \) only contains monomials \( x^{m-j}y^j \) with \( m \leq n \) and \( j \leq k \). Furthermore, in these five cases there is a second order differential operator \( L_2 \) commuting with \( L \) which has the \( p_{n,k} \) as eigenfunctions with eigenvalue only depending on \( k \).

The OP’s \( p_{n,k} \) for case 4 (on the triangular region), as explicitly given above, were introduced by Proriol [10] in 1967. They were mentioned in the survey paper by Koornwinder [7] in 1975. Their special case \( \alpha = \beta = \gamma = 0 \) (constant weight function) was rediscovered by Dubiner [2] in 1991, who was motivated by applications to finite elements. Dubiner’s paper was much quoted in this context. For a while, the special functions and finite elements communities were not aware that they had a joint interest. But in 2000 Hesthaven & Teng [4] referred to Proriol’s paper, while later Karniadakis & Sherwin in their book [6] had ample references to papers on special functions. Conversely, in 2001 Dunkl & Xu referred in their book [3] to Dubiner’s paper.

Another important orthogonal system for case 5 on the disk is as follows.

\( R_{m,n}^\alpha(z) := \text{const.} \left\{ \begin{array}{ll}
P_n^{(\alpha,m-n)}(2|z|^2 - 1)z^{m-n}, & m \geq n, \\
P_m^{(\alpha,n-m)}(2|z|^2 - 1)\overline{z}^{m-n}, & n \geq m
\end{array} \right. 
((m, n) \in (\mathbb{Z}_{\geq 0})^2, z \in \mathbb{C}, \alpha > -1). \)

Then \( R_{m,n}^\alpha(z) = \text{const.} z^m\overline{z}^n + \text{polynomial in } z, \overline{z} \text{ of lower degree.} \)

and \( \int_{x^2+y^2<1} R_{m,n}^\alpha(x+iy) \overline{R}_{k,l}^\alpha(x+iy) (1 - x^2 - y^2)^\alpha dx dy = 0 \) \((m, n) \neq (k, l)) \).

For \( \alpha = 0 \) these polynomials are called Zernike polynomials. They were introduced by Zernike [11] in 1934 for applications in optics and are still much used there. The polynomials \( R_{m,n}^\alpha \) for general \( \alpha \) first occurred in Zernike & Brinkman [12].
For numerical applications it is important that Jacobi polynomials can be approximated by polynomials which are orthogonal on finitely many equidistant points. These are the \textit{Hahn polynomials} \(Q_n(x; \alpha, \beta, N) (n = 0, 1, \ldots, N)\) satisfying
\[
\sum_{x=0}^{N} (Q_n(x; \alpha, \beta, N) \left(\frac{\alpha + x}{x}\right) \left(\frac{\beta + N - x}{N - x}\right) = 0 \quad (n \neq m).
\]
The approximation is: \(\lim_{N \to \infty} Q_n(Nx; \alpha, \beta, N) = \text{const.} P_n^{(\alpha, \beta)}(1 - 2x)\).

From the Hahn polynomials we can build polynomials (Karlin & McGregor [5])
\[
Q_{n,k}(x, y; \alpha, \beta, \gamma, N) := Q_{n-k}(x; \alpha, \beta+\gamma+2k+1, N-k) \left(\frac{N-x}{k}\right) Q_k(y; \beta, \gamma, N-x)
\]
which are orthogonal on the set \(\{(x, y) \in \mathbb{Z}^2 | x, y \geq 0, x + y \leq N\}\) with respect to the weights
\[
w(x, y; \alpha, \beta, \gamma, N) := \left(\frac{\alpha + x}{x}\right) \left(\frac{\beta + y}{y}\right) \left(\frac{\gamma + N - x - y}{N - x - y}\right).
\]
They approximate the polynomials of class 4 on the triangle:
\[
\lim_{N \to \infty} Q_{n,k}(Nx, Ny; \alpha, \beta, \gamma, N) = \text{const.} p_n^{(\alpha, \beta, \gamma)}(x, y),
\]
which looks promising for applications.

\textbf{References}