Curie-Weiss model of the quantum measurement process,

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Quantum measurement as driven phase transition: An exactly solvable model,

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Setup

- The model: test spin + magnet + bath
- Classical measurement of $s_z = \pm 1$
- Statistical interpretation of quantum mechanics
- Selection of collapse basis and collapse
- Registration of measurement
- Summary

- **The model of ABN '03**:
- System S: s_z of spin-$\frac{1}{2}$ particle

$\hat{H}_{\mathsf{S}} = 0$ (ideal measurement)

- Full Hamiltonian: $\hat{H} = \hat{H}_{\mathsf{S}} + \hat{H}_{\mathsf{SA}} + \hat{H}_{\mathsf{A}}$

$$\hat{H}_{\mathsf{SA}} = -g\hat{s}_z \sum_{n=1}^{N} \hat{\sigma}_z^{(n)} = -gN\hat{s}_z\hat{m}, \quad \hat{m} = \frac{1}{N}\sum_{n=1}^{N} \hat{\sigma}_z^{(n)}$$

- Apparatus: A=M+B:

M: Curie-Weiss-type ferromagnet; B: bath.

$$\hat{H}_{\mathsf{A}} = \hat{H}_{\mathsf{M}} + \hat{H}_{\mathsf{MB}} + \hat{H}_{\mathsf{B}}$$

$$\hat{H}_{\mathsf{M}} = -\frac{1}{4}JN\hat{m}^4, \qquad (\text{Curie-Weiss: } -\frac{1}{2}JN\hat{m}^2)$$

- Bath: bilinear coupling to harmonic oscillators,

$$\hat{H}_{\mathsf{MB}} = -\sum_{n=1}^{N} \sum_{a=x,y,z} \hat{\sigma}_a^{(n)} \left\{ \sum_i c_i (\hat{a}_{i;n,a}^{\dagger} + \hat{a}_{i;n,a}) \right\}$$

$$\hat{H}_{\mathsf{B}} = \sum_{n=1}^{N} \sum_{a=x,y,z} \sum_i \hbar\omega_i \hat{a}_{i;n,a}^{\dagger} \hat{a}_{i;n,a}$$

- Coupling parameter: $c_i^2 \sim \gamma \ll \frac{1}{\hbar}$

- **Classical measurement**: measure $s_z = \pm 1$

- of a single spin
- of ensemble of spins, probabilities $p_\uparrow$, $p_\downarrow$

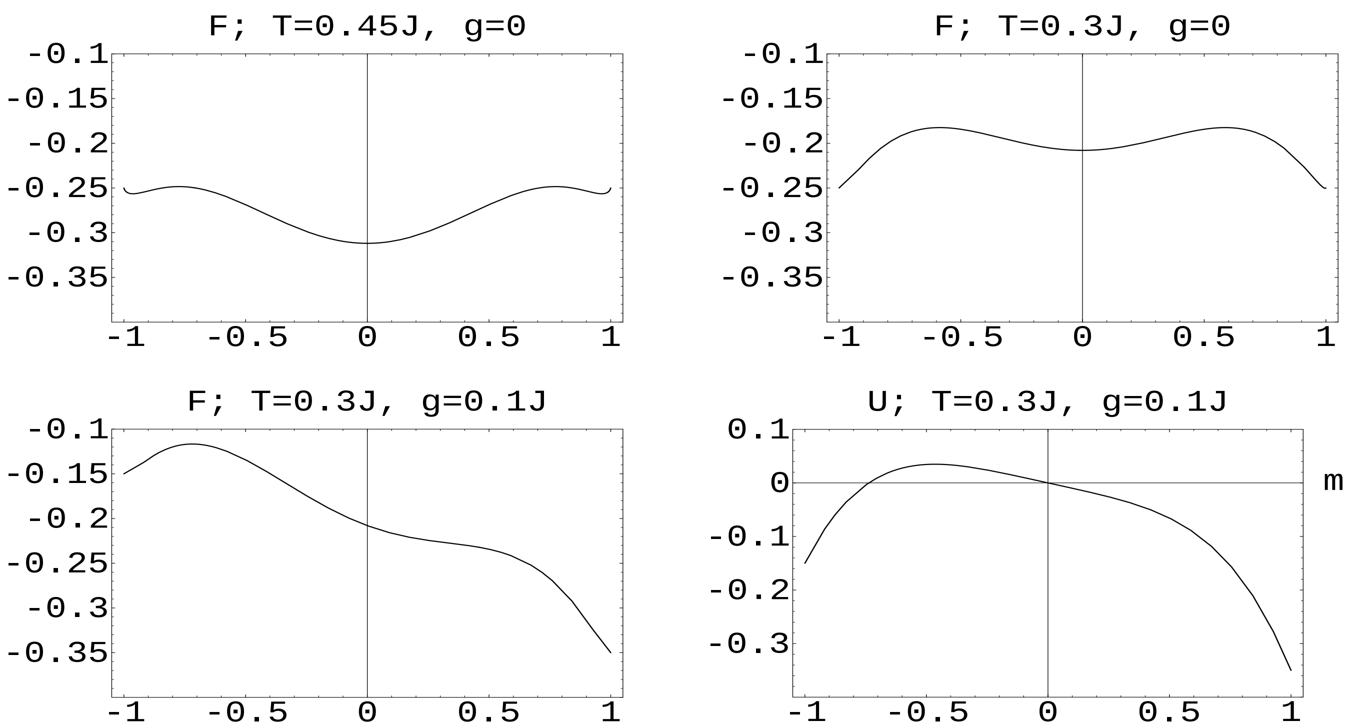
$$H_{\text{classical}} = -gNs_z\underline{m} - \frac{JN}{4}\underline{m}^4, \qquad \underline{m} \equiv \frac{1}{N}\sum_{n=1}^{N}\sigma_{z,n}$$

- Statistical description of apparatus. Initially:

M is paramagnetic (non-biased); B Gibbsian

- Free energy: $F = U - TS$

$$F = -gs_z m - \frac{Jm^4}{4} - T\left[\frac{1+m}{2}\ln\frac{2}{1+m} + \frac{1-m}{2}\ln\frac{2}{1-m}\right]$$



- T low enough $\mapsto$ first order phase transition

- g large enough: registration of $s_z = \pm 1$,

since dynamics will drive $m = 0 \mapsto \pm m^*(g)$

- switch off A: put $g = 0$.

m goes to minimum of $g = 0$ and stays there

- tunneling time $\sim \exp(N)$:

up to then a unique registration of m

- Dynamics:

- guess it, (Suzuki-Kubo, Goldstein-Scully, $\cdots$)
- work out role of bath $\mapsto$ new result

$$\dot{m} = \gamma h \left(1 - \frac{m}{\tanh \beta h}\right), \qquad h = g s_z + J m^{p-1}$$

- Characteristic timescale: $\tau_{\text{meas}} = \frac{1}{\gamma g}$

- The statistical interpretation of quantum mechanics: (Einstein, Ballentine,...)

- A single system does not have "its own" wavefunction.

- Each quantum state describes an ensemble of systems.

- A pure state $|\psi\rangle\langle\psi|$ describes an ensemble of identically prepared systems.

$\mapsto$ Quantum measurement = an ensemble of measurements on an ensemble of systems.

No underlying wavefunction of total system
$\mapsto$ Description in terms of full density matrix.

- Consistency:
All possible outcomes with Born probabilities??

- Full density matrix $\mathcal{D} = \left(\begin{array}{cc} \mathcal{D}_{\uparrow\uparrow} & \mathcal{D}_{\uparrow\downarrow} \\ \mathcal{D}_{\downarrow\uparrow} & \mathcal{D}_{\downarrow\downarrow} \end{array} \right)$

At $t = 0$: $\mathcal{D}(0) = r(0) \otimes R_M(0) \otimes R_{\mathrm{B}}(0)$

- Initial state spin: arbitrary,

$$r(0) = \left(\begin{array}{cc} r_{\uparrow\uparrow}(0) & r_{\uparrow\downarrow}(0) \\ r_{\downarrow\uparrow}(0) & r_{\downarrow\downarrow}(0) \end{array} \right)$$

- Initial paramagnet: $R_M(0) = \Pi_{n=1}^{N} \rho^{(n)}(0)$

$$\rho^{(n)}(0) = \frac{1}{2} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

- Initial state bath: Gibbsian $R_{\mathrm{B}}(0) = \frac{e^{-\beta \hat{H}_{\mathrm{B}}}}{Z_{\mathrm{B}}}$

- Dynamics: von Neumann eqn. $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} = [\hat{H}, \mathcal{D}]$

- Selection of the collapse basis:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} r_{\mathrm{ij}} &\equiv \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{tr}_{\mathrm{M,B}}\, \mathcal{D}_{\mathrm{ij}} = -gN(s_i - s_j)\mathrm{tr}_{\mathrm{M,B}}[\hat{m}, \mathcal{D}_{\mathrm{ij}}] \\ &= 0 \qquad\qquad \text{iff } s_i = s_j \end{aligned}$$

$\mapsto$ Collapse on diagonal basis of $\hat{H}_{\mathrm{SA}}$.

$p_\uparrow = r_{\uparrow\uparrow}$ and $p_\downarrow = r_{\downarrow\downarrow}$ conserved in time;
$r_{\uparrow\downarrow} = r^*_{\downarrow\uparrow}$: possibly decay.

• Short times: bath is inactive.

Test spin + N independent spins of magnet

$$D_{\uparrow\downarrow} = \mathrm{tr}_\mathrm{B}\mathcal{D}_{\uparrow\downarrow} = \frac{r_{\uparrow\downarrow}(0)}{2^N}\Pi_{n=1}^{n}[\cos\frac{2gt}{\hbar} + i\hat{\sigma}_z^{(n)}\sin\frac{2gt}{\hbar}]$$

$$\begin{aligned} r_{\uparrow\downarrow}(t) = \mathrm{tr}_\mathrm{M} D_{\uparrow\downarrow}(t) &= r_{\uparrow\downarrow}(0)[\cos\frac{2gt}{\hbar}]^N \\ &\approx r_{\uparrow\downarrow}(0)\exp(-\frac{t^2}{{\tau_{\mathrm{collapse}}}^2}) \end{aligned}$$

$\tau_{\mathrm{collapse}} = \frac{\hbar}{g\sqrt{2N}}$: Not decoherence.

• Revivals at $t_k = k\,\frac{\pi\hbar}{2g}$ $(k = 1, 2, \cdots)$:

- suppressed by small spread in g's

$$r_{\uparrow\downarrow}(t_k) = (-1)^k r_{\uparrow\downarrow}(0)\exp(-\pi^2 k^2 \frac{\langle g^2\rangle - \langle g\rangle^2}{2\langle g\rangle^2}N);$$

- suppressed by bath $|r_{\uparrow\downarrow}(t_k)| \sim e^{-N\gamma\hbar}$.

Formal solution:

- go to sectors $\mathcal{D}_{\uparrow\uparrow}$, $\mathcal{D}_{\uparrow\downarrow}$, $\mathcal{D}_{\downarrow\uparrow}$ and $\mathcal{D}_{\downarrow\downarrow}$.

- Mean field: $\hat{m}^4 \to m_{ij}^4 + 4\, m_{ij}^3(\hat{m} - m_{\mathrm{ij}})$; self-consistent m_{ij}.

- Weak coupling to bath:
Expand to order $(\hat{H}_{\mathrm{MB}})^2$ = first order in γ.
Trace over bath: $2^N \times 2^N$ matrix $D_{ij} = \mathrm{tr}_B \mathcal{D}_{ij}$

- Solution: N apparatus spins equivalent,

$$D_{ij}(t) = r_{ij}(0) \times \rho_{ij}(t) \otimes \cdots \otimes \rho_{ij}(t).$$

- **Result**:

$\uparrow\downarrow$, $\downarrow\uparrow$: Collapse confirmed. $\tau_{\mathrm{collapse}} \sim \frac{1}{\sqrt{N}}$.

$\uparrow\uparrow$, $\downarrow\downarrow$: Registration as in “classical treatment”.
Characteristic time $\tau_{\mathrm{meas}} = \frac{1}{\gamma g} \gg \frac{\hbar}{T}$
(Energy must be dumped in bath).

Summary

- Q-measurement = collapse + registration

- Collapse occurs due to quantum dynamics of spin+magnet ($\neq$ decoherence)
Very small correlations between test spin and n apparatus spins ($n = 1, 2, \cdots$).

- Registration of the measurement = classical.
Common state of test spin and apparatus:

$$D(t_\mathrm{f}) = p_\uparrow \times |\uparrow\rangle\langle\uparrow| \otimes \rho_{\uparrow\uparrow}(t_\mathrm{f}) \otimes \cdots \otimes \rho_{\uparrow\uparrow}(t_\mathrm{f}) \\ + p_\downarrow \times |\downarrow\rangle\langle\downarrow| \otimes \rho_{\downarrow\downarrow}(t_\mathrm{f}) \otimes \cdots \otimes \rho_{\downarrow\downarrow}(t_\mathrm{f}).$$

$$\rho_{\uparrow\uparrow}(t_\mathrm{f}) = \frac{1}{2}\begin{pmatrix} 1 + m_\uparrow & 0 \\ 0 & 1 - m_\uparrow \end{pmatrix},$$

Gibbs for apparatus spin in classical field $+g$.

- Born rule: $p_\uparrow = r_{\uparrow\uparrow}(0)$, $p_\downarrow = r_{\downarrow\downarrow}(0)$.

- QM alone describes quantum measurements.
N-dependence of apparatus can be tested

- Consistency with statistical interpretation QM