# Foreword

The present book is about the Askey scheme and the $q$-Askey scheme, which are graphically displayed right before chapter 9 and chapter 14, respectively. The families of orthogonal polynomials in these two schemes generalize the classical orthogonal polynomials (Jacobi, Laguerre and Hermite polynomials) and they have properties similar to them. In fact, they have properties so similar that I am inclined (following Andrews & Askey [34]) to call all families in the ($q$-)Askey scheme classical orthogonal polynomials, and to call the Jacobi, Laguerre and Hermite polynomials *very classical* orthogonal polynomials.

These very classical orthogonal polynomials are good friends of mine since almost the beginning of my mathematical career. When I was a fresh PhD student at the Mathematical Centre (now CWI) in Amsterdam, Dick Askey spent a sabbatical there during the academic year 1969–1970. He lectured to us in a very stimulating way about hypergeometric functions and classical orthogonal polynomials. Even better, he gave us problems to solve which might be worth a PhD. He also pointed out to us that there was more than just Jacobi, Laguerre and Hermite polynomials, for instance Hahn polynomials, and that it was one of the merits of the Higher Transcendental Functions (Bateman project) that it included some newer stuff like the Hahn polynomials (see [198, §10.23]). Note that the emphasis in this section of the Bateman project is on Chebyshev's (or Tchebichef's) polynomials of a discrete variable, the special case of Hahn polynomials where we have equal weights on equidistant points. This special case is very important for applications, in particular in numerical analysis, see for instance Savitzky & Golay [468] (this paper from 1964 has now 2778 citations in Google Scholar) and Meer & Weiss [404]. Ironically, as Askey later wrote in his comments on [494], Chebyshev already published in 1875 on what we now call the Hahn polynomials of general parameters.

Of course, Askey told us during 1969–1970 also about the limit transitions Jacobi → Laguerre, Jacobi → Hermite and Laguerre → Hermite (formulas (9.8.16), (9.8.18) and (9.12.13) in this volume). During the seventies there grew a greater awareness that these three limit relations were part of a larger system of such limits, for instance also involving some discrete orthogonal polynomials like Hahn and Meixner polynomials. The idea to present these limits graphically was born at an

Oberwalfach meeting in 1977 on “Combinatorics and Special Functions” organized by George Andrews and Dominique Foata, also attended by me. In Dick Askey’s words (personal communication):

“I gave a talk about many of the classical type orthogonal polynomials and it fell flat. Few there appreciated it. Later in the week Michael Hoare, a physicist then at Bedford College, talked about some very nice work [155] he had done with R.D. Cooper and Mizan Rahman. In this talk he had an overhead of the polynomials they had dealt with, starting with Hahn polynomials at the top and moving down to limiting cases with arrows illustrating the limits which they had used. The audience did not seem to care much about the probability problem, but they were very excited about the chart[1] he had shown and wanted copies. If there was that much interest in his chart, I thought that it should be extended to include all of the classical type polynomials which had been found. This was included in the Memoir [72, Appendix] Jim Wilson and I wrote. We missed one case, since we had found the symmetric continuous Hahn polynomials, but had not realized that the symmetry was not needed. That was done by Atakishiyev and Suslov [81].”

During the conference *Polynômes Orthogonaux et Applications* in Bar-le-Duc, France in 1984, Jacques Labelle presented a poster of size $89 \times 122$ cm containing what he called *Tableau d’Askey* [362]. For some years I had it hanging on the wall of my office, but it gradually faded away.

At this 1984 Bar-le-Duc conference Andrews & Askey [34] talked about the $q$-analogues of the polynomials in the Askey scheme, which had already been around for some seven years, starting with the work of Askey together with his PhD student Jim Wilson. This culminated into their joint memoir [72] in 1985. As Andrews & Askey wrote in [34]:

“A set of orthogonal polynomials is *classical* if it is a special case or a limiting case of the Askey-Wilson polynomials or $q$-Racah polynomials.”

It took some time before also the $q$-Askey scheme was graphically displayed. As Labelle wrote in [362], one would need a 3-dimensional chart, because there are both arrows within the $q$-Askey scheme and from the $q$-Askey scheme to the Askey scheme. But if one is satisfied with just the arrows of the first type, then one can find the $q$-Askey scheme just before chapter 14 in the present volume.

The present book is a merger of the report *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue* by Koekoek and Swarttouw (1994, and thoroughly revised and updated in 1998) and of a series of papers by Lesky on the classification of these polynomials. The report of Koekoek and Swarttouw had its roots in a regular seminar on orthogonal polynomials at Delft University of Technology in 1990 or so, when Koekoek and Swarttouw were PhD students there, and where I also participated. The Koekoek-Swarttouw report, in its various versions, has become very well-known, a real standard reference although it was lacking during all those years the status of a publication at a recognized publisher. At present, Google Scholar gives 686 citations for the version at arXiv (`arXiv:math/9602214v1`

[1] An extension of [155, p.285, Figure 2], called *The seven-fold way of orthogonal polynomials* and *The seven-fold way of probability distributions*