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1. Classical orthogonal polynomials

1.1. Shift operators. Let (a,b) be an open interval and let two systems of monic orthogonal polynomials $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ be defined with respect to strictly positive weight functions w respectively w_1 on (a,b). Suppose that w is continuous and w_1 is continuously differentiable on (a,b). Under suitable boundary assumptions on w and w_1, integration by parts yields

$$\int_a^b p_n'(x)\,q_{m-1}(x)\,w_1(x)\,dx = -\int_a^b p_n(x)\,w(x)^{-1}\,\frac{d}{dx}\big(w_1(x)\,q_{m-1}(x)\big)\,w(x)\,dx \tag{1.1}$$

without stock terms. Suppose that

$$w(x)^{-1}\,\frac{d}{dx}\big(w_1(x)\,x^{n-1}\big) = a_n\,x^n + \text{polynomial of degree} < n \tag{1.2}$$

for certain $a_n \neq 0$. Then (1.1) and (1.2) together with the orthogonality properties of $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ yield that

$$p_n'(x) = n\,q_{n-1}(x) \tag{1.3}$$

and

$$w(x)^{-1}\,\frac{d}{dx}\big(w_1(x)\,q_{n-1}(x)\big) = a_n\,p_n(x). \tag{1.4}$$

The pair of operators D_- and D_+ defined by

$$(D_-f)(x) := f'(x), \tag{1.5}$$

$$(D_+f)(x) := \left(w(x)^{-1}\,\frac{d}{dx}\circ w_1(x)\right) f(x) = \frac{w_1(x)}{w(x)}\,f'(x) + \frac{w_1'(x)}{w(x)}\,f(x), \tag{1.6}$$

will be called a pair of *shift operators*. So we have:
adjointness:

$$\int_a^b (D_-f)(x)\,g(x)\,w_1(x)\,dx = -\int_a^b f(x)\,(D_+g)(x)\,w(x)\,dx, \quad f,\ g \text{ polynomials}, \tag{1.7}$$

shift formulas:

$$D_-p_n = n\,q_{n-1}, \quad D_+q_{n-1} = a_n\,p_n, \tag{1.8}$$

second order differential equation:

$$(D_+\circ D_-)p_n = na_n\,p_n, \tag{1.9}$$

relation between squared L^2-norms:

$$n\int_a^b (q_{n-1}(x))^2\,w_1(x)\,dx = -a_n\int_a^b (p_n(x))^2\,w(x)\,dx. \tag{1.10}$$

The following cases are examples for which the above considerations are valid:

(i) *Jacobi:* $(a,b)=(-1,1)$, $w(x)=(1-x)^\alpha(1+x)^\beta$, $w_1(x)=(1-x^2)\,w(x)$, $\alpha,\beta>-1$.
(ii) *Laguerre:* $(a,b)=(0,\infty)$, $w(x)=x^\alpha e^{-x}$, $w_1(x)=x\,w(x)$, $\alpha>-1$.
(iii) *Hermite:* $(a,b)=(-\infty,\infty)$, $w(x)=w_1(x)=e^{-x^2}$.

Exercise. Show that these three cases are essentially the only cases which satisfy the above conditions.

Note that in each of the three cases we can iterate, i.e., we can associate with the weight function w_1 again another weight function such that the new pair saisfies the above conditions.

1.2. Jacobi polynomials. Let $\alpha, \beta > -1$. Let a, b, w and w_1 be as in case (i) above. Note that the integral of w over $(-1,1)$ is a variant of the beta integral:

$$\int_{-1}^{1} (1-x)^\alpha\,(1+x)^\beta\,dx = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,. \tag{1.11}$$

Denote the monic orthogonal polynomials on $(-1,1)$ with respect to the weight function w by $p_n^{(\alpha,\beta)}$ (monic *Jacobi polynomials*). Then the raising shift operator $D_+ = D_+^{(\alpha,\beta)}$ is given by

$$(D_+^{(\alpha,\beta)} f)(x) = \left((1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d}{dx}\circ(1-x)^{\alpha+1}(1+x)^{\beta+1}\right) f(x) \tag{1.12}$$

$$= (1-x^2)\,f'(x) + (\beta-\alpha-(\alpha+\beta+2)x)\,f(x). \tag{1.13}$$

Hence $a_n = -(n+\alpha+\beta+1)$ and the shift relations become:

$$\frac{d}{dx}\,p_n^{(\alpha,\beta)}(x) = n\,p_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{1.14}$$

$$\begin{aligned}
&\left((1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d}{dx}\circ(1-x)^{\alpha+1}(1+x)^{\beta+1}\right) p_{n-1}^{(\alpha+1,\beta+1)}(x)\\
&\quad= \left((1-x^2)\,\frac{d}{dx} + (\beta-\alpha-(\alpha+\beta+2)x)\right) p_{n-1}^{(\alpha+1,\beta+1)}(x)\\
&\qquad= -(n+\alpha+\beta+1)\,p_n^{(\alpha,\beta)}(x).
\end{aligned} \tag{1.15}$$

The second order differential equation becomes

$$\left((1-x^2)\,\frac{d^2}{dx^2} + (\beta-\alpha-(\alpha+\beta+2)x)\,\frac{d}{dx}\right) p_n^{(\alpha,\beta)}(x) = -n(n+\alpha+\beta+1)\,p_n^{(\alpha,\beta)}(x). \tag{1.16}$$

Iteration of (1.15) yields the *Rodrigues formula*

$$p_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{(n+\alpha+\beta+1)_n}\,(1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}\,(1+x)^{\beta+n}\right\}. \tag{1.17}$$

Formula (1.10) yields the recurrence

$$\begin{aligned}
&\int_{-1}^{1} \left(p_n^{(\alpha,\beta)}(x)\right)^2 (1-x)^\alpha(1+x)^\beta\,dx\\
&\qquad= \frac{n}{n+\alpha+\beta+1}\int_{-1}^{1}\left(p_{n-1}^{(\alpha+1,\beta+1)}(x)\right)^2 (1-x)^{\alpha+1}(1+x)^{\beta+1}\,dx.
\end{aligned} \tag{1.18}$$

Iteration of (1.18) and combination with (1.11) yields

$$\frac{\int_{-1}^{1}\bigl(p_n^{(\alpha,\beta)}(x)\bigr)^2\,(1-x)^\alpha(1+x)^\beta\,dx}{\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\,dx}=\frac{2^{2n}\,n!\,(\alpha+1)_n\,(\beta+1)_n}{(\alpha+\beta+2)_{2n}\,(n+\alpha+\beta+1)_n}\,. \tag{1.19}$$

Consider (1.15) for $x=1$. This yields the recurrence

$$-2(\alpha+1)\,p_{n-1}^{(\alpha+1,\beta+1)}(1)=-(n+\alpha+\beta+1)\,p_n^{(\alpha,\beta)}(1). \tag{1.20}$$

By iteration we obtain

$$p_n^{(\alpha,\beta)}(1)=\frac{2^n\,(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\,. \tag{1.21}$$

By Taylor expansion and by use of (1.14) and (1.21) we obtain

$$\begin{aligned}
p_n^{(\alpha,\beta)}(x)&=\sum_{k=0}^{n}\frac{(x-1)^k}{k!}\left(\frac{d}{dx}\right)^k p_n^{(\alpha,\beta)}(x)\Big|_{x=1}\\
&=\sum_{k=0}^{n}\frac{(x-1)^k}{k!}\,\frac{n!}{(n-k)!}\,p_{n-k}^{(\alpha+k,\beta+k)}(1)\\
&=\sum_{k=0}^{n}\frac{n!\,2^{n-k}\,(\alpha+k+1)_{n-k}\,(x-1)^k}{k!\,(n-k)!\,(n+\alpha+\beta+k+1)_{n-k}}\\
&=\frac{2^n\,(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!}\left(\frac{1-x}{2}\right)^k. \qquad (1.22)
\end{aligned}$$

From (1.22) and (1.21) we obtain

$$\begin{aligned}
\frac{p_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(1)}&=\sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!}\left(\frac{1-x}{2}\right)^k \qquad (1.23)\\
&={}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;(1-x)/2). \qquad (1.24)
\end{aligned}$$

The standard normalization of Jacobi polynomials is different from the monic normalization. Write $P_n^{(\alpha,\beta)}$ for the constant multiple of $p_n^{(\alpha,\beta)}$ such that

$$P_n^{(\alpha,\beta)}(1)=\frac{(\alpha+1)_n}{n!}\,. \tag{1.25}$$

Then, by (1.21) and (1.25),

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x)&=\frac{(n+\alpha+\beta+1)_n}{2^n\,n!}\,p_n^{(\alpha,\beta)}(x) \qquad (1.26)\\
&=\frac{(n+\alpha+\beta+1)_n}{2^n\,n!}\,x^n+\text{lower degree terms}. \qquad (1.27)
\end{aligned}$$

From (1.25) and (1.24) we obtain:

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;(1-x)/2). \tag{1.28}$$

One might now rewrite all previous formulas in terms of these renormalized Jacobi polynomials. See Temme, vraagstuk 5.4, (5.29), (5.18), vraagstuk 5.11 for the rewriting of (1.14), (1.16), (1.17), (1.19), respectively.

1.3. Laguerre and Hermite polynomials. Consider now the Laguerre example (ii) and the Hermite example (iii) in §1.1. These suggest to introduce monic *Laguerre polynomials* l_n^α which are orthogonal on $(0,\infty)$ with respect to the weight function $x \mapsto x^\alpha\, e^{-x}$ $(\alpha > -1)$, and monic *Hermite polynomials* h_n which are orthogonal on $(-\infty,\infty)$ with respect to the weight function $x \mapsto e^{-x^2}$. For these two classes of polynomials we may now imitate what we have done for Jacobi polynomials in §1.2. In particular, the raising shift formulas now become

$$\left(x\,\frac{d}{dx} + (\alpha+1-x)\right) l_{n-1}^{\alpha+1}(x) = -l_n^\alpha(x), \tag{1.29}$$

$$\left(\frac{d}{dx} - 2x\right) h_{n-1}(x) = -2h_n(x). \tag{1.30}$$

The Laguerre analogues of (1.21), (1.23) and (1.24) are

$$l_n^\alpha(0) = (-1)^n\,(\alpha+1)_n, \tag{1.31}$$

$$\frac{l_n^\alpha(x)}{l_n^\alpha(0)} = \sum_{k=0}^n \frac{(-n)_k\,x^k}{(\alpha+1)_k\,k!} \tag{1.32}$$

$$= {}_1F_1(-n;\alpha+1;x). \tag{1.33}$$

In (1.33) appears an ${}_1F_1$ hypergeometric function, a so-called *confluent hypergeometric function*. When we compare (1.21) and (1.23) with (1.31) and (1.32) then we can conclude that the following limit relation holds:

$$\lim_{\beta\to\infty} (-\beta/2)^n\, p_n^{(\alpha,\beta)}(1-2x/\beta) = l_n^\alpha(x). \tag{1.34}$$

This limit transition can also be read off from the weight functions, at least formally. The polynomials $x \mapsto p_n^{(\alpha,\beta)}(1-2x/\beta)$ are orthogonal on the interval $(0,\beta)$ with respect to the weight function $x \mapsto x^\alpha(1-x/\beta)^\beta$. This tends formally, as $\beta\to\infty$, to an orthogonality on $(0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$.

Laguerre polynomials are usually normalized as $L_n^\alpha = \text{const.}\, l_n^\alpha$ such that

$$L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}\,. \tag{1.35}$$

Then, by (1.31) and (1.35),

$$L_n^\alpha(x) = \frac{(-1)^n}{n!}\, l_n^\alpha(x). \tag{1.36}$$

There is no analogue of (1.20) and (1.21) for Hermite polynomials. The usual normalization of Hermite polynomials is by the definition

$$H_n(x) := 2^n\, h_n(x). \tag{1.37}$$

1.4. Quadratic transformations. We formulate two propositions which follow very easily from the definition of a system of orthogonal polynomials.

Proposition 1.1 Let $\{p_n\}_{n=0}^{\infty}$ be a system of monic orthogonal polynomials with respect to a weight function w on an interval $(-a,a)$. Put $v(x) := w(-x)$. Then the polynomials $x \mapsto (-1)^n\,p_n(-x)$ are monic orthogonal polynomials with respect to the weight function v on the interval $(-a,a)$. If the weight function w is even (i.e., $w(x) = w(-x)$) then p_n is an even or odd function according to whether n is even or odd.

This proposition immediately implies that

$$p_n^{(\alpha,\beta)}(-x) = (-1)^n\,p_n^{(\beta,\alpha)}(x). \tag{1.38}$$

Proposition 1.2 Let $\{p_n\}_{n=0}^{\infty}$ be a system of monic orthogonal polynomials with respect to an even weight function w on an interval $(-a,a)$. Put

$$q_n(x^2) := p_{2n}(x), \quad r_n(x^2) := x^{-1}\,p_{2n+1}(x). \tag{1.39}$$

Define weight functions v_1 and v_2 on $(0,a^2)$ by

$$v_1(x^2) := x^{-1}\,w(x), \quad v_2(x^2) := x\,w(x). \tag{1.40}$$

Then the polynomials q_n are monic orthogonal polynomials on $(0,a^2)$ with respect to the weight function v_1 and the polynomials r_n are monic orthogonal polynomials on $(0,a^2)$ with respect to the weight function v_2.

As a corollary we obtain the following quadratic transformations:

$$p_n^{(\alpha,-\frac12)}(2x^2-1) = 2^n\,p_{2n}^{(\alpha,\alpha)}(x), \tag{1.41}$$

$$x\,p_n^{(\alpha,\frac12)}(2x^2-1) = 2^n\,p_{2n+1}^{(\alpha,\alpha)}(x), \tag{1.42}$$

$$l_n^{-\frac12}(x^2) = h_{2n}(x), \tag{1.43}$$

$$x\,l_n^{\frac12}(x^2) = h_{2n+1}(x). \tag{1.44}$$

Exercise. Combination of (1.31) and (1.32) with (1.43) and (1.44) yields an expansion of $h_n(x)$ in terms of powers of x. Derive this. In a similar way, derive an expansion of $p_n^{(\alpha,\alpha)}(x)$ in terms of powers of x.

It follows by combination of (1.41)–(1.44), (1.38) and (1.34) that there is the limit transition

$$\lim_{\alpha\to\infty} \alpha^{\frac12 n}\,p_n^{(\alpha,\alpha)}(\alpha^{-\frac12}x) = h_n(x). \tag{1.45}$$

2. Standard formuals for classical orthogonal polynomials

2.1. Jacobi polynomials.

Definition: Let $\alpha, \beta > -1$. The Jacobi polynomial $P_n^{(\alpha,\beta)}$ is the polynomial of degree n such that

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x)\, x^k\, (1-x)^\alpha\, (1+x)^\beta\, dx = 0 \quad \text{for } k = 0, 1, \ldots, n-1, \tag{2.1}$$

and

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}\,. \tag{2.2}$$

Explicit expression:

$$P_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(1)\ {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1}{2}(1-x)) \tag{2.3}$$

$$= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\, k!} \left(\frac{1-x}{2}\right)^k. \tag{2.4}$$

$$= \frac{(n+\alpha+\beta+1)_n}{2^n\, n!}\, x^n + \text{terms of lower degree.} \tag{2.5}$$

Shift operators:

$$\frac{d}{dx}\, P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1)\, P_{n-1}^{(\alpha+1,\beta+1)}(x) \text{ if } n > 0 \quad \text{and } = 0 \text{ if } n = 0, \tag{2.6}$$

$$\frac{d}{dx}\left[(1-x)^{\alpha+1}\,(1+x)^{\beta+1}\, P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] = -2n\,(1-x)^\alpha\,(1+x)^\beta\, P_n^{(\alpha,\beta)}(x). \tag{2.7}$$

Differential equation:

$$\left((1-x^2)\left(\frac{d}{dx}\right)^2 + (\beta-\alpha-(\alpha+\beta+2)x)\,\frac{d}{dx}\right) P_n^{(\alpha,\beta)}(x) = -n(n+\alpha+\beta+1)\, P_n^{(\alpha,\beta)}(x). \tag{2.8}$$

Rodrigues formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!}\,(1-x)^{-\alpha}\,(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n \left[(1-x)^{n+\alpha}\,(1+x)^{n+\beta}\right]. \tag{2.9}$$

Orthogonality relations:

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)} \int_{-1}^{1} P_m^{\alpha,\beta}(x)\, P_n^{\alpha,\beta}(x)\,(1-x)^\alpha\,(1+x)^\beta\, dx$$

$$= \delta_{m,n}\, \frac{(\alpha+1)_n\,(\beta+1)_n\,(n+\alpha+\beta+1)_n}{(\alpha+\beta+2)_{2n}\, n!}\,. \tag{2.10}$$

Symmetry relation and value at −1:

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n\, P_n^{(\beta,\alpha)}(x), \qquad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n\,(\beta+1)_n}{n!}\,. \tag{2.11}$$

Three-term recurrence relation:

$$\begin{aligned} x\,P_n^{(\alpha,\beta)}(x) =& \frac{2(n+1)\,(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta+2)}\,P_{n+1}^{(\alpha,\beta)}(x) \\ &+ \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+2)}\,P_n^{(\alpha,\beta)}(x) \\ &+ \frac{2(n+\alpha)\,(n+\beta)}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+1)}\,P_{n-1}^{(\alpha,\beta)}(x). \end{aligned} \tag{2.12}$$

Quadratic transformations:

$$\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-\frac12)}(2x^2-1)}{P_n^{(\alpha,-\frac12)}(1)}, \tag{2.13}$$

$$\frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)} = \frac{x\,P_n^{(\alpha,\frac12)}(2x^2-1)}{P_n^{(\alpha,\frac12)}(1)}. \tag{2.14}$$

Gegenbauer or ultraspherical polynomials:

$$C_n^\lambda(x) := \frac{(2\lambda)_n}{(\lambda+\frac12)_n}\,P_n^{(\lambda-\frac12,\lambda-\frac12)}(x), \quad \lambda \neq 0. \tag{2.15}$$

Legendre polynomials:

$$P_n(x) := P_n^{(0,0)}(x) = C_n^{\frac12}(x). \tag{2.16}$$

Chebyshev polynomials of first and second kind:

$$T_n(\cos\theta) := \cos(n\theta) = \frac{P_n^{(-\frac12,-\frac12)}(\cos\theta)}{P_n^{(-\frac12,-\frac12)}(1)}\,, \tag{2.17}$$

$$U_n(\cos\theta) := \frac{\sin((n+1)\theta)}{\sin\theta} = \frac{(n+1)\,P_n^{(\frac12,\frac12)}(\cos\theta)}{P_n^{(\frac12,\frac12)}(1)}\,. \tag{2.18}$$

Power series in x for Gegenbauer polynomials:

$$P_n^{(\alpha,\alpha)}(x) = \frac{(n+2\alpha+1)_n}{2^n\,n!}\,x^n\,{}_2F_1\left[\begin{matrix}-\frac12 n, -\frac12 n+\frac12\\ -n-\alpha+\frac12\end{matrix};x^{-2}\right] \tag{2.19}$$

$$= \frac{(\alpha+1)_n}{(2\alpha+1)_n}\sum_{k=0}^{[n/2]}\frac{(-1)^k\,(\alpha+\frac12)_{n-k}}{k!\,(n-2k)!}\,(2x)^{n-2k}. \tag{2.20}$$

Generating functions:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,w^n = 2^{\alpha+\beta}\,R^{-1}\,(1-w+R)^{-\alpha}\,(1+w+R)^{-\beta}, \quad x\in[-1,1],\ |w|<1, \tag{2.21}$$

where

$$R := (1-2xw+w^2)^{\frac{1}{2}}, \tag{2.22}$$

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\,P_n^{(\alpha,\beta)}(x)\,w^n = (1-w)^{-\alpha-\beta-1} \\ \times {}_2F_1\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2) \\ \alpha+1\end{matrix}; \frac{2w(x-1)}{(1-w)^2}\right], \quad x\in[-1,1],\ |w|<1, \tag{2.23}$$

$$\sum_{n=0}^{\infty} \frac{(2\alpha+1)_n}{(\alpha+1)_n}\,P_n^{(\alpha,\alpha)}(x)\,w^n = \sum_{n=0}^{\infty} C_n^{\alpha+\frac{1}{2}}(x)\,w^n = (1-2wx+w^2)^{-\alpha-\frac{1}{2}}, \\ x\in[-1,1],\ |w|<1. \tag{2.24}$$

Limit formula:

$$\lim_{\alpha\to\infty} \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \left(\frac{1+x}{2}\right)^n. \tag{2.25}$$

2.2. Laguerre polynomials.

Definition: Let $\alpha > -1$. The Laguerre polynomial L_n^α is the polynomial of degree n such that

$$\int_0^\infty L_n^\alpha(x)\,x^k\,x^\alpha\,e^{-x}\,dx = 0 \quad \text{for } k=0,1,\ldots,n-1, \tag{2.26}$$

and

$$L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}\,. \tag{2.27}$$

Explicit expression:

$$L_n^\alpha(x) = L_n^\alpha(0)\ {}_1F_1(-n;\alpha+1;x) \tag{2.28}$$

$$= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k\,k!}\,x^k. \tag{2.29}$$

$$= \frac{(-1)^n}{n!}\,x^n + \text{terms of lower degree.} \tag{2.30}$$

Shift operators:

$$\frac{d}{dx}\,L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) \text{ if } n>0 \quad \text{and } = 0 \text{ if } n=0, \tag{2.31}$$

$$\frac{d}{dx}\left[x^{\alpha+1}\,e^{-x}\,L_{n-1}^{\alpha+1}(x)\right] = n\,x^\alpha\,e^{-x}\,L_n^\alpha(x). \tag{2.32}$$

Differential equation:

$$\left(x\left(\frac{d}{dx}\right)^2+(\alpha+1-x)\,\frac{d}{dx}\right)L_n^\alpha(x)=-n\,L_n^\alpha(x). \tag{2.33}$$

Rodrigues formula:

$$L_n^\alpha(x)=\frac{1}{n!}\,x^{-\alpha}\,e^x\,\left(\frac{d}{dx}\right)^n\left[x^{n+\alpha}\,e^{-x}\right]. \tag{2.34}$$

Orthogonality relations:

$$\frac{1}{\Gamma(\alpha+1)}\int_0^\infty L_m^\alpha(x)\,L_n^\alpha(x)\,x^\alpha\,e^{-x}\,dx=\delta_{m,n}\,\frac{(\alpha+1)_n}{n!}\,. \tag{2.35}$$

Three-term recurrence relation:

$$x\,L_n^\alpha(x)=-(n+1)\,L_{n+1}^\alpha(x)+(2n+\alpha+1)\,L_n^\alpha(x)-(n+\alpha)\,L_{n-1}^\alpha(x). \tag{2.36}$$

Limit formula:

$$\lim_{\beta\to\infty}P_n^{(\alpha,\beta)}(1-2\beta^{-1}x)=L_n^\alpha(x). \tag{2.37}$$

Generating functions:

$$\sum_{n=0}^\infty L_n^\alpha(x)\,w^n=(1-w)^{-\alpha-1}\,\exp\frac{xw}{w-1}\,,\quad |w|<1, \tag{2.38}$$

$$\sum_{n=0}^\infty\frac{L_n^\alpha(x)\,w^n}{(\alpha+1)_n}=e^w\,{}_0F_1(-;\alpha+1;-wx). \tag{2.39}$$

2.3. Hermite polynomials.

Definition: The Hermite polynomial H_n is the polynomial of degree n such that

$$\int_{-\infty}^\infty H_n(x)\,x^k\,e^{-x^2}\,dx=0\quad\text{for }k=0,1,\ldots,n-1, \tag{2.40}$$

and

$$H_n(x)=2^n\,x^n+\text{terms of lower degree}. \tag{2.41}$$

Quadratic transformations

$$H_{2n}(x)=(-1)^n\,2^{2n}\,n!\,L_n^{-\frac{1}{2}}(x^2), \tag{2.42}$$

$$H_{2n+1}(x)=(-1)^n\,2^{2n+1}\,n!\,x\,L_n^{\frac{1}{2}}(x^2). \tag{2.43}$$

Explicit expression:

$$H_n(x) = (2x)^n \, {}_2F_0(-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}; -; -x^{-2}) \tag{2.44}$$

$$= n! \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k \, (2x)^{n-2k}}{k! \, (n-2k)!} \,. \tag{2.45}$$

Shift operators:

$$\frac{d}{dx} \, H_n(x) = 2n \, H_{n-1}(x) \text{ if } n>0 \quad \text{and } = 0 \text{ if } n=0, \tag{2.46}$$

$$\frac{d}{dx} \big[e^{-x^2} \, H_{n-1}(x)\big] = -e^{-x^2} \, H_n(x). \tag{2.47}$$

Differential equation:

$$\left(\left(\frac{d}{dx} \right)^2 - 2x \, \frac{d}{dx} \right) H_n(x) = -2n \, H_n(x). \tag{2.48}$$

Rodrigues formula:

$$H_n(x) = (-1)^n \, e^{x^2} \left(\frac{d}{dx} \right)^n \left[e^{-x^2} \right]. \tag{2.49}$$

Orthogonality relations:

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} H_m(x) \, H_n(x) \, e^{-x^2} \, dx = \delta_{m,n} \, 2^n \, n! \,. \tag{2.50}$$

Three-term recurrence relation:

$$x \, H_n(x) = \tfrac{1}{2} \, H_{n+1}(x) + n \, H_{n-1}(x). \tag{2.51}$$

Limit formulas:

$$\lim_{\alpha \to \infty} \alpha^{-\frac{1}{2}n} \, P_n^{(\alpha,\alpha)}(\alpha^{-\frac{1}{2}} x) = \frac{H_n(x)}{2^n \, n!} \,, \tag{2.52}$$

$$\lim_{\alpha \to \infty} (2\alpha)^{-\frac{1}{2}n} \, L_n^{\alpha}((2\alpha)^{\frac{1}{2}} \, x + \alpha) = \frac{(-1)^n}{2^n \, n!} \, H_n(x). \tag{2.53}$$

Generating function:

$$\sum_{n=0}^{\infty} \frac{H_n(x) \, w^n}{n!} = \exp(2xw - w^2). \tag{2.54}$$

3. Bessel functions

Consider the Fourier transform pair for functions in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$:

$$g(\xi) = (2\pi)^{-\frac{1}{2}d} \int_{\mathbb{R}^d} f(x)\, e^{-i\, x.\xi}\, dx, \tag{3.1}$$

$$f(x) = (2\pi)^{-\frac{1}{2}d} \int_{\mathbb{R}^d} g(\xi)\, e^{i\, x.\xi}\, d\xi. \tag{3.2}$$

Observe that f is moreover a radial function iff g is moreover a radial function and that then $f(x) = F(|x|)$, $g(\xi) = G(|\xi|)$ for certain even functions $F, G \in \mathcal{S}(\mathbb{R})$. A computation shows that then (3.1) can be rewritten in the form

$$G(\rho) = (2\pi)^{-\frac{1}{2}d} \int_0^\infty F(r)\, K(r,\rho)\, r^{d-1}\, dr,$$

where

$$\begin{aligned} K(r,\rho) &:= \int_{S^{d-1}} e^{-ir\rho\, x'.e_1}\, d\omega(x') \\ &= \mathrm{vol}(S^{d-2}) \int_{-1}^{1} e^{-ir\rho t}\, (1-t^2)^{\frac{1}{2}(d-3)}\, dt, \end{aligned} \tag{3.3}$$

S^{d-1} is de unit sphere in $\mathbb{R}^d$, and the measure $d\omega$ is the volume element on S^{d-1}, and

$$\mathrm{vol}(S^{d-2}) = \frac{2\pi^{\frac{1}{2}(d-1)}}{\Gamma(\frac{1}{2}(d-1))}\,.$$

Put

$$\mathcal{J}_\alpha(z) := {}_0F_1(\alpha+1; -z^2/4), \quad z \in \mathbb{C}.$$

Apart from some elementary factors this is a Bessel function. An easy computation shows that

$$\mathcal{J}_\alpha(z) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{-izt}\, (1-t^2)^{\alpha-\frac{1}{2}}\, dt, \quad \mathrm{Re}\,\alpha > -\tfrac{1}{2}.$$

Hence,

$$K(r,\rho) = \frac{(2\pi)^{\frac{1}{2}d}}{2^{\frac{1}{2}(d-1)}\,\Gamma(\frac{1}{2}d)}\, \mathcal{J}_{\frac{1}{2}d-1}(r\rho). \tag{3.4}$$

Thus, in the radial case, we can rewrite the transform pair (3.1)–(3.2) in the form

$$G(\rho) = \frac{1}{2^{\frac{1}{2}(d-1)}\,\Gamma(\frac{1}{2}d)} \int_0^\infty F(r)\, \mathcal{J}_{\frac{1}{2}d-1}(r\rho)\, r^{d-1}\, dr, \tag{3.5}$$

$$F(r) = \frac{1}{2^{\frac{1}{2}(d-1)}\,\Gamma(\frac{1}{2}d)} \int_0^\infty G(\rho)\, \mathcal{J}_{\frac{1}{2}d-1}(r\rho)\, \rho^{d-1}\, d\rho. \tag{3.6}$$

This transform pair is known as the *Hankel transform* pair, and it remains valid for arbitrary real $d>0$, but this will not be proved here.

It follows from (3.3), (3.4) that

$$\mathcal{J}_{\frac{1}{2}d-1}(|x|)=\frac{1}{\mathrm{vol}(S^{d-1})}\int_{S^{d-1}}e^{-i\,x.y'}\,d\omega(y'),\quad x\in\mathbb{R}^d. \tag{3.7}$$

Hence

$$(\Delta+1)\big[\mathcal{J}_{\frac{1}{2}d-1}(|x|)\big]=0, \tag{3.8}$$

where Δ is the Laplace operator on $\mathbb{R}^d$. By taking radial parts we find that

$$\left(\frac{d^2}{dr^2}+\frac{d-1}{r}\,\frac{d}{dr}+1\right)\mathcal{J}_{\frac{1}{2}d-1}(r)=0. \tag{3.9}$$

It is easily verified that (3.9) remains valid for more general real or complex values of d.

Consider now the operator in (3.8) for $d=2$, but acting on a function which is not necessarily radial:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+1\right)F(x,y)=0.$$

If $F(r\,\cos\theta, r\,\sin\theta)=f(r)\,e^{in\theta}$ $(n\in\mathbb{Z})$ then we find the following differential equation for $f(r)$:

$$r^2\,f''(r)+r\,f'(r)+(r^2-n^2)\,f(r)=0.$$

A possible solution is

$$f(r)=r^{|n|}\,{}_0F_1(|n|+1;-r^2/4)=r^{|n|}\,\mathcal{J}_{|n|}(r).$$

This outcome is a possible historical explanation for the following convention of defining the *Bessel function* of order α:

$$J_\alpha(z):=\frac{z^\alpha}{2^\alpha\Gamma(\alpha+1)}\,{}_0F_1(\alpha+1;-z^2/4) \tag{3.10}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k\,(z^2/4)^{k+\frac{1}{2}\alpha}}{\Gamma(\alpha+k+1)\,k!}\,. \tag{3.11}$$

For unique determination of z^α in case of non-integer α we take $|\arg z|<\pi$. Observe that (3.11) remains meaningful if $\alpha=-n\in\{-1,-2,\ldots\}$. The summation then starts with $k=n$ and we obtain

$$J_{-n}(z)=(-1)^n\,J_n(z).$$

In the case $d=2$ formula (3.7) can be rewritten as

$$J_0(r)=\frac{1}{2\pi}\int_0^{2\pi}e^{ir\,\sin\theta}\,d\theta.$$

Hence, in the Fourier series

$$e^{ir\,\sin\theta} = \sum_{n=-\infty}^{\infty} e^{in\theta}\, f_n(r)$$

we have $f_0(r) = J_0(r)$. It is easily seen that $\Delta+1$ acting on the function $(r\,\cos\theta, r\,\sin\theta) \mapsto e^{in\theta}\, f_n(r)$ yields zero. Thus possibly $f_n(r) = \mathrm{const.}\, J_n(r)$. The constant is even equal to 1. Indeed, straightforward computation yields that

$$\exp(\tfrac{1}{2}z(t-t^{-1})) = \sum_{n=-\infty}^{\infty} t^n\, J_n(z). \tag{3.12}$$

This formula can be viewed as a generating function for Bessel functions of integer order. It gives a possible historical explanation for the choice of the constant factor in (3.10).

For Gaussian hypergeometric functions we have seen the quadratic transformation

$${}_2F_1(a,b;2b;z) = (1-z)^{-\frac{1}{2}a}\, {}_2F_1\left(\tfrac{1}{2}a, b-\tfrac{1}{2}a; b+\tfrac{1}{2}; \frac{z^2}{4(z-1)}\right), \quad z \notin [1,\infty).$$

Replace z by z/a and let $a \to \infty$ in the above formula. We find at least formally, by taking termwise limits, that

$${}_1F_1(b;2b;z) = e^{\frac{1}{2}z}\, {}_0F_1(b+\tfrac{1}{2}; z^2/16). \tag{3.13}$$

Formula (3.13) can be considered as a quadratic transformation between special ${}_1F_1$-functions and general ${}_0F_1$ functions. The formula can also be seen as a corollary of a quadratic transformation between the second order differential equation of ${}_1F_1$ and of ${}_0F_1$. In this way, special solutions around 0 or ∞ for the ${}_0F_1$ differential equation can be related to similar solutions for the ${}_1F_1$ differential equation, and known integral representations for solutions in the ${}_1F_1$ case will yield integral representations fro solutions in the ${}_0F_1$ case. Formula (3.13) implies that

$$\begin{aligned} J_\alpha(z) &= \frac{z^\alpha\, e^{-iz}}{2^\alpha\,\Gamma(\alpha+1)}\, {}_1F_1(\alpha+\tfrac{1}{2}; 2\alpha+1; 2iz) \\ &= \frac{z^\alpha\, e^{iz}}{2^\alpha\,\Gamma(\alpha+1)}\, {}_1F_1(\alpha+\tfrac{1}{2}; 2\alpha+1; -2iz). \end{aligned}$$

Define the *modified Bessel function* by

$$\begin{aligned} I_\alpha(z) :=& \frac{z^\alpha}{2^\alpha\Gamma(\alpha+1)}\, {}_0F_1(\alpha+1; z^2/4), \quad |\arg z| < \pi, \\ =& e^{-\frac{1}{2}\alpha\pi i}\, J_\alpha(iz), \quad -\pi < \arg z < \tfrac{1}{2}\pi. \end{aligned} \tag{3.14}$$

Then $I_\alpha(z)$ is a solution $f(z)$ of the differential equation

$$z^2\, f''(z) + z\, f'(z) - (z^2+\alpha^2)\, f(z) = 0. \tag{3.15}$$

Note that solutions f of (3.15) correspond to solutions g of the differential equation

$$w\, g''(w) + (2\alpha+1-w)\, g'(w) - (\alpha+\tfrac{1}{2})\, g(w) = 0$$

under the identification

$$f(z) = \mathrm{const.}\, z^\alpha\, e^{-z}\, g(2z).$$

We now discuss some special cases of this correspondence.

Case 1. If

$$\begin{aligned} g(w) &= {}_1F_1(\alpha+\tfrac{1}{2}; 2\alpha+1; w) \\ &= 1+\mathcal{O}(|w|), \quad |w|\to 0, \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\, e^w\, w^{-\alpha-\frac{1}{2}}(1+\mathcal{O}(|w|^{-1})), \quad |w|\to\infty,\ |\arg w|<\pi/2, \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac{1}{2})^2}\int_0^1 e^{sw}\, s^{\alpha-\frac{1}{2}}\,(1-s)^{\alpha-\frac{1}{2}}\, ds, \quad \operatorname{Re}\alpha>-\tfrac{1}{2}, \end{aligned}$$

then

$$\begin{aligned} f(z) &= \frac{z^\alpha\, e^{-z}}{2^\alpha\,\Gamma(\alpha+1)}\, g(2z) \\ &= I_\alpha(z) \\ &= \frac{z^\alpha}{2^\alpha\Gamma(\alpha+1)}\,(1+\mathcal{O}(|z|)), \quad |z|\to 0,\ |\arg z|<\pi, \\ &= (2\pi)^{-\frac{1}{2}}\, z^{-\frac{1}{2}}\, e^z\,(1+\mathcal{O}(|z|^{-1})), \quad |z|\to\infty,\ |\arg z|<\tfrac{1}{2}\pi \\ &= \frac{z^\alpha}{\pi^{\frac{1}{2}}\, 2^\alpha\,\Gamma(\alpha+\frac{1}{2})}\int_{-1}^1 e^{tz}\,(1-t^2)^{\alpha-\frac{1}{2}}\, dt, \quad \operatorname{Re}\alpha>-\tfrac{1}{2}. \end{aligned}$$

Case 2. If

$$\begin{aligned} g(w) &= w^{-2\alpha}\, {}_1F_1(\alpha+\tfrac{1}{2}; -2\alpha+1; w), \quad |\arg w|<\pi, \\ &= w^{-2\alpha}(1+\mathcal{O}(|w|)), \quad |w|\to 0,\ |\arg w|<\pi, \\ &= \frac{\Gamma(-2\alpha+1)}{\Gamma(-\alpha+\frac{1}{2})}\, e^w\, w^{-\alpha-\frac{1}{2}}\,(1+\mathcal{O}(|w|^{-1})), \quad |w|\to\infty,\ |\arg w|<\tfrac{1}{2}\pi, \end{aligned}$$

then

$$\begin{aligned} f(z) &= \frac{2^{3\alpha}}{\Gamma(1-\alpha)}\, z^\alpha\, e^{-z}\, g(2z) \\ &= I_{-\alpha}(z) \\ &= \frac{2^\alpha\, z^{-\alpha}}{\Gamma(1-\alpha)}\,(1+\mathcal{O}(|z|)), \quad |z|\to 0,\ |\arg z|<\pi, \\ &= (2\pi)^{-\frac{1}{2}}\, z^{-\frac{1}{2}}\, e^z\,(1+\mathcal{O}(|z|^{-1})), \quad |\arg z|<\tfrac{1}{2}\pi. \end{aligned}$$

Case 3. If

$$\begin{aligned} g(w) &= U(\alpha+\tfrac{1}{2}, 2\alpha+1, w) \\ &= w^{-\alpha-\frac{1}{2}}\,(1+\mathcal{O}(|w|^{-1})), \quad |w|\to\infty,\ |\arg w|<3\pi/2, \\ &= \frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^\infty e^{-sw}\, s^{\alpha-\frac{1}{2}}\,(1+s)^{\alpha-\frac{1}{2}}\, ds, \quad \operatorname{Re}\alpha>-\tfrac{1}{2},\ \operatorname{Re} w>0, \end{aligned}$$

then

$$
\begin{aligned}
f(z) &= \pi^{\frac{1}{2}}\,(2z)^{\alpha}\,e^{-z}\,g(2z) \\
&= \pi^{\frac{1}{2}}\,(2z)^{\alpha}\,e^{-z}\,U(\alpha+\tfrac{1}{2},2\alpha+1,2z) && (3.16)\\
&= K_\alpha(z) && (3.17)\\
&= (\pi/2)^{\frac{1}{2}}\,z^{-\frac{1}{2}}\,e^{-z}\,(1+\mathcal{O}(|z|^{-1})), \quad |z|\to\infty,\ |\arg z|<3\pi/2. && (3.18)\\
&= \frac{\pi^{\frac{1}{2}}\,z^{\alpha}}{2^{\alpha}\,\Gamma(\alpha+\frac{1}{2})}\int_1^\infty e^{-tz}\,(t^2-1)^{\alpha-\frac{1}{2}}\,dt.
\end{aligned}
$$

Here, in (3.17), the function K_α is the *Macdonald function* or *modified Bessel function of the third kind*, which is defined either by (3.16) in terms of a $U(a,c,\,.\,)$ function or characterized as the solution f of the differential equation (3.15) with asymptotic behaviour (3.18).

Case 4. Now observe that the differential equation (3.15) is invariant under the transformation $z\mapsto -z$, i.e., if f is a solution then so are the functions $z\mapsto f(e^{i\pi}z)$ and $z\mapsto f(e^{-i\pi}z)$.)Note that we cannot simply write $f(-z)$, since a solution f of (3.15) may be multi-valued on $\mathbb{C}\backslash\{0\}$.) Hence we obtain from Case 3 the following solution f of (3.15).

$$
\begin{aligned}
f(z) &= K_\alpha(e^{i\pi}z)\\
&= \pi^{\frac{1}{2}}\,e^{i\pi\alpha}\,2^{\alpha}\,z^{\alpha}\,e^{z}\,U(\alpha+\tfrac{1}{2},2\alpha+1,2e^{i\pi}z)\\
&= (\pi/2)^{\frac{1}{2}}\,i^{-1}\,z^{-\frac{1}{2}}\,e^{z}\,(1+\mathcal{O}(|z|^{-1})), \quad |z|\to\infty,\ -5\pi/2<\arg z<\pi/2.
\end{aligned}
$$

In order to express I_α in terms of K_α and $K_{-\alpha}$ and to give the asymptotics of $K_\alpha(z)$ as $|z|\to\infty$ for a wider region than $|\arg z|<\pi/2$, we will extend some of the earlier results for solutions f of the confluent hypergeometric differential equation

$$
z\,f''(z)+(c-z)\,f'(z)-a\,f(z)=0. \tag{3.19}
$$

Remember the formula

$$
U(a,c,z)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a;c;z)+\frac{\Gamma(c-1)}{\Gamma(a)}\,z^{1-c}\,{}_1F_1(a-c+1;2-c;z), \quad |\arg z|<\pi. \tag{3.20}
$$

Replace in this formula z by $e^{i\pi}z$, replace a by $c-a$, multiply both sides next by e^z, and then apply the transformation formula

$$
{}_1F_1(a;c;z)=e^{z}\,{}_1F_1(c-a;c;-z)
$$

to the right-hand side twice. We obtain

$$
\begin{aligned}
e^{z}\,U(c-a,c,e^{i\pi}z) &= \frac{\Gamma(1-c)}{\Gamma(1-a)}\,{}_1F_1(a;c;z)\\
&\quad+\frac{\Gamma(c-1)}{\Gamma(c-a)}\,e^{i\pi(1-c)}\,z^{1-c}\,{}_1F_1(a-c+1;2-c;z), \quad -2\pi<\arg z<0.
\end{aligned} \tag{3.21}
$$

Since the left-hand side of (3.21) is written as a linear combination of two solutions of (3.19), it is a solution of (3.19) itself. Now eliminate $z^{1-c}\,{}_1F_1(a-c+1;2-c;z)$ from (3.20) and (3.21). We obtain

$$\begin{aligned}
&{}_1F_1(a;c;z)\\
&=\frac{e^{-i\pi a}\,\Gamma(c)}{\Gamma(c-a)}\,U(a,c,z)+\frac{e^{i\pi(c-a)}\,\Gamma(c)}{\Gamma(a)}\,e^z\,U(c-a,c,e^{i\pi}z),\quad -\pi<\arg z<0, &(3.22)\\
&=\frac{e^{-i\pi a}\,\Gamma(c)}{\Gamma(c-a)}\,z^{-a}\,(1+\mathcal{O}(|z|^{-1}))\\
&\qquad\qquad+\frac{\Gamma(c)}{\Gamma(a)}\,z^{c-a}\,e^z\,(1+\mathcal{O}(|z|^{-1})),\quad |z|\to\infty,\ -\pi<\arg z<0. &(3.23)
\end{aligned}$$

For the asymptotic formula (3.23) we have used the earlier result that

$$U(a,c,z)=z^{-a}\,(1+\mathcal{O}(|z|^{-1})),\quad |z|\to\infty,\ |\arg z|<3\pi/2.$$

Note that (3.23) gives an extension to the earlier result

$${}_1F_1(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)}\,z^{a-c}\,e^z\,(1+\mathcal{O}(|z|^{-1})),\quad |z|\to\infty,\ |\arg z|<\pi/2.$$

Formulas (3.20), (3.21) and (3.22) only hold for generic values of a and c. If certain parameters or parameter differences become integer then solutions of (3.19) with logarithmic terms will enter.

Substitution of the above cases 1, 2 and 3 in formula (3.20) will yield

$$K_\alpha(z)=\frac{\pi}{2\sin(\pi\alpha)}\,(I_{-\alpha}(z)-I_\alpha(z)),$$

while substitution of cases 1 and 3 in formula (3.12) yield

$$I_\alpha(z)=(i\pi)^{-1}\,(e^{-i\pi\alpha}\,K_\alpha(z)-K_\alpha(e^{i\pi}z)),\quad -\pi<\arg z<0. \tag{3.24}$$

Now define the *Hankel functions* or *Bessel functions of the third kind* by

$$H_\alpha^{(1)}(z):=\frac{2e^{-\frac12\alpha\pi i}}{i\pi}\,K_\alpha(e^{-\frac12\pi i}z),\quad -\tfrac12\pi<z<\tfrac12\pi, \tag{3.25}$$

$$H_\alpha^{(2)}(z):=\frac{-2e^{\frac12\alpha\pi i}}{i\pi}\,K_\alpha(e^{\frac12\pi i}z),\quad -\tfrac12\pi<z<\tfrac12\pi. \tag{3.26}$$

Combination of (3.24) with (3.25), (3.26) and (3.14) yields

$$J_\alpha(z)=\tfrac12\,H_\alpha^{(1)}(z)+\tfrac12\,H_\alpha^{(2)}(z),\quad -\tfrac12\pi<z<\tfrac12\pi. \tag{3.27}$$

From (3.25), (3.26) and (3.18) we obtain the asymptotic estimates

$$H_\alpha^{(1)}(z)=(\tfrac12\pi z)^{-\frac12}\,e^{i(z-\frac12\pi\alpha-\frac14\pi)}\,(1+\mathcal{O}(|z|^{-1})),\quad |z|\to\infty,\ -\pi<\arg z<2\pi, \tag{3.28}$$

$$H_\alpha^{(2)}(z)=(\tfrac12\pi z)^{-\frac12}\,e^{-i(z-\frac12\pi\alpha-\frac14\pi)}\,(1+\mathcal{O}(|z|^{-1})),\quad |z|\to\infty,\ -2\pi<\arg z<\pi. \tag{3.29}$$

Hence, combination of (3.27), (3.28), (3.29) yields the following asymptotic estimate for the Bessel function:

$$J_\alpha(x) = (\tfrac{1}{2}\pi x)^{-\frac{1}{2}}\, \cos(x - \tfrac{1}{2}\pi\alpha - \tfrac{1}{4}\pi)\, (1 + \mathcal{O}(|x|^{-1})), \quad x \to \infty,\ x > 0. \tag{3.30}$$

The functions J_α, $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are solutions of the following differential equation (which we gave earlier in the case of integer α).

$$z^2\, f''(z) + z\, f'(z) + (z^2 - \alpha^2)\, f(z) = 0. \tag{3.31}$$

One other solution of (3.31) is the *Neumann function* or *Bessel function of the second kind* given by

$$Y_\alpha(z) := \frac{1}{2i}\, (H_\alpha^{(1)}(z) - H_\alpha^{(2)}(z)). \tag{3.32}$$

For Y_α there is by (3.28), (3.29), (3.32) an asymptotic estimate similar to (3.30):

$$Y_\alpha(x) = (\tfrac{1}{2}\pi x)^{-\frac{1}{2}}\, \sin(x - \tfrac{1}{2}\pi\alpha - \tfrac{1}{4}\pi)\, (1 + \mathcal{O}(|x|^{-1})), \quad x \to \infty,\ x > 0. \tag{3.33}$$

Since the differential equation (3.31) is invariant under the transformation $\alpha \mapsto -\alpha$, the function $J_{-\alpha}$ is also a solution. We have seen earlier that $J_{-\alpha}$ is a constant multiple of J_α if $\alpha \in \mathbb{Z}$. For non-integer α there is the following relationship between the three solutions J_α, $J_{-\alpha}$ and Y_α.

$$Y_\alpha(z) = \frac{\cos(\alpha\pi)\, J_\alpha(z) - J_{-\alpha}(z)}{\sin(\alpha\pi)}\,.$$

This can be derived from (3.30) and (3.33).

Bessel functions can be obtained as limit cases of Jacobi polynomials:

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}(1 - x^2/(2n^2))}{P_n^{(\alpha,\beta)}(1)} = \mathcal{J}_\alpha(x). \tag{3.34}$$

Indeed, this follows from

$$\lim_{n\to\infty} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; x^2/(4n^2)) = {}_0F_1(\alpha+1; -x^2/4).$$

We will give an indication how the orthogonality relations for Jacobi polynomials tend to the Hankel transform pair. First observe that a refinement of (3.34) is given by

$$\lim_{N\to\infty} \frac{P_{n_N}^{(\alpha,\beta)}(1 - x^2/(2N^2))}{P_{n_N}^{(\alpha,\beta)}(1)} = \mathcal{J}_\alpha(\lambda x) \quad \text{if} \quad \lim_{N\to\infty} \frac{n_N}{N} = \lambda.$$

Now let $n_N \to \lambda$, $m_N \to \mu$ as $N \to \infty$ and assume that $\lambda \neq \mu$. Then the orthogonality relations for Jacobi polynomials imply that

$$\int_0^{2N} P_{n_N}^{(\alpha,\beta)}(1 - x^2/(2N^2))\, P_{m_N}^{(\alpha,\beta)}(1 - x^2/(2N^2))\, x^{2\alpha+1}\, (1 - x^2/(4N^2))^\beta\, dx = 0.$$

A formal limit transition as $N \to \infty$ would yield

$$\int_0^\infty \mathcal{J}_\alpha(\lambda x)\, \mathcal{J}_\alpha(\mu x)\, x^{2\alpha+1}\, dx = 0, \quad \lambda \neq \mu.$$

The above integral no longer converges, but formally it is directly related to the Hankel transform pair.