More on Fourier integrals

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This note gives a minor extension to Chap. 5 of the book Fourier analysis, an introduction by E. M. Stein and R. Shakarchi.

Dense subspaces of $L^p(\mathbb{R})$

By a simple function on $\mathbb{R}$ we mean a finite linear combination of characteristic functions $\chi_E$ of measurable subsets $E$ of $\mathbb{R}$. In particular, a simple function on $\mathbb{R}$ is integrable iff it is a finite linear combination of characteristic functions $\chi_E$ with $\lambda(E) < \infty$.

The Lebesgue measure $\lambda$ on $\mathbb{R}$ is regular, i.e., for every measurable set $E \subset \mathbb{R}$ we have:

$$\lambda(E) = \inf \{ \lambda(V) : E \subset V \text{ and } V \text{ open} \},$$

$$\lambda(E) = \sup \{ \lambda(K) : K \subset E \text{ and } K \text{ compact} \}.$$

This follows from Theorem 2.18 in Rudin, Real and complex analysis.

Proposition Let $1 \leq p < \infty$. The following spaces are dense in $L^p(\mathbb{R})$:

1. The space of integrable simple functions on $\mathbb{R}$.

2. The linear span of the characteristic functions of bounded intervals in $\mathbb{R}$.

3. The space of continuous functions on $\mathbb{R}$ of compact support, i.e., which vanish outside some bounded interval.

Proof We will prove these results for $p = 1$. The proof for other $p$ is similar.

Proof of 1. Every $f \in L^1(\mathbb{R})$ can be written as $f = f_1 - f_2 + if_3 - if_4$ with $f_1, f_2, f_3, f_4$ nonnegative $L^1$ functions. So it is sufficient to prove that every nonnegative $L^1$ function $f$ can be approximated in $L^1$ norm by integrable simple functions. There is an increasing sequence of nonnegative simple functions $t_n(x)$ which tend pointwise to $f$ as $n \to \infty$. Then $\int t_n$ tends to $\int f$ as $n \to \infty$, so $\|f - t_n\|_1 \to 0$.

Proof of 2. By 1. it is sufficient to prove that, if $E \subset \mathbb{R}$ is measurable with $\lambda(E) < \infty$ then $\chi_E$ can be approximated in $L^1$ norm by finite linear combinations of characteristic functions of bounded intervals. Let $\varepsilon > 0$. By regularity of $\lambda$ there is an open set $V \supset E$ such that $\lambda(V) < \lambda(E) + \frac{1}{2}\varepsilon < \infty$. Since $V$ is a countable disjoint union of open intervals, there is a finite union $W \subset V$ of bounded open intervals such that $\lambda(W) > \lambda(V) - \frac{1}{2}\varepsilon$. Hence $\|\chi_E - \chi_W\| < \varepsilon$.

Proof of 3. Every characteristic function of a bounded interval can be approximated in $L^1$ norm by continuous functions of compact support. Now use 2.
We can use part 2. of this Proposition in order to prove the Riemann-Lebesgue Lemma for the Fourier transform:

If \( f \in L^1(\mathbb{R}) \) then \( \hat{f}(\xi) \to 0 \) as \( \xi \to \pm\infty \).

Just observe that the statement is true for \( f = \chi_{[a,b]} \).

**Exercises**

**Exercise 1.** (For this exercise use results from both Fourier series and Fourier integrals.) Below define \( x^{-1}\sin x \) for \( x = 0 \) by continuity.

a) Let \( t \in \mathbb{R} \). Show that for each \( x \in (-\pi, \pi) \) we have

\[
\sum_{n=-\infty}^{\infty} \frac{\sin(\pi(t-n))}{\pi(t-n)} e^{inx} = e^{ixt}
\]

with pointwise convergence. What is the evaluation of the sum on the left-hand side for other real values of \( x \)?

b) Show that, for all \( n, m \in \mathbb{Z} \), we have

\[
\int_{-\infty}^{\infty} \frac{\sin(\pi(t-n))}{\pi(t-n)} \frac{\sin(\pi(t-m))}{\pi(t-m)} dt = \delta_{n,m},
\]

where the integral converges absolutely.

c) Does there exist \( f \in L^2(\mathbb{R}) \) with \( f \neq 0 \) such that

\[
\int_{-\infty}^{\infty} f(t) \frac{\sin(\pi(t-n))}{\pi(t-n)} dt = 0 \quad \text{for all } n \in \mathbb{Z} ?
\]

d) Let \( f \in L^2([-\pi, \pi]) \). Define \( \hat{f} \) as a function on \( \mathbb{R} \) by

\[
\hat{f}(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ixt} \, dx \quad (t \in \mathbb{R}).
\]

(For \( t \in \mathbb{Z} \) this defines the Fourier coefficients of \( f \); for general \( t \in \mathbb{R} \) this defines the Fourier transform of a function on \( \mathbb{R} \) which vanishes outside \([-\pi, \pi]\).) Show that

\[
\hat{f}(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \frac{\sin(\pi(t-n))}{\pi(t-n)} \quad (t \in \mathbb{R})
\]

with absolutely convergent sum.

(This shows in particular the following. Let \( g \) be an \( L^2 \) function on \( \mathbb{R} \) which is the Fourier transform \( g = \hat{f} \) of an \( L^2 \) function \( f \) on \( \mathbb{R} \) vanishing outside \([-\pi, \pi]\) (see (1)). So \( g \) is also continuous. Then \( g \) is completely determined by its restriction to \( \mathbb{Z} \), with reconstruction formula given by (2).)
Hints to Problem 7 in Chapter 5 of Stein & Shakarchi
As for (b), show that, if \( f \) is continuous and of moderate decrease and if for all \( k \in \mathbb{Z}_{\geq 0} \) we have
\[
\int_{-\infty}^{\infty} f(y) e^{-y^2} y^k dy = 0 \quad (3)
\]
then we have for all \( x \in \mathbb{R} \) that
\[
\int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy} dy = 0,
\]
and hence, by Ch.5, Exercise 8, \( f = 0 \). Now try to go from the assumption in (b) of Problem 7 to equation (3) above.

As for (c), a result from (a) is:
\[
h_k(x) = (-1)^k e^{x^2/2} \left( \frac{d}{dx} \right)^k e^{-x^2}.
\]
Show, by once differentiating this formula, that
\[
h_{k+1}(x) = \left( x - \frac{d}{dx} \right) h_k(x). \quad (4)
\]
Use (4) in (c) in order to prove the result there by induction with respect to \( k \).

Now show, by applying the operator \( x + \frac{d}{dx} \) to both sides of (4) and by using induction with respect to \( k \), that
\[
\left( x + \frac{d}{dx} \right) h_{k+1}(x) = 2(k + 1) h_k(x). \quad (5)
\]
Now it has to be proved in (d) that
\[
(L h_k)(x) := \left( x^2 - \frac{d^2}{dx^2} \right) h_k(x) = (2k + 1) h_k(x).
\]
Show this by expressing the operator \( L \) in terms of \( x - \frac{d}{dx} \) and \( x + \frac{d}{dx} \) and by using (4) and (5).