More on Fourier series

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1 Some extensions to Chaps. 2 and 3 of the book *Fourier analysis, an introduction* by E. M. Stein and R. Shakarchi

Remark 1. (Alternative method for Exercise 16 in Ch.2)
We use the *Chebyshev polynomials* (of the first kind)

$$T_n(\cos\theta) := \cos(n\theta) \qquad (\theta \in \mathbb{R},\ n \in \mathbb{Z}_{\geq 0}).$$

The above definition determines $T_n(x)$ uniquely for $x \in [-1,1]$. We also see that $T_n(x)$ is a polynomial of degree n in x because

$$\cos\theta\,\cos n\theta = \tfrac{1}{2}\cos(n+1)\theta + \tfrac{1}{2}\cos(n-1)\theta,$$

hence

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) \qquad (n \in \mathbb{Z}_{>0}),$$

while $T_0(x) = 1$. Now the claim follows by induction w.r.t. n.

Now we prove the Weierstrass approximation theorem for $f \in C([a,b])$. Without loss of generality we may assume that $[a,b] = [-1,1]$. Put $g(\theta) := f(\cos\theta)$. Then g is continuous, even and 2π-periodic on $\mathbb{R}$. Hence $\widehat{g}(n) = \widehat{g}(-n)$ and

$$\sigma_N(g)(\theta) = \widehat{g}(0) + 2\sum_{n=1}^{N-1} \frac{N-n}{N}\,\widehat{g}(n)\,\cos n\theta.$$

Then $\sigma_N(g) \to g$ uniformly, certainly on $[0,\pi]$, as $N \to \infty$ (see Ch.2, Theorem 5.2). Put

$$f_{N-1}(x) := \widehat{g}(0) + 2\sum_{n=1}^{N-1} \frac{N-n}{N}\,\widehat{g}(n)\,T_n(x).$$

Then $f_{N-1}(\cos\theta) = \sigma_N(g)(\theta)$ and $f_{N-1}(x)$ is a polynomial of degree $\leq N-1$ in x. Then $f_{N-1} \to f$, uniformly on $[-1,1]$, as $N \to \infty$.

Remark 2. (Extension of Exercise 12 in Ch.2)
(b) Let $(c_n)_{n=1}^{\infty}$ be a sequence of real numbers, put $s_n := \sum_{k=1}^{n} c_k$ and $\sigma_n := n^{-1}\sum_{k=1}^{n} s_k$. Show that $\lim_{n\to\infty} s_n = \infty$ implies that $\lim_{n\to\infty} \sigma_n = \infty$. So a series diverging to $+\infty$ is not Cesàro summable.

(c) Show (by the same method as on p.84, Ch.3) that

$$\sum_{n=2}^{\infty} \frac{1}{n\log n} = \infty.$$

(d) Show that the trigonometric series

$$\sum_{|n|\geq 2} \frac{1}{|n|\log|n|}\, e^{inx}$$

cannot be the Fourier series of a 2π-periodic continuous function.
Hint Otherwise the series would be Cesàro summable for all x, certainly for $x = 0$.

(e) Conclude that $c_n = o(|n|^{-1})$ as $|n| \to \infty$ is not a sufficient condition in order that $\sum_{n\in\mathbb{Z}} c_n\, e^{inx}$ is the Fourier series of a 2π-periodic continuous function.

Remark 3. (Extension of Exercise 18 in Ch.3)
For a sequence $(c_n)_{n\in\mathbb{Z}}$ there is a hierarchy of its behaviour as $|n| \to \infty$ given by $c_n = \mathcal{O}(|n|^{-\alpha})$ or $c_n = o(n^{-\alpha})$ ($\alpha \in \mathbb{R}$). Then $c_n = o(|n|^{-\alpha}) \Longrightarrow c_n = \mathcal{O}(|n|^{-\alpha})$ and, with $\alpha > \beta$, $c_n = \mathcal{O}(|n|^{-\alpha}) \Longrightarrow c_n = o(|n|^{-\beta})$, but the converses of these implications are not valid.

Now let f be an arbitrary 2π-periodic function on $\mathbb{R}$, integrable over bounded intervals. Replace f by a (unique) 2π-periodic continuous function if the difference h of f with that function has $\int_{-\pi}^{\pi} |h(x)|\, dx = 0$. Then there are the following implications, and these implications are sharpest for the estimate for $\widehat{f}(n)$ in the above hierarchy:

$$\begin{aligned}
f \text{ is continuous } &\Longrightarrow \widehat{f}(n) = o(1);\\
f \text{ is continuous } &\Longleftarrow \widehat{f}(n) = O(|n|^{-1-\varepsilon}) \text{ for some } \varepsilon > 0;\\
f \text{ is continuously differentiable } &\Longrightarrow \widehat{f}(n) = o(|n|^{-1});\\
f \text{ is continuously differentiable } &\Longleftarrow \widehat{f}(n) = O(|n|^{-2-\varepsilon}) \text{ for some } \varepsilon > 0;\\
f \text{ is } C^k &\Longrightarrow \widehat{f}(n) = o(|n|^{-k});\\
f \text{ is } C^k &\Longleftarrow \widehat{f}(n) = O(|n|^{-k-1-\varepsilon}) \text{ for some } \varepsilon > 0;\\
f \text{ is } C^\infty &\Longleftrightarrow \widehat{f}(n) = O(|n|^{-k}) \text{ for all } k > 0.
\end{aligned}$$

Theorem 4 (Extension of Ch.3, Theorem 2.1**).**
Let f be an integrable function on the circle which has a **jump discontinuity** *at θ_0 in the sense that the two limits*

$$f(\theta_0^+) := \lim_{h\downarrow 0} f(\theta_0 + h), \quad f(\theta_0^-) := \lim_{h\uparrow 0} f(\theta_0 + h)$$

exist, and which is **right and left differentiable** *at θ_0 in the sense that the two limits*

$$f'(\theta_0^+) := \lim_{h\downarrow 0} \frac{f(\theta_0 + h) - f(\theta_0^+)}{h}, \quad f'(\theta_0^-) := \lim_{h\uparrow 0} \frac{f(\theta_0 + h) - f(\theta_0^-)}{h}$$

exist. Then $S_N(f)(\theta_0) \to \frac{1}{2}\left(f(\theta_0^+) + f(\theta_0^-)\right)$ as N tends to infinity.

See also Exercise 17 in Chapter 2, which formulates similar theorems for the Abel means and the Cesàro means.

Theorem 5 (Extension of Ch.3, Theorem 2.2**).**
Suppose f and g are two integrable functions defined on the circle, and for some θ_0 there exists an open interval I containing θ_0 such that $f(\theta) = g(\theta)$ for all $\theta \in I$. Then either $S_N(f)(\theta_0)$ and $S_N(g)(\theta_0)$ both converge as $N \to \infty$, while tending to the same limit, or both diverge as $N \to \infty$.

Remark 6. (Extension of Exercise 12 in Ch.3)
Observe that

$$\begin{aligned} D_N(x) &= \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)} \\ &= \frac{\sin(Nx)\cos(\frac{1}{2}x)+\cos(Nx)\sin(\frac{1}{2}x)}{\sin(\frac{1}{2}x)} \\ &= \sin(Nx)\cot(\tfrac{1}{2}x)+\cos(Nx) \\ &= \frac{2\sin(Nx)}{x}+\cos(Nx)+\big(\cot(\tfrac{1}{2}x)-2x^{-1}\big)\sin(Nx). \end{aligned}$$

Put

$$\phi(x) := \cot(\tfrac{1}{2}x)-2x^{-1} \qquad (0<|x|<2\pi), \tag{1}$$

and put $\phi(0) := 0$. Then

$$D_N(x) = \frac{2\sin(Nx)}{x}+\cos(Nx)+\phi(x)\sin(Nx). \tag{2}$$

It can be proved, as an exercise, that:

a) ϕ is continuous on $(-2\pi, 2\pi)$;

b) $\phi'(0) = -\frac{1}{6}$;

c) ϕ is C^1 on $(-2\pi, 2\pi)$ and strictly decreasing;

d) $\phi(\pi) = -2\pi^{-1}$, $\phi(-\pi) = 2\pi^{-1}$, $\max_{|x|\le\pi}|\phi(x)| = 2\pi^{-1}$.

Integration of (2) yields for $0<x\le 2\pi$:

$$\begin{aligned} \tfrac{1}{2}\int_0^x D_N(t)\,dt &= \int_0^x \frac{\sin(Nt)}{t}\,dt+\frac{\sin(Nx)}{2N}+\tfrac{1}{2}\int_0^x \phi(t)\,\sin(Nt)\,dt \\ &= \int_0^{Nx}\frac{\sin s}{s}\,ds+\frac{\sin(Nx)}{2N}-\frac{\phi(x)\cos(Nx)}{2N}+\frac{1}{2N}\int_0^x \phi'(t)\cos(Nt)\,dt. \end{aligned}$$

Hence, if $0<a<\pi$ then

$$\tfrac{1}{2}\int_0^x D_N(t)\,dt = \int_0^{Nx}\frac{\sin s}{s}\,ds+\mathcal{O}(N^{-1}), \quad \text{uniformly as } N\to\infty \text{ for } 0<x\le a. \tag{3}$$

In particular,

$$\tfrac{1}{2}\pi = \tfrac{1}{2}\int_0^\pi D_N(t)\,dt = \int_0^{N\pi}\frac{\sin s}{s}\,ds+\mathcal{O}(N^{-1}) = \int_0^\infty \frac{\sin s}{s}\,ds. \tag{4}$$

Remark 7. (Concerning Exercise 20 in Ch.3)
Let f be the sawtooth function, for which the Fourier series was computed in Ch.2, Exercise 8:

$$f(x) \sim \sum_{n\neq 0}(2in)^{-1}\,e^{inx}.$$

Then (see Ch.3, Exercise 20)

$$S_N(f)(x) = \sum_{0<|n|\le N} (2in)^{-1}\, e^{inx} = \tfrac{1}{2}\int_0^x \Big(\textstyle\sum_{0<|n|\le N} e^{inx}\Big)\, dx = \tfrac{1}{2}\int_0^x D_N(t)\, dt - \tfrac{1}{2}x.$$

Now let $0 < a < \pi$, use (3) and observe that $f(x) = \frac{1}{2}\pi - \frac{1}{2}x$ on $(0, 2\pi)$. Thus

$$S_N(f)(x) - f(x) = \int_0^{Nx} \frac{\sin s}{s}\, ds - \tfrac{1}{2}\pi + \mathcal{O}(N^{-1}), \quad \text{uniformly as } N \to \infty \text{ for } 0 < x \le a. \tag{5}$$

Define the function Si (*integral sine*) (see also (4)) by

$$\operatorname{Si}(y) := \int_0^y \frac{\sin t}{t}\, dt \quad (y \ge 0), \qquad \operatorname{Si}(\infty) := \lim_{y\to\infty} \operatorname{Si}(y) = \tfrac{1}{2}\pi. \tag{6}$$

Then Si is increasing on intervals $(2k\pi, (2k+1)\pi)$ $(k \in \mathbb{Z}_{\ge 0})$ and Si is decreasing on intervals $((2k+1)\pi, (2k+2)\pi)$ $(k \in \mathbb{Z}_{\ge 0})$, and

$$\operatorname{Si}(\pi) > \operatorname{Si}(3\pi) > \operatorname{Si}(5\pi) > \ldots > \operatorname{Si}(\infty) = \tfrac{1}{2}\pi > \ldots > \operatorname{Si}(4\pi) > \operatorname{Si}(2\pi) > \operatorname{Si}(0) = 0.$$

Thus Si is positive on $(0, \infty)$ and it attains its absolute maximum on $[0, \infty)$ at π. A numerical computation yields that

$$\int_0^\pi \frac{\sin t}{t}\, dt = \operatorname{Si}(\pi) \approx 1.18 \operatorname{Si}(\infty) = 1.18\, \pi/2.$$

We obtain from (5) that

$$\begin{aligned} \max_{0<x\le\pi} \big(S_N(f)(x) - f(x)\big) &= S_N(f)(\pi/N) - f(\pi/N) + \mathcal{O}(N^{-1}) \\ &= \operatorname{Si}(\pi) - \operatorname{Si}(\infty) + \mathcal{O}(N^{-1}) \approx 0.09\,\pi \quad \text{as } N \to \infty. \end{aligned} \tag{7}$$

This is the *Gibbs phenomenon*: for large N the partial Fourier sum of the sawtooth function f fastly increases from 0 at $x = 0$ to approximately $1.18\, f(0^+) = 1.18\,\pi/2$ at $x = \pi/N$, and it oscillates for $x > \pi/N$ around f with decreasing local maxima and minima for $S_N(f) - f$. See the Mathematica notebook `gibbs.nb` for pictures.

Remark 8. (Extension of Exercise 14 in Ch.3)
Let f be a 2π-periodic C^1-function. The absolute convergence of the Fourier series of f (to be proved in this exercise), together with the pointwise convergence of $S_n(f)$ to f (Theorem 2.1 in Ch.3), implies the uniform convergence of $S_n(f)$ to f. Prove this uniform convergence also in a different way, by a slight adaptation of the proof of Theorem 2.1 in Ch.3.

These conclusions about absolute and uniform convergence remain valid if f is continuous and the derivative of f is only piecewise continuous. A *piecewise continuous derivative* means that f on any finite interval is continuously differentiable outside finitely many points $x_1, \ldots, x_n$, and that at x_i the right derivative $f'(x_i^+)$ and left derivative $f'(x_i^-)$ exist, and that $\lim_{x\downarrow x_i} f'(x) = f'(x_i^+)$ and $\lim_{x\uparrow x_i} f'(x) = f'(x_i^-)$.

Now let g be a 2π periodic function which is C^1 outside $x_0+2\pi\mathbb{Z}$, and which behaves near $x=x_0$ such that the four limits

$$g(x_0^+) := \lim_{h\downarrow 0} g(x_0+h), \quad g(x_0^-) := \lim_{h\uparrow 0} g(x_0+h),$$

$$g'(x_0^+) := \lim_{h\downarrow 0} \frac{g(x_0+h)-g(x_0^+)}{h}, \quad g'(x_0^-) := \lim_{h\uparrow 0} \frac{g(x_0+h)-g(x_0^-)}{h}$$

exist. For convenience assume that $x_0=0$ and that $g(x_0^+)>g(x_0^-)$. Let f be the sawtooth function. Then $p(x) := g(x)-\pi^{-1}(g(0^+)-g(0^-))f(x)$ is a 2π-periodic continuous function with a derivative which is continuous except for a possible jump at 0 (and at integer multiples of 2π). Hence, in combination with the results for Exercise 20 in Ch.3 above, we see the Gibbs phenomenon for g:

$$\begin{aligned}\lim_{N\to\infty}\left(\max_{0<x\le\pi}\left(S_N(g)(x)-g(x)\right)\right) &= \lim_{N\to\infty}\left(S_N(g)(\pi/N)-g(\pi/N)\right)\\ &= \lim_{N\to\infty}\pi^{-1}(g(0^+)-g(0^-))(\mathrm{Si}(\pi)-\mathrm{Si}(\infty)) \approx 0.09\,(g(0^+)-g(0^-)).\end{aligned}$$

The case of finitely many jumps in g can be handled in a similar way.

2 The isoperimetric inequality

Below we write

$$\|f\|_2 := \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Theorem 9. *The area A of a region in the plane which is enclosed by a closed non-selfintersecting C^1-curve of length L satisfies $A\le L^2/(4\pi)$. Equality holds iff the curve is a circle.*

Proof Without loss of generality we may assume that $L=2\pi$, and that the curve is positively oriented and parametrized by its arc length. We may also identify the plane with $\mathbb{C}$. Then the curve has the form $t\mapsto f(t)$ with f a 2π-periodic C^1-function and with $|f'(t)|=1$ for all t. Furthermore we may assume without loss of generality that $\widehat{f}(0)=(2\pi)^{-1}\int_0^{2\pi}f(t)\,dt=0$. Then we have to show that $A\le\pi$ with equality iff $f(t)=e^{i(t+t_0)}$ for some $t_0\in\mathbb{R}$. Now we have

$$\begin{aligned}A &\overset{(1)}{=} \tfrac{1}{2}\mathrm{Im}\int_0^{2\pi} f'(t)\,\overline{f(t)}\,dt = \pi\,\mathrm{Im}\,\langle f',f\rangle \le \pi\,|\langle f',f\rangle| \overset{(2)}{\le} \pi\,\|f'\|_2\,\|f\|_2\\ &\overset{(3)}{=} \pi\,\|f\|_2 \overset{(4)}{=} \pi\,\|f-\widehat{f}(0)\|_2 \overset{(5)}{\le} \pi\,\|f'\|_2 \overset{(6)}{=} \pi. \quad (8)\end{aligned}$$

Equality (1) follows from Vrst 1. Inequality (2) is the Cauchy-Schwarz inequality. Equalities (3) and (6) use that $\|f'\|_2=1$ by the assumption $|f'(t)|=1$. Equality (4) uses the assumption $\widehat{f}(0)=0$. Equality (5) follows from Vrst 2. The proof of the last part of the theorem is in Vrst 3. ☐

Exercises

Vrst 1. Let $t \mapsto f(t)$ be a positively oriented closed non-selfintersecting C^1-curve in $\mathbb{C}$. Show that the area of the enclosed region equals $\frac{1}{2}\mathrm{Im}\int_0^{2\pi} f'(t)\,\overline{f(t)}\,dt$.

Vrst 2. Let f be a 2π-periodic C^1-function. Show that $\|f - \widehat{f}(0)\|_2 \leq \|f'\|_2$ with equality iff $\widehat{f}(n) = 0$ for $n \neq -1, 0, 1$.

Vrst 3. Show that equality everywhere in formula (8) implies that $f(t) = e^{i(t+t_0)}$ for some $t_0 \in \mathbb{R}$.