Okounkov’s $BC$-type interpolation Macdonald polynomials and their $q = 1$ limit

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T. H. Koornwinder,

*Okounkov’s BC-type interpolation Macdonald polynomials and their $q = 1$ limit*,

http://arxiv.org/abs/1408.5993
Prelude: one variable, $q = 1$

<table>
<thead>
<tr>
<th></th>
<th>$P_m(x)$</th>
<th>$x^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>monomial</td>
<td>$P_m^i(x)$</td>
<td>$(-1)^m(-x)_m$</td>
</tr>
<tr>
<td>factorial</td>
<td>$P_m^i(x; \alpha)$</td>
<td>$(-1)^m(\alpha - x)_m(\alpha + x)_m$</td>
</tr>
<tr>
<td>quadr. factorial</td>
<td>$P_m^i(x; \alpha)$</td>
<td>$(1-\alpha)_m(\alpha + m - 1)_m$</td>
</tr>
<tr>
<td>Jacobi</td>
<td>$P_m^i(x; \alpha, \beta)$</td>
<td>$\frac{(1-\alpha)_m(\alpha + m - 1)_m}{(m+\alpha+\beta+1)_m} \binom{\alpha}{\alpha+1}^{-m} \binom{1}{\alpha+1}^m 2F_1 \left( -m, m+\alpha+\beta+1 \right; x)$</td>
</tr>
</tbody>
</table>

$$P_m^i(x) = x(x-1) \ldots (x-m+1),$$
$$P_m^i(x; \alpha) = (x^2 - \alpha^2)(x^2 - (\alpha + 1)^2)(x^2 - (\alpha + m - 1)^2),$$

(even) monic polynomials determined by vanishing properties.

Limits:

$$P_m(x; \alpha, \beta) + P_m^i(x) = P_m(x) + \text{lower degree},$$
$$P_m^i(x) = P_m(x^2) + \text{lower degree};$$

$$P_m^i(x + \alpha, \alpha) \xrightarrow{\alpha \to \infty} P_m(x),$$
$$P_m(x; \alpha, \beta) \xrightarrow{\alpha \to \infty} P_m(x - 1).$$
Prelude: one variable, \( q = 1 \), binomial formulas

### Monomial

\[ P_m(x) \]

### Factorial

\[ P^\text{ip}_m(x) \]

### Quadratic Factorial

\[ P^\text{ip}_m(x; \alpha) \]

### Jacobi

\[ P_m(x; \alpha, \beta) \]

\[
\frac{P_m(x; \alpha, \beta)}{P_m(0; \alpha, \beta)} = \sum_{k=0}^{m} \frac{(-m)_k(m + \alpha + \beta + 1)_k}{(\alpha + 1)_k k!} x^k
\]

\[
\alpha' := \frac{1}{2}(\alpha + \beta + 1) \sum_{k=0}^{m} \frac{P^\text{ip}_k(m + \alpha'; \alpha')}{P^\text{ip}_k(k + \alpha'; \alpha')} \frac{P_k(x)}{P_k(0; \alpha)} \quad \alpha \to \infty
\]

\[
(1 - x)^m = \sum_{k=0}^{m} \binom{m}{k} (-x)^k, \quad \text{i.e.,}
\]

\[
\frac{P_m(1 - x)}{P_m(1)} = \sum_{k=0}^{m} \frac{P^\text{ip}_k(m)}{P^\text{ip}_k(k)} \frac{P_k(-x)}{P_k(1)}.
\]
Throughout $0 < q < 1$. Four classes of polynomials:

<table>
<thead>
<tr>
<th>Type</th>
<th>Polynomial</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>monomial</td>
<td>$P_m(x; q)$</td>
<td>$x^m$</td>
</tr>
<tr>
<td>$q$-factorial</td>
<td>$P_{ip}^m(x; q)$</td>
<td>$x^m(x^{-1}; q)_m$</td>
</tr>
<tr>
<td>quadr. $q$-factorial</td>
<td>$P_{ip}^m(x; q, a)$</td>
<td>$\frac{(ax, ax^{-1}; q)_m}{(-1)^m q^{\frac{1}{2}m(m-1)} a^m}$</td>
</tr>
<tr>
<td>Askey-Wilson</td>
<td>$\frac{P_m(x; q_1, q_2, q_3, q_4)}{P_m(a_1, q_1, a_2, a_3, a_4)}$</td>
<td>$4\phi_3 \left( q^{-n}, q^{n-1} a_1 a_2 a_3 a_4, a_1 x, a_1 x^{-1} ; q, q \right)$</td>
</tr>
</tbody>
</table>

Put $\langle x; y \rangle := x + x^{-1} - y - y^{-1}$.

\[
P_{ip}^m(x; q) = (x - 1)(x - q) \ldots (x - q^{m-1}),
\]

\[
P_{ip}^m(x; q, a) = \prod_{j=0}^{m-1} (x + x^{-1} - a q^j - a^{-1} q^{-j}) = \prod_{j=0}^{m-1} \langle x; a q^j \rangle,
\]

monic polynomials resp. monic symmetric Laurent polynomials determined by their vanishing properties.
Prelude: one variable, $q$-case, limits

monomial $P_m(x; q)$
$q$-factorial $P_{m}^{ip}(x; q)$
quadr. $q$-factorial $P_{m}^{ip}(x; q, a)$
Askey-Wilson $P_{m}^{ip}(x; q, q, a, a_1, a_2, a_3, a_4)$

\[
P_{m}^{ip}(x; q, a) \quad \begin{cases} 
P_{m}(x; q, a) \\ P_{m}(x; q) 
\end{cases} = P_{m}(x; q) + \text{lower degree};
\]

\[
a^{-m} P_{m}^{ip}(ax; q, a) \xrightarrow{a \to \infty} P_{m}^{ip}(x; q), \\
a_1^{-m} P_{m}(a_1 x; q, a_1, a_2, a_3, a_4) \xrightarrow{a_1 \to \infty} P_{m}(x; q).
\]
Prelude: one variable, $q$-case, binomial formulas

\[ a^{-m}P_m(ax; q, a) \xrightarrow{a \to \infty} P_m(x; q), \]
\[ a_1^{-m}P_m(a_1x; q, t; a_1, a_2, a_3, a_4) \xrightarrow{a_1 \to \infty} P_m(x; q). \]

\[
\frac{P_m(x; q; a_1, a_2, a_3, a_4)}{P_m(a_1; q; a_1, a_2, a_3, a_4)} = \sum_{k=0}^{m} \frac{q^k}{(a_1 a_2, a_1 a_3, a_1 a_4, q; q)_k} \times (q^{-m}, q^{m-1} a_1 a_2 a_3 a_4; q)_k \ (a_1 x, a_1 x^{-1}; q)_k
\]
\[
a' := (q^{-1} a_1 a_2 a_3 a_4)^{1/2} \sum_{k=0}^{m} \frac{P_{k}^{ip}(q^m a'_1; q, a'_1)}{P_{k}^{ip}(q^k a'_1; q, a'_1)} \frac{P_{k}^{ip}(x; q, a_1)}{P_k(a_1; q; a_1, a_2, a_3, a_4)}
\]

\[
x := a_1 x, a_1 \to \infty
\]

\[
x^m = 2\phi_0\left(\frac{q^{-m}, x^{-1}}{q}; q, q^m x\right) = \sum_{k=0}^{m} \frac{(q^{-m}, x^{-1}; q)_{k}}{(-1)^k q^{1/2} k(k-1)} (q^m x)^{k},
\]

i.e., \[
\frac{P_m(x; q)}{P_m(1; q)} = \sum_{k=0}^{m} \frac{P_{k}^{ip}(q^m; q)}{P_{k}^{ip}(q^k; q)} \frac{P_{k}^{ip}(x; q)}{P_k(1; q)}.\]
**Prelude: one variable, limits for \( q \uparrow 1 \)**

<table>
<thead>
<tr>
<th>Class</th>
<th>( P_m(x; q) )</th>
<th>( x^m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>monomial</td>
<td>( P_m(x; q) )</td>
<td>( x^m(1/q; q)_m )</td>
</tr>
<tr>
<td>( q )-factorial</td>
<td>( P_m(x; q) )</td>
<td>( x^m(1/q; q)_m )</td>
</tr>
<tr>
<td>quadr. ( q )-factorial</td>
<td>( P_m(x; q, a) )</td>
<td>( (ax, ax^{-1}; q)_m )</td>
</tr>
<tr>
<td>Askey-Wilson</td>
<td>( \frac{P_m(x; q, a_1, a_2, a_3, a_4)}{P_m(a_1; q, a_2, a_3, a_4)} )</td>
<td>( 4\phi_3 \left( \frac{q^{-n}, q^{n-1} a_1 a_2 a_3 a_4, a_1 x, a_1 x^{-1}}{a_1 a_2, a_1 a_3, a_1 a_4}; q, q \right) )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
P_m(x; q) & \to P_m(x), \\
(q - 1)^{-m} P_m^i p(q^x; q) & \to P_m^i p(x), \\
P_m^i p(x; q) & \to P_m(x - 1), \\
(1 - q)^{-2m} P_m^i p(q^x; q, q^\alpha) & \to P_m^i p(x; \alpha), \\
P_m^i p(x; q, q^\alpha) & \to (-4)^m P_m(\frac{1}{4}(2 - x - x^{-1})), \\
P_m(x; q, q^\alpha, q^{\alpha+1}, -q^{\beta+1}, 1, -1) & \to (-4)^m P_m(\frac{1}{4}(2 - x - x^{-1}); \alpha, \beta).
\end{align*}
\]
Fixed $n$. $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$) is a partition. Set of all such $\lambda$ is denoted by $\Lambda_n$. $\lambda$ has length $\ell(\lambda) := |\{j \mid \lambda_j > 0\}|$ and weight $|\lambda| := \lambda_1 + \cdots + \lambda_n$.

Example $\lambda = (5, 3, 3, 1)$ has Young diagram

```
  0 0 0
  0 0 1
  0 0 0
```

dominance partial ordering:

$\mu \leq \lambda$ iff $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ ($i = 1, \ldots, n$).

inclusion partial ordering:

$\mu \subset \lambda$ iff $\mu_i \leq \lambda_i$ ($i = 1, \ldots, n$).

$W_n := S_n \ltimes (\mathbb{Z}_2)^n$.

$x = (x_1, \ldots, x_n)$, $x^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n}$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ partition.

$m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu$, $\tilde{m}_\lambda(x) := \sum_{\mu \in W_n \lambda} x^\mu$. 
Throughout $n \geq 2, \ 0 < q < 1, \ 0 < t < 1, \ \tau > 0$.

<table>
<thead>
<tr>
<th></th>
<th>Type</th>
<th>Polynomial</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A$</td>
<td>Macdonald</td>
<td>$P_\lambda(x; q, t)$</td>
</tr>
<tr>
<td>2</td>
<td>$A$</td>
<td>interpolation Macdonald</td>
<td>$P_{\lambda}^{ip}(x; q, t)$</td>
</tr>
<tr>
<td>3</td>
<td>$BC$</td>
<td>$BC_n$ interpolation Macdonald</td>
<td>$P_{\lambda}^{ip}(x; q, t, a)$</td>
</tr>
<tr>
<td>4</td>
<td>$BC$</td>
<td>Koornwinder</td>
<td>$P_\lambda(x; q, t; a_1, a_2, a_3, a_4)$</td>
</tr>
<tr>
<td>5</td>
<td>$A$</td>
<td>Jack</td>
<td>$P_\lambda(x; \tau)$</td>
</tr>
<tr>
<td>6</td>
<td>$A$</td>
<td>interpolation Jack</td>
<td>$P_{\lambda}^{ip}(x; \tau)$</td>
</tr>
<tr>
<td>7</td>
<td>$BC$</td>
<td>$BC_n$ interpolation Jack</td>
<td>$P_{\lambda}^{ip}(x; \tau, a)$ (new)</td>
</tr>
<tr>
<td>8</td>
<td>$BC$</td>
<td>Jacobi</td>
<td>$P_\lambda(x; \tau; \alpha, \beta)$</td>
</tr>
</tbody>
</table>

For 1, 2, 5, 6, 8 and for $P_{\lambda}^{ip}(x^{\frac{1}{2}}; \tau, \alpha)$ in 7: 
$$= m_\lambda + \sum_{\mu < \lambda} u_{\lambda, \mu} m_\mu.$$ 
For 3, 4: 
$$= \tilde{m}_\lambda + \sum_{\mu < \lambda} u_{\lambda, \mu} \tilde{m}_\mu.$$ 

Definition by orthogonality for 1, 4, 5, 8.
Inventors of interpolation functions

interpolation Jack:

interpolation Macdonald:
Sahi (1996); Knop (1997); Okounkov (1998)

$BC_n$ interpolation Macdonald:
Okounkov (1998)
Orthogonality

\[ \Delta(z) := \Delta_+(z)\Delta_+(z^{-1}), \quad \oint \frac{dz}{z} := \prod_{j=1}^{n} \oint \frac{dz_j}{z_j}. \]

**Macdonald and Jack:** \[ \oint P_\lambda(z) \, m_\mu(z^{-1}) \, \Delta(z) \, \frac{dz}{z} = 0 \text{ if } \mu < \lambda. \]

\[ \Delta_+(z) = \prod_{1 \leq i < j \leq n} \frac{(z_i z_j^{-1}; q)_\infty}{(z_i z_j^{-1}; q)_\infty} \text{ resp. } \prod_{1 \leq i < j \leq n} (1 - z_i z_j^{-1})^\tau. \]

Homogeneous of degree \(|\lambda|\).

**Koornwinder:** \[ \oint P_\lambda(z) \, \tilde{m}_\mu(z) \, \Delta(z) \, \frac{dz}{z} = 0 \text{ if } \mu < \lambda. \]

\[ \Delta_+(z) := \prod_{j=1}^{n} \frac{(z_j^2; q)_\infty}{(a_1 z_j, a_2 z_j, a_3 z_j, a_4 z_j; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(z_i z_j, z_i z_j^{-1}; q)_\infty}{(t z_i z_j, t z_i z_j^{-1}; q)_\infty}. \]

**Jacobi:** \[ \int_{[0,1]^n} P_\lambda(x) \, m_\mu(x) \, \Delta(x) \, dx = 0 \text{ if } \mu < \lambda. \]

\[ \Delta(x) := \prod_{j=1}^{n} x_j^\alpha (1 - x_j)^\beta \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\tau}. \]

In all cases **full orthogonality** can be proved.
Interpolation

Let \( \delta := (n - 1, n - 2, \ldots, 1, 0) \).

\[
\begin{aligned}
P^\text{ip}_\lambda (\mu + \tau \delta; \tau) \\
P^\text{ip}_\lambda (\mu + \tau \delta + \alpha; \tau, \alpha) \\
P^\text{ip}_\lambda (q^\mu t^\delta; q, t) \\
P^\text{ip}_\lambda (q^\mu t^\delta a; q, t, a)
\end{aligned}
\right\} = 0 \quad \text{if not } \lambda \subset \mu,
\]

in particular

\[= 0 \quad \text{if } |\mu| \leq |\lambda|, \mu \neq \lambda.\]
Young diagram of partition $\lambda$ consists of boxes $(i, j)$ with $i = 1, \ldots, \ell(\lambda)$ and $j = 1, \ldots, \lambda_i$. The conjugate partition $\lambda'$ has transposed diagram. Then $\ell(\lambda') = \lambda_1$.

Example $\lambda = (7, 5, 5, 2, 2), \lambda' = (5, 5, 3, 3, 3, 1, 1)$:

\[\begin{array}{c}
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \\
\cdot
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \\
\cdot
\end{array}
\end{array}\]

If $\mu \subset \lambda$ then $\lambda - \mu := \{s \in \lambda \mid s \notin \mu\}$ is a skew diagram.

$\lambda - \mu$ is a horizontal strip if $\lambda'_j - \mu'_j \leq 1$ ($j = 1, \ldots, \lambda_1$).

Then write $\mu \preceq \lambda$.

Example $\mu = (5, 5, 3, 2, 1), \lambda - \mu =$

\[\begin{array}{c}
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \\
\cdot
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \\
\cdot
\end{array}
\end{array}\]

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BC-type interpolation Macdonald polynomials
The diagram of $\lambda \in \Lambda_n$ becomes a \textit{tableau} $T$ if the boxes $(i,j)$ of $\lambda$ are filled by numbers $T(i,j)$. Then $T$ is called a \textit{semistandard} tableau of \textit{shape} $\lambda$ with entries in $\{1, \ldots, n\}$ if $T(i,j) \in \{1, 2, \ldots, n\}$ is weakly increasing in $j$ and strongly increasing in $i$.

Similarly $T$ is a \textit{reverse semistandard} tableau if $T(i,j)$ is weakly decreasing in $j$ and strongly decreasing in $i$. If $T$ is semistandard then $S(i,j) := n + 1 - T(i,j)$ gives a reverse semistandard tableau $S$. 

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\textit{BC}-type interpolation Macdonald polynomials
Example (semistandard tableau)
\(\lambda = (7, 5, 5, 2, 2), \ell(\lambda) = 5, n = 6:\)

\[
\begin{array}{cccccc}
1 & 1 & 1 & 3 & 4 & 6 6 \\
2 & 2 & 2 & 5 & 5  \\
3 & 3 & 5 & 6 & 6  \\
4 & 5  \\
5 & 6
\end{array}
\]

In general, let:
\(\lambda^{(i)} := \{ s \in \lambda \mid T(s) \leq i \}\), this is a partition.

Then: \(0^n = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(n-1)} \subset \lambda^{(n)} = \lambda\) and \(\lambda^{(i)} - \lambda^{(i-1)} = \{ s \in \lambda \mid T(s) = i \}\).

\(T\) restricted to the diagram of \(\lambda^{(i)}\) is a semistandard tableau of shape \(\lambda^{(i)}\) with entries in \(\{1, \ldots, i\}\). In the example:

\((\emptyset) \subset (3) \subset (3, 3) \subset (4, 3, 2) \subset (5, 3, 2, 1) \subset (5, 5, 3, 2, 1) \subset (7, 5, 5, 2, 2)\).
Combinatorial formula for Macdonald polynomials

\[ P_\lambda(x; q, t) = \sum_T \psi_T(q, t) \prod_{s \in \lambda} x_T(s), \]

sum over all semistandard tableaux \( T \) of shape \( \lambda \) with entries in \( \{1, \ldots, n\} \), so \( T(s) = i \) for \( s \) in the horizontal strip \( \lambda^{(i)} - \lambda^{(i-1)} \).

\[ \psi_T(q, t) = \prod_{i=1}^n \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q, t). \] Hence

\[ P_\lambda(x; q, t) = \sum_T \prod_{i=1}^n \left( \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q, t) x_i^{\left|\lambda^{(i)}\right| - \left|\lambda^{(i-1)}\right|} \right) \]

\[ = \sum_T \prod_{i=1}^n P_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i; q, t), \] where

\[ P_{\lambda/\mu}(z; q, t) := \psi_{\lambda/\mu}(q, t) z^{\left|\lambda\right| - \left|\mu\right|} \quad \text{if } \lambda \in \Lambda_n, \mu \in \Lambda_{n-1}, \mu \preceq \lambda, \]

and \( := 0 \) otherwise.

Hence we have the \textit{branching formula}

\[ P_\lambda(x_1, \ldots, x_{n-1}, x_n; q, t)) = \sum_{\mu} P_{\lambda/\mu}(x_n; q, t) P_\mu(x_1, \ldots, x_{n-1}; q, t). \]
Combinatorial formula (cntd.)

\[ P_\lambda(x; q, t) = \sum_T \prod_{i=1}^n \left( \psi_{\lambda(i)/\lambda(i-1)}(q, t) x_i^{|\lambda(i)|-|\lambda(i-1)|} \right). \]

Then \( \psi_{\lambda/\mu}(q, t) = \psi'_{\lambda'/\mu'}(t, q) \), where

\[ e_r(x) P_\lambda(x; q, t) = \sum_\mu \psi'_{\mu/\lambda}(q, t) P_\mu(x; q, t) \quad (\text{Pieri formula}). \]

\[ \psi_{\mu/\nu}(q, t) = \prod_{s \in (R \setminus C)_{\mu/\nu}} \frac{b_\nu(s; q, t)}{b_\mu(s; q, t)}. \]

Here \( s \in (R \setminus C)_{\mu/\nu} \) iff \( s \in \nu \) in a row of \( \mu \) intersecting with \( \mu - \nu \) but outside each column of \( \mu \) intersecting with \( \mu - \nu \).

\[ b_\lambda(s; q, t) := \frac{1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}}{1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}}, \]

\[ a_\lambda(i, j) := \lambda_i - j, \quad l_\lambda(i, j) := |\{ k > i \mid \lambda_k \geq j \}|. \]
Combinatorial formula (cntd.)

For a reverse semistandard tableau $T$ with entries in $\{1, \ldots, n\}$ put $\lambda^{(i)} := \{s \in T \mid T(s) > i\}$. Then
$$\lambda^{(i-1)} - \lambda^{(i)} = \{s \in T \mid T(s) = i\}.$$  

Now write the combinatorial formula for Macdonald polynomials as a sum over reverse semistandard tableaux:

$$P_\lambda(x; q, t) = \sum_T \psi_T(q, t) \prod_{s \in \lambda} x_{T(s)},$$

where

$$\psi_T(q, t) = \prod_{i=1}^n \psi_{\lambda^{(i-1)}/\lambda^{(i)}}(q, t).$$
Macdonald polynomials:

\[ P_\lambda(x; q, t) = \sum_T \psi_T(q, t) \prod_{s \in \lambda} x_{T(s)}. \]

Interpolation Macdonald polynomials:

\[ P^{ip}_\lambda(x; q, t) = \sum_T \psi_T(q, t) \prod_{s \in \lambda} (x_{T(s)} - q^{a'_{\lambda}(s)} t^{n - T(s) - l'_{\lambda}(s)}). \]

BC\(_n\) interpolation Macdonald polynomials:

\[ P^{ip}_\lambda(x; q, t, a) = \sum_T \psi_T(q, t) \prod_{s \in \lambda} \langle x_{T(s)}; q^{a'_{\lambda}(s)} t^{n - T(s) - l'_{\lambda}(s)} a \rangle. \]

\[ a'_{\lambda}(i, j) := j - 1, \quad l'_{\lambda}(i, j) := i - 1, \quad \langle x; y \rangle := x + x^{-1} - y - y^{-1}. \]

The \(T\)-sums are over all reverse tableaux \(T\) of shape \(\lambda\) with entries in \(\{1, \ldots, n\}\).
Combinatorial formulas ($q = 1$)

\[
\psi_T(\tau) := \prod_{i=1}^{n} \prod_{s \in (R \setminus C)_{\lambda(i-1)/\lambda(i)}} \frac{b_{\lambda(i)}(s; \tau)}{b_{\lambda(i-1)}(s; \tau)}.
\]

\[
b_{\lambda}(s; \tau) := \frac{a_{\lambda}(s) + \tau(l_{\lambda}(s) + 1)}{a_{\lambda}(s) + \tau l_{\lambda}(s) + 1}.
\]

Jack polynomials: \( P_{\lambda}(x; \tau) = \sum_T \psi_T(\tau) \prod_{s \in \lambda} x_{T(s)} \).

Interpolation Jack polynomials:

\[
P_{\lambda}^{ip}(x; \tau) = \sum_T \psi_T(\tau) \prod_{s \in \lambda} \left( x_{T(s)} - a'_{\lambda}(s) - \tau(n - T(s) - l'_{\lambda}(s)) \right).
\]

\( BC_n \) interpolation Jack polynomials: (new)

\[
P_{\lambda}^{ip}(x; \tau, \alpha) := \lim_{q \uparrow 1} (1 - q)^{-2|\lambda|} P_{\lambda}^{ip}(q^x; q, q^\tau, q^\alpha) = \sum_T \psi_T(\tau) \times \prod_{s \in \lambda} \left( x_{T(s)}^2 - (a'_{\lambda}(s) + \tau(n - T(s) - l'_{\lambda}(s)) + \alpha)^2 \right).
\]

The \( T \)-sums over all reverse tableaux \( T \) of shape \( \lambda \) with entries in \( \{1, \ldots, n\} \).
\[ P_\lambda(x; q, q^\tau; q^{\alpha+1}, -q^{\beta+1}, 1, -1) \to (-4)^{|\lambda|} P_\lambda(\frac{1}{4}(2 - x - x^{-1}); \tau; \alpha, \beta), \]

\[ (1 - q)^{-2|\lambda|} P_\lambda^{ip}(q^x; q, q^\tau, q^{\alpha}) \to P_\lambda^{ip}(x; \tau, \alpha), \]

\[ P_\lambda^{ip}(x; q, q^\tau, q^{\alpha}) \to (-4)^{|\lambda|} P_\lambda(\frac{1}{4}(2 - x - x^{-1}); \tau), \]

\[ (q - 1)^{-|\lambda|} P_\lambda^{ip}(q^x; q, q^\tau) \to P_\lambda^{ip}(x; \tau), \]

\[ P_\lambda^{ip}(x; q, q^\tau) \to P_\lambda(x - 1^n, \tau), \]

\[ P_\lambda(x; q, q^\tau) \to P_\lambda(x; \tau). \]
Limits, $q$ fixed

**Highest term:**

\[
\begin{align*}
P_{\lambda}(x; q, t; a_1, a_2, a_3, a_4) & \end{align*}
\]

\[
\begin{align*}
P_{\lambda}(x; q, t) & = P_{\lambda}(x; q, t) + \text{degree lower than } |\lambda|; \\
P_{\lambda}^{\text{ip}}(x; q, t, a) & = P_{\lambda}(x; q, t) + \text{degree lower than } |\lambda|,
\end{align*}
\]

and

\[
\begin{align*}
P_{\lambda}(x; \tau; \alpha, \beta) & = P_{\lambda}(x; \tau) + \text{degree lower than } |\lambda|,
\end{align*}
\]

\[
\begin{align*}
P_{\lambda}^{\text{ip}}(x; \tau, \alpha) & = P_{\lambda}(x^2; \tau) + \text{degree in } x^2 \text{ lower than } |\lambda|.
\end{align*}
\]

**Parameter to $\infty$:**

\[
\begin{align*}
a^{-|\lambda|} P_{\lambda}^{\text{ip}}(ax; q, t, a) & \xrightarrow{a \to \infty} P_{\lambda}^{\text{ip}}(x; q, t), \\
a_1^{-|\lambda|} P_{\lambda}(a_1 x; q, t; a_1, a_2, a_3, a_4) & \xrightarrow{a_1 \to \infty} P_{\lambda}(x; q, t) \quad \text{(new)},
\end{align*}
\]

\[
\begin{align*}
(2\alpha)^{-|\lambda|} P_{\lambda}^{\text{ip}}(x + \alpha, \alpha, \tau) & \xrightarrow{\alpha \to \infty} P_{\lambda}^{\text{ip}}(x, \tau), \\
P_{\lambda}(x; \tau; \alpha, \beta) & \xrightarrow{\alpha \to \infty} P_{\lambda}(x - 1^n; \tau).
\end{align*}
\]
Binomial formula for Koornwinder polynomials

\[ a' := (q^{-1} a_1 a_2 a_3 a_4)^{1/2}, \]
\[ a_1 a_2 = a_1 a_2, \quad a_1 a_3 = a_1 a_3, \quad a_1 a_4 = a_1 a_4. \]

**Binomial formula** (Okounkov, Transf. Groups, 1998):

\[
\frac{P_\lambda(x; q, t; a_1, a_2, a_3, a_4)}{P_\lambda(t^\delta a_1; q, t; a_1, a_2, a_3, a_4)} = \sum_{\mu \subset \lambda} \frac{P^\text{ip}_{\mu}(q^\lambda t^\delta a'_1; q, t, a'_1)}{P^\text{ip}_{\mu}(q^\mu t^\delta a'_1; q, t, a'_1)} \frac{P^\text{ip}_\mu(x; q, t, a_1)}{P^\mu(t^\delta a_1; q, t; a_1, a_2, a_3, a_4)}.
\]

This implies the duality

\[
\frac{P_\lambda(q^\nu t^\delta a_1; q, t; a_1, a_2, a_3, a_4)}{P_\lambda(t^\delta a_1; q, t; a_1, a_2, a_3, a_4)} = \frac{P^\nu(t^\delta a'_1; q, t; a'_1, a'_2, a'_3, a'_4)}{P^\nu(t^\delta a'_1; q, t; a'_1, a'_2, a'_3, a'_4)}
\]

if

\[
\frac{P^\text{ip}_\mu(t^\delta a_1; q, t; a_1, a_2, a_3, a_4)}{P^\text{ip}_\mu(q^\mu t^\delta a_1; q, t, a_1)} = \frac{P^\mu(t^\delta a'_1; q, t; a'_1, a'_2, a'_3, a'_4)}{P^\mu(q^\mu t^\delta a'_1; q, t, a'_1)}.
\]

Clear if \( a'_1 = a_1 \). In general clear from evaluation formulas.

\[
P_\lambda(x; q, t) \frac{P_\lambda(t^\delta; q, t)}{P_\lambda(t^\delta; q, t)} = \sum_{\mu \subset \lambda} \frac{P^\mu (q^\lambda t^\delta; q, t)}{P^\mu (q^\mu t^\delta; q, t)} \frac{P^\mu (x; q, t)}{P^\mu (t^\delta; q, t)}.
\]

This implies the duality

\[
\frac{P_\lambda(q^\nu t^\delta; q, t)}{P_\lambda(t^\delta; q, t)} = \frac{P_\nu(q^\lambda t^\delta; q, t)}{P_\nu(t^\delta; q, t)}.
\]

Compare with binomial formula for Koornwinder polynomials.

\[
P_\lambda(x; q, t; a_1, a_2, a_3, a_4) \frac{P_\lambda(t^\delta a_1; q, t; a_1, a_2, a_3, a_4)}{P_\lambda(t^\delta a_1; q, t; a_1, a_2, a_3, a_4)} = \sum_{\mu \subset \lambda} \frac{P^\mu (q^\lambda t^\delta a'_1; q, t, a'_1)}{P^\mu (q^\mu t^\delta a'_1; q, t, a'_1)} \frac{P^\mu (x; q, t, a_1)}{P^\mu (t^\delta a_1; q, t; a_1, a_2, a_3, a_4)}.
\]

**Theorem**

\[
\lim_{a_1 \to \infty} a_1^{-|\lambda|} P_\lambda(a_1 x; q, t; a_1, a_2, a_3, a_4) = P_\lambda(x; q, t)
\]

**Proof** Verify for \( x = q^\delta \) in evaluation formulas.
Binomial formula for $BC_n$ Jacobi polynomials

Let $\alpha' := \frac{1}{2}(\alpha + \beta + 1)$.

**Theorem**

$$
\frac{P_\lambda(x; \tau; \alpha, \beta)}{P_\lambda(0; \tau; \alpha, \beta)} = \sum_{\mu \subseteq \lambda} \frac{P^\text{ip}_\mu(\lambda + \tau \delta + \alpha'; \tau, \alpha')}{P^\text{ip}_\mu(\mu + \tau \delta + \alpha'; \tau, \alpha')} \frac{P_\mu(x; \tau)}{P_\mu(0; \tau; \alpha, \beta)}.
$$

**Proof**

Use

$$
\frac{P_\lambda(z; q, q^\tau; q^{\alpha+1}, -q^{\beta+1}, 1, -1)}{P_\lambda(q^{\tau \delta+\alpha+1}; q, q^\tau; q^{\alpha+1}, -q^{\beta+1}, 1, -1)} = \sum_{\mu \subseteq \lambda} \frac{P^\text{ip}_\mu(q^{\lambda+\tau \delta+\alpha'}; q, q^\tau, q^{\alpha'})}{P^\text{ip}_\mu(q^{\mu+\tau \delta+\alpha'}; q, q^\tau, q^{\alpha'})} \frac{P^\text{ip}_\mu(z; q, q^\tau, q^{\alpha+1})}{P^\text{ip}_\mu(q^{\tau \delta+\alpha+1}; q, q^\tau; q^{\alpha+1}, -q^{\beta+1}, 1, -1)}
$$

together with (for $q \uparrow 1$) the limits

\[
\begin{align*}
P_\lambda(z; q, q^\tau; q^{\alpha+1}, -q^{\beta+1}, 1, -1) &\to (-4)^{|\lambda|} P_\lambda\left(\frac{1}{4}(2 - z - z^{-1}); \tau; \alpha, \beta\right), \\
P^\text{ip}_\lambda(z; q, q^\tau, q^{\alpha}) &\to (-4)^{|\lambda|} P_\lambda\left(\frac{1}{4}(2 - z - z^{-1}); \tau\right), \\
(1 - q)^{-2|\lambda|} P^\text{ip}_\lambda(q^x; q, q^\tau, q^{\alpha}) &\to P^\text{ip}_\lambda(x; \tau, \alpha).
\end{align*}
\]
So \( P_\lambda(x; \tau; \alpha, \beta) = \sum_{\mu \subset \lambda} c_{\lambda,\mu} P_\mu(x; \tau) \)

with \( c_{\lambda,\mu} \), up to products of elementary factors, given by

\[ P_{\lambda}^{ip}(\lambda + \tau \delta + \alpha'; \tau, \alpha') \]

which can be expressed by a certain sum over all reverse tableaux of shape \( \mu \) with entries in \( \{1, \ldots, n\} \).

On the other hand Macdonald (manuscript, 1987; arXiv:1309.4568) gives this as a certain sum over all standard tableaux of shape \( \lambda/\mu \) and with entries in \( 1, \ldots, |\lambda - \mu| \).

The relationship between both expressions is not clear.
Binomial formula for Jack polynomials

\[
P_\lambda(1 + x, \tau) = \frac{P_\lambda(1; \tau)}{P_\lambda(1; \tau)} = \sum_{\mu \subset \lambda} \frac{P_\mu^{ip}(\lambda + \tau \delta; \tau)}{P_\mu^{ip}(\mu + \tau \delta; \tau)} \frac{P_\mu(x; \tau)}{P_\mu(1; \tau)}.
\]

Hence

\[
\frac{P_\mu^{ip}(\lambda + \tau \delta; \tau)}{P_\mu^{ip}(\mu + \tau \delta; \tau)} = \binom{\lambda}{\mu}_\tau \quad \text{(Lassalle, 1990)}.
\]

Theorem (Beerends & K., unpublished; Rösler, K. & Voit, 2013)

\[
\lim_{\alpha \to \infty} P_\lambda(x; \tau; \alpha, \beta) = P_\lambda(x - 1; \tau).
\]

Proof Let \(\alpha \to \infty\) in

\[
\frac{P_\lambda(x; \tau; \alpha, \beta)}{P_\lambda(0; \tau; \alpha, \beta)} = \sum_{\mu \subset \lambda} \frac{P_\mu^{ip}(\lambda + \tau \delta + \alpha'; \tau, \alpha')}{P_\mu^{ip}(\mu + \tau \delta + \alpha'; \tau, \alpha')} \frac{P_\mu(x; \tau)}{P_\mu(0; \tau; \alpha, \beta)}
\]

and use that \((2\alpha)^{-|\lambda|} P_\lambda^{ip}(x + \alpha; \tau, \alpha) \to P_\lambda^{ip}(x; \tau), \quad \alpha \to \infty.\)

It is now sufficient to verify the Theorem for \(x = 0\) by using the evaluation formulas.
Reduction formula for $BC_n$ interpolation Macdonald polynomials

$$P_{\mu}^{ip}(z; q, t, a) = (-a)^n \mu_n q^{-\frac{1}{2}n \mu_n (\mu_n-1)} \prod_{j=1}^{n} ((zja; q)_{\mu_n} (z_j^{-1}a; q)_{\mu_n})$$

$$\times P_{\mu_1 n, \mu_2 n}^{ip}(z; q, t, q^{\mu_n} a).$$

In particular for $\mu = (\mu_1, \mu_2)$.

Also use that a sum over all reverse tableaux $T$ of shape $(m, 0)$ with entries in $\{1, 2\}$ is a single sum.

Thus the combinatorial formula for $P_{m,0}^{ip}(z_1, z_2; q, t, a)$ gives an explicit expression for $P_{m_1, m_2}^{ip}(z_1, z_2; q, t, a)$. 
Explicit formula for $BC_2$ interpolation Macdonald

\[
P_{m_1, m_2}^{ip}(z_1, z_2; q, t, a) = \frac{(-1)^{m_1-m_2}}{tm_1-m_2 a^{m_1+m_2} q^{\frac{1}{2} m_1 (m_1-1)} q^{\frac{1}{2} m_2 (m_2-1)}} \times (z_1 a, z_1^{-1} a, z_2 a, z_2^{-1} a; q) \times (q^{m_2 z_1 t a}, q^{m_2 z_1^{-1} t a}; q) \times 4 \phi_3\left(\begin{array}{c}
q^{-m_1+m_2}, t, q^{m_2 z_2 a}, q^{m_2 z_2^{-1} a} \\
q^{1-m_1+m_2} t^{-1}, q^{m_2 z_1 t a}, q^{m_2 z_1^{-1} t a}
\end{array} ; q, q\right).
\]

Thus, by Okounkov’s binomial formula, this gives an explicit formula for Macdonald-Koornwinder polynomials $P_{n_1, n_2}(z_1, z_2; q, t; a_1, a_2, a_3, a_4)$. 
Explicit formula for $BC_2$ interpolation

\[
P_{m_1,m_2}^{ip}(x_1, x_2; \tau, \alpha) = (\alpha + x_1, \alpha - x_1, \alpha + x_2, \alpha - x_2)_{m_2} \times (\tau + \alpha + x_1 + m_2, \tau + \alpha - x_1 + m_2)_{m_1-m_2} \times 4F_3\left(\begin{array}{c}
-m_1 + m_2, \tau, m_2 + \alpha + x_2, m_2 + \alpha - x_2 \\
1 - m_1 + m_2 - \tau, \tau + \alpha + x_1 + m_2, \tau + \alpha - x_1 + m_2
\end{array}; 1\right).
\]

Together with ($C_{m_1-m_2}^\tau$ is Gegenbauer polynomial)

\[
P_{m_1,m_2}(x_1, x_2; \tau) = \frac{(m_1 - m_2)!}{(\tau)_{m_1-m_2}} (x_1 x_2)^{1/2} (m_1 + m_2) \, C_{m_1-m_2}^\tau \left(\frac{x_1 + x_2}{2(x_1 x_2)^{1/2}}\right)
\]

we have by the binomial formula for $BC_2$ Jacobi polynomials an explicit expression for these polynomials.

Much earlier (1978) in a very different way obtained by K. & Sprinkhuizen.
Van Diejen & Emsiz, arXiv:1408.2280, obtained a branching formula and hence a combinatorial formula for Koornwinder polynomials.

\[ P_\lambda(x_1, \ldots, x_n; q, t; a_1, a_2, a_3, a_4) = \sum_{\mu} P_{\lambda/\mu}(x_n; q, t; a_1, a_2, a_3, a_4) \times P_\mu(x_1, \ldots, x_{n-1}; q, t; a_1, a_2, a_3, a_4). \]

The sum is over all \( \mu \) such that \( \ell(\mu) \leq n - 1 \) and \( \exists \nu \) with \( \mu \preceq \nu \preceq \lambda \).

\[ P_{\lambda/\mu}(x; q, t; a_1, a_2, a_3, a_4) = \sum_{k=0}^{d} B^k_{\lambda/\mu}(q, t; a_1, a_2, a_3, a_4) \langle x; a_1 \rangle_{q,k}, \]

\[ d = |\{ j \mid \lambda_j' = \mu_j' + 1 \}|, \quad \langle x; a_1 \rangle_{q,k} := \prod_{j=0}^{k-1} \langle x; q^j a_1 \rangle. \]
Branching formula for Koornwinder polynomials (cntd.)

For \( \mu \subset m^n \) let \((m^n - \mu)_j := m - \mu_{n+1-j} \) \((j = 1, \ldots, n)\).

\[
B_{\lambda/\mu}^k(q, t; a_1, a_2, a_3, a_4) = (-1)^{k + |\lambda| - |\mu|} C^{n\lambda_1 - \lambda', \lambda_1}_{(n-1)^\lambda_1 - \mu', \lambda_1 - k}(t; q; a_1, a_2, a_3, a_4),
\]
where \( C^{\mu, n}_{\lambda, r} \) are Pieri coefficients:

\[
E_r(x; t, a) := \sum_{1 \leq j_1 < \ldots < j_r \leq n} \prod_{i=1}^r \langle x_{j_i}; t^{j_i-1} a \rangle,
\]

\[
E_r(x; t, a) P_\lambda(x; q, t; a_1, a_2, a_3, a_4) = \sum_{\mu} C^{\mu, n}_{\lambda, r}(q, t; a_1, a_2, a_3, a_4) P_\mu(x; q, t; a_1, a_2, a_3, a_4).
\]

Here \( \lambda \in \Lambda_n \) and the sum runs over all \( \mu \in \Lambda_n \) such that there exists \( \nu \subset \lambda \cap \mu \) with \( \lambda - \nu \) and \( \mu - \nu \) vertical strips and

\[
|\lambda - \nu| + |\mu - \nu| \leq r.
\]
The branching formula for Koornwinder polynomials is much more complicated than the branching formula for Macdonald polynomials, but the latter one is obtained from the first one by taking the highest degree part at both sides of the identity.

There is no obvious relation between the Van Diejen-Emsiz branching formula for Koornwinder polynomials and Okounkov’s branching formula for $BC$ interpolation Macdonald polynomials.
Further literature

Further literature (cntd.)