Structure relation and raising/lowering operators for orthogonal polynomials

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Orthogonal polynomials $\{p_n(x)\}$: three-term recurrence relation

$$x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x).$$

Classical OP’s (Jacobi, Laguerre, Hermite):

- eigenfunctions of 2nd order differential operator
- derivative again OP
- Rodrigues formula

These three properties are generated by a pair of shift operators:

One lowers the degree and raises the parameters.
The other raises the degree and lowers the parameters.
Example

Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, orthogonal with respect to $(1 - x)^\alpha (1 + x)^\beta \, dx$ on $(-1, 1)$. Shift operator equations:

$$\frac{d}{dx} P_n^{(\alpha, \beta)}(x) = \mu_n P_{n-1}^{(\alpha+1, \beta+1)}(x), \quad (1)$$

$$(1 - x)^{-\alpha} (1 + x)^{-\beta} \frac{d}{dx} (1 - x)^{\alpha+1} (1 + x)^{\beta+1} P_{n-1}^{(\alpha+1, \beta+1)}(x)$$

$$\quad = \nu_n P_n^{(\alpha, \beta)}(x). \quad (2)$$

- *eigenfunctions of 2nd order differential operator*: compose (1) and (2).
- *derivative again OP*: by (1).
- *Rodrigues formula*: iterate (2).

**Wanted**: operators lowering or raising degree without parameter shift.
The classical OP’s \( \{p_n(x)\} \) satisfy:

- **three-term recurrence relation**
  \[
  x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x),
  \]  
  \( (3) \)

- **structure relation** (\( \pi(x) \) polynomial of degree \( \leq 2 \))
  \[
  \pi(x)p'_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x),
  \]  
  \( (4) \)

- **lowering relation**
  \[
  \pi(x)p'_n(x) - (\alpha_n x + \beta_n)p_n(x) = \gamma_n p_{n-1}(x),
  \]  
  \( (5) \)

- **raising relation**
  \[
  \pi(x)p'_n(x) - (\bar{\alpha}_n x + \bar{\beta}_n)p_n(x) = \bar{\gamma}_n p_{n+1}(x).
  \]  
  \( (6) \)

(5) and (6) obtained by eliminating a term from (3) and (4). However, lowering and raising operators dependent on \( n \).
Al-Salam & Chihara (1972) characterized classical OP’s as OP’s with structure relation or, equivalently, with lowering relation or raising relation.

Semi-classical orthogonal polynomials are OP’s \( \{ p_n(x) \} \) which satisfy the more general structure relation

\[
\pi(x) p'_n(x) = \sum_{j=n-s}^{n+t} a_{n,j} p_j(x)
\]

(\( \pi(x) \) a polynomial; \( s, t \) independent of \( n \)).
Conceptual generation of structure relation

Let \( \{p_n(x)\} \) be system of OP's, orthogonal w.r.t. measure \( d\mu \), satisfying three-term recurrence relation

\[
Xp_n = A_n p_{n+1} + B_n p_n + C_n p_{n-1} \quad ((Xf)(x) := x f(x)),
\]

and being eigenfunctions of some explicit operator \( D \), symmetric w.r.t. \( d\mu \): \( Dp_n = \lambda_n p_n \). Put \( \gamma_n := \lambda_{n+1} - \lambda_n \).

**Definition**

structure operator \( L := [D, X] = DX - XD \).

**Theorem**

\( L \) is skew-symmetric w.r.t. \( d\mu \), and we have structure equation

\[
Lp_n = \gamma_n A_n p_{n+1} - \gamma_{n-1} C_n p_{n-1}.
\]
By elimination of term from

\[ Xp_n = A_n p_{n+1} + B_n p_n + C_n p_{n-1}, \]

\[ Lp_n = \gamma_n A_n p_{n+1} - \gamma_{n-1} C_n p_{n-1}, \]

we get a lowering and raising relation:

\[-\gamma_n (x - B_n) p_n(x) + (Lp_n)(x) = - (\gamma_n + \gamma_{n-1}) C_n p_{n-1}(x), \]

\[ \gamma_{n-1} (x - B_n) p_n(x) + (Lp_n)(x) = (\gamma_n + \gamma_{n-1}) A_n p_{n+1}(x). \]
Example, Hermite

Example

Hermite polynomials $H_n(x)$, orthogonal w.r.t. $e^{-x^2} \, dx$ on $(-\infty, \infty)$.

$$(DH_n)(x) := \frac{1}{2} H''_n(x) - x H'_n(x) = -n H_n(x),$$

$$X H_n = \frac{1}{2} H_{n+1} + n H_{n-1},$$

$$(LH_n)(x) := ([D, X] H_n)(x) =$$

$$H'_n(x) - x H_n(x) = -\frac{1}{2} H_{n+1}(x) + n H_{n-1}(x).$$
Laguerre polynomials $L_n^\alpha(x)$, orthogonal w.r.t. $e^{-x}x^\alpha \, dx$ on $(0, \infty)$.

$$(DL_n^\alpha)(x) := x \frac{d^2}{dx^2} L_n^\alpha(x) + (\alpha + 1 - x) \frac{d}{dx} L_n^\alpha(x) = -n L_n^\alpha(x),$$

$$X L_n^\alpha = - (n + 1)L_{n+1}^\alpha + (2n + \alpha + 1)L_n^\alpha - (n + \alpha)L_{n-1}^\alpha,$$

$$(LL_n^\alpha)(x) := ([D, X]L_n^\alpha)(x)$$

$$= 2x \frac{d}{dx} L_n^\alpha(x) + (\alpha + 1 - x) L_n^\alpha(x)$$

$$= (n + 1)L_{n+1}^\alpha(x) - (n + \alpha)L_{n-1}^\alpha(x).$$
Example

Jacobi polynomials $P_{n}^{(\alpha,\beta)}(x)$.

\[ D := \frac{1}{2}(1 - x^2)\frac{d^2}{dx^2} + \frac{1}{2}(\beta - \alpha - (\alpha + \beta + 2)x)\frac{d}{dx}, \]

\[ \lambda_n = -\frac{1}{2}n(n + \alpha + \beta + 1), \quad \gamma_n = -\frac{1}{2}(2n + \alpha + \beta + 2). \]

Structure operator:

\[ (Lf)(x) := (1 - x^2)f'(x) - \frac{1}{2}(\alpha - \beta + (\alpha + \beta + 2)x)f(x) \]

Structure relation:

\[ \left( (1 - x^2)\frac{d}{dx} - \frac{1}{2}(\alpha - \beta + (\alpha + \beta + 2)x) \right)P_{n}^{(\alpha,\beta)}(x) = \]

\[ -\frac{(n + 1)(n + \alpha + \beta + 1)}{2n + \alpha + \beta + 1} P_{n+1}^{(\alpha,\beta)}(x) + \frac{(n + \alpha)(n + \beta)}{2n + \alpha + \beta + 1} P_{n-1}^{(\alpha,\beta)}(x). \]
Askey scheme

Wilson

cont. dual Hahn

Meixner-Pollaczek

Laguerre

Hermite

Jacobi

Meixner

Krawtchouk

dual Hahn

Hahn

Charlier
There was in 1977 a meeting at Oberwolfach on combinatorics run by Foata. I gave a talk about many of classical type orthogonal polynomials and it fell flat. Few there appreciated it.

Later in the week Mike Hoare, a physicist then at Bedford College, talked about some very nice work he and Mizan Rahman had done. In this talk he had an overhead of the polynomials they had dealt with, starting with Hahn polynomials at the top and moving down to limiting cases with arrows illustrating the limits which they had used. The audience did not seem to care much about the probability problem, but they were very excited about the chart he had shown and wanted copies. If there was that much interest in his chart, I thought that it should be extended to include all of the classical type polynomials which had been found.
Jacques Labelle’s Askey tableau poster

Tom H. Koornwinder  Structure relation for orthogonal polynomials
Eigenfunctions of structure operator

OP’s \( \{ p_n(x) \} \), \( \int_a^b p_n(x) p_m(x) \, d\mu(x) = \omega_n^{-1} \delta_{n,m} \).

Write structure relation as

\[ L_x(p_n(x)) = M_n(p_n(x)), \]

\( L_x \) skew symmetric operator on \( L^2([a, b], d\mu) \), \( M_n \) skew symmetric operator on \( l^2(\mathbb{N}, \omega_n) \).

Formally we expect eigenfunctions \( \phi_\lambda(x) \) of \( L_x \) and \( q_n(\lambda) \) of \( M_n \):

\[ L_x(\phi_\lambda(x)) = i\lambda \phi_\lambda(x), \quad M_n(q_n(\lambda)) = i\lambda q_n(\lambda) \quad (q_n/q_0 \text{ OP’s in } \lambda) \]

such that

\[ \int_a^b p_n(x) \phi_{-\lambda}(x) \, d\mu(x) = q_n(\lambda). \]
This works fine for Hermite, Laguerre and Jacobi:

- **Hermite:**
  \[ \phi_\lambda(x) = e^{\frac{1}{2}x^2 + i\lambda x}, \]
  \[ q_n(\lambda) = (2\pi)^{\frac{1}{2}} i^{-n} e^{-\frac{1}{2} \lambda^2} H_n(\lambda). \]

- **Laguerre:**
  \[ \phi_\lambda(x) = e^{\frac{1}{2}x} x^{\frac{1}{2}} (i\lambda - \alpha - 1), \]
  \[ q_n(\lambda) = i^{-n} 2^{\frac{1}{2} \alpha + 1} (i\lambda - \alpha - 1) \Gamma\left(\frac{1}{2}(\alpha + 1 - i\lambda)\right) \]
  \[ P_n^{\frac{1}{2}(\alpha + 1)}(1; \frac{1}{2}\lambda; \frac{1}{2}\pi) \]
  (Meixner-Pollaczek)

- **Jacobi:**
  \[
  \int_{-1}^{1} P_n^{(\alpha, \beta)}(x)(1 - x)^{\frac{1}{2}(\alpha - 1 + i\lambda)}(1 + x)^{\frac{1}{2}(\beta - 1 - i\lambda)} \, dx
  \]
  \[= \text{stuff} \times p_n\left(\frac{1}{2}\lambda; \frac{\alpha + 1}{2}, \frac{\beta + 1}{2}, \frac{\alpha + 1}{2}, \frac{\beta + 1}{2}\right) \]
  (continuous Hahn)

Possibly related to K (LNM 1171, 1985) and Groenevelt (IMRN, 2003).
Askey-Wilson polynomials

$q$-Analogue of Askey scheme, with Askey-Wilson polynomials $p_n(x; a, b, c, d \mid q)$ on top.

\[ p_n[z] = p_n\left(\frac{1}{2}(z + z^{-1})\right) \]

\[ := \frac{(ab, ac, ad; q)_n}{a^n} 4\phi_3\left(\frac{q^{-n}, q^{n-1}abcd, az, az^{-1}}{ab, ac, ad}; q, q\right). \]

Orthogonal w.r.t. inner product

\[ \langle f, g \rangle := \frac{1}{4\pi i} \oint_C f[z] g[z] w(z) \frac{dz}{z}, \]

\[ w(z) := \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty}, \]

where $C$ is unit circle traversed in positive direction.
Second order $q$-difference operator $D$:

$$Dp_n = \lambda_n p_n,$$

where

$$\frac{1}{2}(1 - q^{-1})(Df)[z] = v(z) f[qz] - (v(z) + v(z^{-1})) f[z]$$

$$+ v(z^{-1}) f[q^{-1}z],$$

$$v(z) = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)},$$

$$\frac{1}{2}(1 - q^{-1})\lambda_n = (q^{-n} - 1)(1 - abcdq^{n-1}).$$

Structure operator $L := [D, X]$, where $(Xf)[z] := \frac{1}{2}(z + z^{-1}) f[z]$:

$$(Lf)[z] = \frac{u[z] f[qz] - u[z^{-1}] f[q^{-1}z]}{z - z^{-1}},$$

where

$$u[z] = (1 - az)(1 - bz)(1 - cz)(1 - dz) z^{-2}.$$
Zhedanov (1991),
“Hidden symmetry” of Askey-Wilson polynomials.
Defines algebra $\mathcal{AW}(3)$ with generators $K_0, K_1, K_2$ and relations

$$[K_0, K_1]_q = K_2,$$
$$[K_1, K_2]_q = C_0 K_0 + B K_1 + D_0,$$
$$[K_2, K_0]_q = B K_0 + C_1 K_1 + D_1,$$

with $q$-commutator $[X, Y]_q := q^{\frac{1}{2}} XY − q^{-\frac{1}{2}} YX$
and with structure constants $B, C_0, D_0, C_1, D_1$.

Constants can be chosen such that relations are realized in terms of operators $D$ and $X$ for Askey-Wilson polynomials:

$$K_0 = \frac{1}{2}(1 − q^{-1}) D + 1 + q^{-1} abcd, \quad K_1 = X.$$

Then $K_2$ is $q$-structure operator.
Suppose that $\mathcal{AW}(3)$ acts on a vector space spanned by one-dimensional eigenspaces of $K_0$: $K_0 \psi_n = \lambda_n \psi_n$.

Then for each $\psi_n$ there are neighbouring eigenvectors $\psi_{n+1}, \psi_{n-1}$ such that

$$K_1 \psi_n = a_n \psi_{n+1} + b_n \psi_n + c_n \psi_{n-1},$$  

$$K_2 \psi_n = (q^2 \lambda_{n+1} - q^{-1} \lambda_n) a_n \psi_{n+1} + (q^2 - q^{-1}) \lambda_n b_n \psi_n$$

$$+ (q^2 \lambda_{n-1} - q^{-1} \lambda_n) c_n \psi_{n-1},$$

and $\lambda_{n+1} + \lambda_{n-1} = (q + q^{-1}) \lambda_n$,

and raising and lowering relations by elimination from (7), (8).

In the Askey-Wilson realization of $\mathcal{AW}(3)$ (7) is three-term recurrence relation and (8) is $q$-structure relation.
If we take Askey-Wilson realization of $AW(3)$ with $K_1 = X$ but $K_0 = \text{const.}$ instead of with special constant term added, and $K_2 := [K_0, K_1]_q$, then quadratic term occurs in right-hand side of second relation:

$$[K_1, K_2]_q$$ is linear combination of $D$, $X$, $X^2$ and $1$.

This is called $q$-string equation (Grünbaum & Haine, Terwilliger & Vidunas).

In this form $q = 1$ limit of $AW(3)$ possible with realization by Jacobi, etc.. Then string equation:


In Hermite case related to matrix models in quantum gravity (Witten, 1991).
Matrix models

\[ V(x) := \sum_{j=1}^{r} t_j x^j. \]

\[ Z := \int_{\mathcal{M}_n} e^{-\text{tr} V(M)} \, dM = \text{const.} \int_{\mathbb{R}^n} \prod_{i<j} (x_i - x_j)^2 \prod_i e^{-V(x_i)} \, dx_1 \ldots dx_n. \]

Gives rise to study of OP’s with respect to measure \( e^{-V(x)} \, dx \).

Suppose \( \{p_n\} \) OP’s on \( L^2(d\mu) \) and \( L \) skew symmetric operator on \( L^2(d\mu) \) with \( Lp_n = a_n p_{n+1} - c_n p_{n-1} \) and \( [L, X] = \pi(X) \) \((\pi \text{ polynomial})\).

Let \( p_n^{(t)} \) OP’s on \( L^2(e^{-V(x)} \, d\mu(x)) \). Then

\[ L^{(t)} := L - \frac{1}{2} \sum_{j \geq 1} j t_j x^{j-1} \text{ skew-symmetric with respect to } e^{-V(x)} \, d\mu(x) \text{ and } L^{(t)} p_n^{(t)} \in \text{Span}\{p_{n-r+1}^{(t)}, \ldots, p_{n+r-1}^{(t)}\} \text{ and } [L^{(t)}, X] = \pi(X) \text{ (string equation)}. \]
Macdonald polynomial $P_\lambda(x; q, t)$, root system $A_{n-1}$, $x = (x_1, \ldots, x_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ partition, $P_\lambda$ symmetric polynomial in $x$, homogeneous of degree $|\lambda|$.

Eigenfunction of $q$-difference operators $D_r$ ($r = 0, 1, \ldots, n$):

$$D_r P_\lambda = e_r(q^{\lambda_1} t^{n-1}, q^{\lambda_2} t^{n-2}, \ldots, q^{\lambda_n}) P_\lambda,$$

$$D_r = t^{\frac{1}{2}} r^{(r-1)} \sum_{|I|=r} \prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, i},$$

$$(T_{q, i} f)(x) := f(x_1, \ldots, qx_i, \ldots, x_n).$$

Elementary cases: $D_0 f = f$, $D_n f(x) = t^{\frac{1}{2}} n^{(n-1)} f(qx)$. 

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Structure relation for orthogonal polynomials
Symmetry and Pieri formula

Normalized Macdonald polynomials
\[ \tilde{P}_\lambda := P_\lambda / P_\lambda (t^{n-1}, t^{n-2}, \ldots, 1). \]

Symmetry:
\[ \tilde{P}_\mu (q_1^{\lambda_1} t^{n-1}, q_2^{\lambda_2} t^{n-2}, \ldots, q_n^{\lambda_n}) = \tilde{P}_\lambda (q_1^{\mu_1} t^{n-1}, q_2^{\mu_2} t^{n-2}, \ldots, q_n^{\mu_n}). \]

Pieri formula:
\[
e_r(x) \tilde{P}_\lambda (x) = \sum_{\substack{0 \leq \theta_j \leq 1 \\ |\theta| = r}} t^{\theta_2 + 2\theta_3 + \ldots + (n-1)\theta_n} \times \prod_{1 \leq i < j \leq n} \frac{1 - q^{\lambda_i - \lambda_j} t^{j-i+\theta_i-\theta_j}}{1 - q^{\lambda_i - \lambda_j} t^{j-i}} \tilde{P}_{\lambda + \theta}(x).
\]
“Casimir” Pieri formula (Pieri for $r = 1$):

$$ (x_1 + \cdots + x_n)\tilde{P}_\lambda(x) = \sum_{k=1,\ldots,n} A_{\lambda,k} \tilde{P}_{\lambda+\varepsilon_k}(x). \quad (9) $$

Structure relations ($r = 1, \ldots, n - 1$):

$$ [D_r, X_1 + \cdots + X_n]\tilde{P}_\lambda \\
= \sum_{k=1,\ldots,n} ((T_{q,k} - 1)e_r)(q^{\lambda_1}t^{n-1}, q^{\lambda_2}t^{n-2}, \ldots, q^{\lambda_n}) A_{\lambda,k} \tilde{P}_{\lambda+\varepsilon_k}. \quad (10) $$

These $n$ equations (9), (10), if linearly independent, will yield by elimination raising relations. $\tilde{P}_\lambda \rightarrow \tilde{P}_{\lambda+\varepsilon_k}$. Can be done explicitly analogous to work in Jack case by García Fuertes, Lorente & Perelomov (2001).
Explicit raising relations

\[ D(w) := \sum_{r=0}^{n} w^r D_r. \]

\[ D(w) \tilde{P}_\lambda = \left( \prod_{i=1}^{n} \left( 1 + wt^{n-i} q^{\lambda_i} \right) \right) \tilde{P}_\lambda. \]

\[ D(w) \left( (X_1 + \cdots + X_n) \tilde{P}_\lambda \right) = \sum_{k=1,\ldots,n} A_{\lambda,k} D(w) \tilde{P}_{\lambda+\varepsilon_k} \]

\[ = \sum_{k=1,\ldots,n} A_{\lambda,k} \prod_{i=1}^{n} \left( 1 + wt^{n-i} q^{\lambda_i+\delta_{i,k}} \right) \tilde{P}_{\lambda+\varepsilon_k} \]

\[ (w := -t^{i-n} q^{-\lambda_j}) = A_{\lambda,j} (1 - q) \prod_{i\neq j} (1 - q^{\lambda_i-\lambda_j} t^{j-i}) \tilde{P}_{\lambda+\varepsilon_j}. \]
Explicit raising relations, continued

Final form of raising relations:

$$\sum_{r=1}^{n-1} (-t^{j-n-\lambda_j})^r [D_r, X_1 + \cdots + X_n] \tilde{P}_\lambda$$

$$= A_{\lambda,j} (1 - q) \prod_{i \neq j} (1 - q^{\lambda_i - \lambda_j} t^{j-i}) \tilde{P}_{\lambda+\varepsilon_j}.$$

Similar results with case $e_{n-1}(X)\tilde{P}_\lambda$ of Pieri relations.

Use $P_{\lambda_1+1, \ldots, \lambda_n+1}(x) = x_1 x_2 \cdots x_n P_\lambda(x)$. Then

$$(x_1^{-1} + \cdots + x_n^{-1}) \tilde{P}_\lambda(x) = \sum_{k=1}^{n} B_{\lambda,k} \tilde{P}_{\lambda-\varepsilon_k}(x).$$

Hence further structure relations and further explicit lowering relations $\tilde{P}_\lambda \rightarrow \tilde{P}_{\lambda-\varepsilon_k}$.

More generally, explicit lowering and raising relations

$$\tilde{P}_\lambda \rightarrow \tilde{P}_{\lambda\pm\varepsilon_I}, \quad \varepsilon_I := \sum_{i \in I} \varepsilon_i.$$
(I thank A. M. Garsia for a comment on an earlier version of this slide.)

Explicit raising and lowering operators independent of $\lambda$, acting on integral form $J_\lambda$ of Macdonald polynomials:

$$K_m J_{\lambda_1,\lambda_2,\ldots,\lambda_m,0,...,0} = J_{\lambda_1+1,...,\lambda_m+1,0,...,0} \quad (m = 1, 2, \ldots, n),$$

$$M_m J_{\lambda_1,\lambda_2,\ldots,\lambda_m,0,...,0} = c_{\lambda,m} J_{\lambda_1-1,...,\lambda_m-1,0,...,0} \quad (m = 1, 2, \ldots, n).$$

These operators are given in several forms. The form of the operators which leads to a proof of part of the Macdonald conjectures for the double Kostka coefficients, does not seem to be expressible in terms of structure operators (except for $m = 1$).

However, Lapointe & Vinet (LMP, 1997) have the operators $K_m$ in a form which fits into the present scheme.
Lapointe-Vinet creation operators

Observe:

\[ D(w) \left( e_m(X) \tilde{P}_{\lambda_1,\ldots,\lambda_m,0,\ldots,0} \right) = \sum_{|\theta|=m, \ 0 \leq \theta_j \leq 1, \ \lambda+\theta \text{ partition}} A_{\lambda,\theta} D(w) \tilde{P}_{\lambda+\theta}. \]

There is a term with \( \theta = (1^m) \) and in all other terms \( \theta_{m+1} = 1 \), and

\[ D(w) \tilde{P}_{\lambda+\theta} = \left( \prod_{i=1}^{n} (1 + wt^{n-i} q^{\lambda_i+\theta_i}) \right) \tilde{P}_{\lambda+\theta}, \]

where the eigenvalue has \((m + 1)\)th factor \(1 + wt^{n-m-1} q^{\theta_{m+1}}\).

Hence

\[ D(-q^{-1} t^{m-n+1}) \left( e_m(X) \tilde{P}_{\lambda_1,\ldots,\lambda_m,0,\ldots,0} \right) = \text{const.} \ P_{\lambda_1+1,\ldots,\lambda_{m+1},0,\ldots,0}. \]

The operator \( D(-q^{-1} t^{m-n+1}) \circ e_m(X) \) only depends on \( m \): it is the same operator for all \( \lambda \) of length \( \leq m \).