Quantum measurement as driven phase transition: An exactly solvable model,
Phys. Rev. A. 64 (2001) 032108.

Curie-Weiss model of the quantum measurement process,
preprint 2002.

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Setup

- Introduction
- The model
- Its dynamics
- Diagonal elements of density matrix
- Off-diagonal elements
- Result of measurement
- Summary, conclusion

The measurement

- Hermitean operator $\widehat{S}$: eigenvalue s_i, eigenfunction ψ_i.
- By measuring $\widehat{S}$, an eigenvalue s_i is observed.

- Pure state: If $\psi = \sum_i c_i \psi_i$, then the probability is $|c_i^2|$.
After measurement: $\psi \to \psi_i$: collaps of wavefunction, reduction of wavepacket.

- General: density matrix $\rho = \sum \rho_{ij} \psi_i^* \psi_j$: prob. is ρ_{ii}.

- Grandmother's tale: collaps is instantaneous,
no Schrödinger cats: $\rho_{i \neq j} = 0$.

- von Neumann:
collaps is non-unitary $\to$ Additional postulate needed:
Full Quant Mech = Quant Mech + collaps postulate.

- Other fairy tales:

- Wigner, and his friend: Mind-body problems
If I perform a measurement but do not look at the outcome, it is not finished.
(But my friend could look at it, and then ...)

- Everett: multi-universe picture:
In each measurement a collection of universes is opened; we go into one of them...

For adults only: van Kampen's 10 theorems (1988); Balian's requirements for quantum measurement (1989).

- Apparatus is macroscopic.

- At $t = 0$ density matrix total system uncorrelated:

$$D(0) = r(0) \otimes R(0)$$

system S: arbitrary $r(0)$
apparatus A: in metastable state $R(0)$

- Apparatus reaches at end of measurement state R_i. The R_i are equally probable, to avoid bias.

- Each R_i is stable, for robust, permanent registration. Pointer variable $\widehat{A}$ has negligible fluctuations around A_i.

- The observable s of S does not change much during the process (fast measurement).

- For ideal measurement: special type of decoherence, depending on measured observable $\hat{s}$, which reduces $r(0)$ its diagonal block $r_i = r_{ii}(0)|i\rangle\langle i|$ associated with s_i.

- Special classical correlations between S and A,

$$D(0) \to D(\infty) = \sum_i p_i \, |i\rangle\langle i| \otimes R_i.$$

Probabilities: $p_i = r_{ii}(0)$ (Born's law).
Final state S: $r_i = p_i \, |i\rangle\langle i|$ (von Neumann's reduction).

• **ABN1**: Apparatus = ideal Bose gas+bath, measures position of test particle.

Apparatus initially close to Bose-Einstein transition; driven into it by the measurement process.

Picture confirmed. Drawback: initial state not metastable.

• **ABN2**: Measure s_z of spin-$\frac{1}{2}$ particle, using Curie-Weiss ferromagnet coupled to bath.

Full Hamiltonian:

$$\hat{H} = \hat{H}_{\mathsf{SA}} + \hat{H}_{\mathsf{A}} + \hat{H}_{\mathsf{AB}} + \hat{H}_{\mathsf{B}}$$

$$\hat{H}_{\mathsf{SA}} = -gN\hat{s}_z\hat{m}, \qquad \hat{m} = \frac{1}{N}\sum_{n=1}^{N}\hat{\sigma}_z^{(n)}$$

$$\hat{H}_{\mathsf{A}} = -\frac{1}{4}JN\hat{m}^4, \qquad \hat{H}_{\mathsf{AB}} = \sum_{n=1}^{N}\sum_{a=x,y,z}\hat{\sigma}_a^{(n)}\mathcal{B}_a^{(n)}$$

• Stationary, quasi-Ohmic bath:

$$\mathrm{tr}_{\mathsf{B}}[\, D_B(0)\mathcal{B}_a^{(m)}(t)\mathcal{B}_b^{(n)}(s)\,] = \delta_{a,b}\,\delta_{m,n}\,K(t-s)$$

$$K(t) = \gamma\hbar^2\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{8\pi}\,e^{\mathrm{i}\omega t}\,[\frac{\omega}{\tanh\frac{1}{2}\beta\hbar\omega} - \omega]\,e^{-|\omega|/\Gamma}$$

$\gamma \ll 1$: dimensionless coupling constant
Γ: large Debye cut-off frequency.

• T not too large, $p > 2$:
Apparatus can start in metastable paramagnet ($m = 0$).

Dynamics:

- Consider full density matrix $\mathcal{D}(0) = r(0) \otimes R(0) \otimes D_B(0)$
- von Neumann eqn. $i\hbar \frac{d}{dt}\mathcal{D} = [\hat{H}, \mathcal{D}]$
- go to sectors with fixed $s_z = \pm 1$: $\mathcal{D}_{ij} = \langle i|\mathcal{D}|j\rangle$.
- Mean field: $\hat{m}^4 \to m_{ij}^4 + 4\, m_{ij}^3 \hat{m}$; self-consistent m_{ij}.
- Weak coupling to bath: Expand to first order in γ. Trace over bath: $2^N \times 2^N$ matrix $D_{ij} = \mathrm{tr}_B \mathcal{D}_{ij}$

- Solution: N apparatus spins equivalent,

$$D_{ij}(t) = r_{ij}(0) \times \rho_{ij}(t) \otimes \cdots \otimes \rho_{ij}(t).$$

- **Result: Bloch equations for spins of apparatus.**

Define $\sigma_{0,ij} = \mathrm{tr}\, \rho_{ij}, \qquad \sigma_{z,ij} = \mathrm{tr}\, \hat{\sigma}_z \rho_{ij}$.

Initial paramagnet:

$$\rho_{ij}(0) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \sigma_{0,ij} = 1;\ \sigma_{z;ij} = 0.$$

No transverse components: $\sigma_{x,ij} = \sigma_{y;ij} = 0$ at all t.

- **Evolution of the diagonal elements.**
- $\sigma_{0,\uparrow\uparrow}(t) = \sigma_{0,\downarrow\downarrow}(t) = 1$ conserved;

$\rightarrow r_{\uparrow\uparrow}(t) \equiv \mathrm{tr}_{\mathsf{A}}\, D_{\uparrow\uparrow}(t) = r_{\uparrow\uparrow}(0)$ conserved.

- magnetization $m_\uparrow = \sigma_{z,\uparrow\uparrow}$: $\quad \dot{m}_\uparrow = \frac{2\gamma g_\uparrow}{\hbar}(1 - \frac{m_\uparrow}{\tanh \beta g_\uparrow})$,

  effective field: $g_\uparrow(t) = g + J m_\uparrow^3(t)$.

- $m_\uparrow$ goes to minimum of Curie-Weiss free energy in field:

$$\frac{F(m)}{N} = -gm - \frac{Jm^4}{4} + T\frac{1+m}{2}\ln\frac{1+m}{2} + T\frac{1-m}{2}\ln\frac{1-m}{2}$$

- T small enough, g large enough $\rightarrow m_\uparrow(\infty)$ near 1. Switch off g after measurement: $m_\uparrow$ goes to solution of $m = \tanh \beta J m^3$ near $m = 1$. Keeps that value for ever.

- Characteristic time for measurement,

$$\tau_{\mathrm{meas}} = \frac{\hbar}{\gamma g}, \qquad \frac{1}{\gamma} : \text{coupling to bath.}$$

- **Off-diagonal elements.**
- $\dot{\sigma}_{0,12} = \frac{2ig}{\hbar}\sigma_{z,12}; \qquad \dot{\sigma}_{z,12} = \frac{2ig}{\hbar}\sigma_{0,12} - 2\Lambda\,\sigma_{z,12}.$

Damping coefficient:

$\Lambda(t) = \frac{1}{\pi}\gamma\Gamma^2 t$ (small t); $\qquad \Lambda(\infty) = \frac{\gamma g}{\hbar \tanh \beta g}$

Time needed for collaps of wavefunction:
(Fate of Schrödinger cats)

- $N \gg \frac{1}{\gamma}: \qquad r_{\uparrow\downarrow}(t) \sim r_{\uparrow\downarrow}(0)\, \exp[-(t/\tau_{\text{collaps}})^2],$

$$\tau_{\text{collaps}} = \frac{\sqrt{\pi}}{\sqrt{\gamma N}}\,\frac{1}{\Gamma} \ll \tau_{\text{meas}} :$$

Collaps solely due to coupling to apparatus.

- $N \ll \frac{1}{\gamma}: \qquad r_{12}(t) = r_{12}(0)\, e^{-N\Lambda t}\left(\cos\frac{2gt}{\hbar} + \frac{\hbar\Lambda}{2g}\sin\frac{2gt}{\hbar}\right)^N,$

$$\tau_{\text{collaps}} = \frac{1}{N}\,\frac{\hbar}{\gamma g}\,\tanh \beta g \sim \frac{1}{N}\tau_{\text{meas}},$$

- **Result of the measurement.**

Proper description of the measurement process:
Common state of tested system and apparatus

$$D(\infty) = p_\uparrow \times |\uparrow\rangle\langle\uparrow| \otimes \rho_{\uparrow\uparrow}(\infty) \otimes \cdots \otimes \rho_{\uparrow\uparrow}(\infty)$$

$$+p_\downarrow \times |\downarrow\rangle\langle\downarrow| \otimes \rho_{\downarrow\downarrow}(\infty) \otimes \cdots \otimes \rho_{\downarrow\downarrow}(\infty),$$

Born rule: $p_\uparrow = r_{\uparrow\uparrow}(0)$, $p_\downarrow = r_{\downarrow\downarrow}(0)$.

$\rho_{\uparrow\uparrow}(\infty)$ and $\rho_{\downarrow\downarrow}(\infty)$: Gibbsian density matrix for apparatus spin in classical field $\pm h$:

$$\rho_{ii}(\infty) = \frac{1}{2}\begin{pmatrix} 1 + m_{ii} & 0 \\ 0 & 1 - m_{ii} \end{pmatrix}, \qquad i = 1, 2.$$

- **Further tests.**

Let this setup be input for larger measurement (possible in spintronics?).

Observing collaps of test particle: $\langle \hat{s}_x \rangle(t) = 2\Re\, r_{12}(t)$.

Observation of test particle & apparatus spin:
$\langle \hat{s}_y \hat{\sigma}_z \rangle(t) \approx \tan(2gt/\hbar)\, \langle \hat{s}_x \rangle(t)$.

Summary

- To measure s_z of test spin, it is coupled to Curie-Weiss ferromagnet, coupled to a bath.

- Initial state of apparatus is metastable paramagnet.

- Collaps of wavefunction is short but finite time;
due to coupling to apparatus, which is noisy due to bath.

- Proper common state of test spin and apparatus.

- Born rule obeyed.

Conclusion

- The quantum measurement described by statistical thermodynamics of the test system and apparatus.

- Good measurement occurs for macroscopic apparatus.

- Collaps of wavefunction is not instantaneous.

- No other explanations (friends, multi-universes) needed.