Quantum measurement as driven phase transition: An exactly solvable model

Armen E. Allahverdyan, Roger Balian and Theo M. Nieuwenhuizen, cond-mat/0102428

von Neumann's *additional postulate*:

For the wavefunction $|\psi\rangle = \sum_n c_n|x_n\rangle$, the measured value of the operator x is x_k, with probability $p_k = |c_k|^2$.

Collaps of wavefunction or **reduction of wavepacket**:
After measurement $|\psi\rangle$ is replaced by the eigenstate $|x_k\rangle$.

Today's view: No extra postulate needed.
The apparatus is also subject to quantum mechanics.

The game: find a (realistic) model where various steps of the measurement process can be derived.

Our approach: Quantum statistical mechanics of a system close a phase transition; it is driven into the condensed phase by the measurement process.

The quantum measurement problem

- System has hermiten operator x with eigenvalues x_k.

- Apparatus has pointer variable X, which takes value X_k in correspondence with x_k.

ρ = density operator of system,
R = density operator of apparatus,
$\mathcal{R}$ = global density operator

Begin measurement: $\mathcal{R}(t=0) = \rho(0) \otimes R(0)$.

End: $\mathcal{R}(t=\theta) \simeq \sum_k p_k \mathcal{R}_k$ with $\mathrm{tr}(\mathcal{R}_k \mathcal{R}_l) \simeq \delta_{kl}$.

Exclusive events: negligible statistical fluctuation:

$\mathrm{tr}(\mathcal{R}_k X) = X_k, \qquad \mathrm{tr}(\mathcal{R}_k X^2) \simeq X_k^2.$

Probability: $p_k = \mathrm{tr}_S\{\rho(0)\Pi_k\}, \quad \Pi_k = |x_k\rangle\langle x_k|$.

Ideal measurement: system weakly perturbed.

$\mathcal{R}_k(\theta) = \rho_k \otimes R_k$. Stronger condition: $\mathrm{tr}(R_k R_l) \simeq \delta_{kl}$, *robustness* of apparatus as information-storing device.

Reduction of wave-packet or *collapse of wavefunction*: after measurement system is in state $|x_k\rangle$: $\rho_k = \frac{1}{p_k}\Pi_k \rho(0) \Pi_k$

- **Requirements for a good apparatus**

1) Has degree of freedom X that can relax to X_k.

2) Is macroscopic to ensure irreversible relaxation.

3) Relaxation selectively triggered by the interaction of X with x.

4) No bias by apparatus: The X_k are equally probable; the R_k have the same entropy.

5) The apparatus is a stable and robust information storing device: the states R_k are nearly in equilibrium; X is a nearly conserved collective variable (after the measurement).

5') Take a macroscopic system that can undergo a phase transition, with *order parameter* X_k. $\rightarrow$ robust and (meta)stable; ergodicity breaking

6) x is coupled to X and the apparatus should *amplify* this signal. Possible near phase transition: x is small source coupling to order parameter.

7) Relaxation of order parameter ensured by coupling of apparatus to a bath.

Our setup

Apparatus: ideal Bose gas coupled to particle reservoir:

$H_{\mathsf{A}} = \sum_i \varepsilon_i a_i^\dagger a_i, \qquad \varepsilon_i = \frac{\hbar^2 k^2}{2M}, \qquad \mathcal{N} = \sum_i a_i^\dagger a_i$

$\rho_{\mathsf{A}}(0) = \frac{1}{Z}\,\mathrm{exp}(-\beta H_{\mathsf{A}} + \beta\mu\mathcal{N}), \quad Z = \mathrm{tr}\,\mathrm{exp}(-\beta H_{\mathsf{A}} + \beta\mu\mathcal{N}).$

System: 1-d particle in potential: $H_{\mathsf{S}} = \frac{p^2}{2m} + V(x)$

Interaction: $H_{\mathrm{I}} = -g\ x\ X, \qquad X = \sqrt{\frac{\hbar}{2}}\ (a_0^\dagger + a_0)$

Bath: $H_{\mathsf{AB}} = \hbar \sum_{i,m} \left(\Omega_m\, \xi_{im}^\dagger\, \xi_{im} + c_m a_i^\dagger \xi_{im} + c_m^* \xi_{im}^\dagger a_i \right)$
allows relaxation of apparatus modes at temperature T.

Bose-Einstein condensation of apparatus

$H_{\mathsf{A}} + H_{\mathrm{I}} = \sum_i \varepsilon_i a_i^\dagger a_i - J(a_0 + a_0^\dagger)$ with $J = gx\sqrt{\hbar/2}$.

So at given μ: $\langle a_0 \rangle = -\frac{J}{\mu}, \qquad \langle a_0^\dagger a_0 \rangle = \frac{1}{\mathrm{e}^{-\beta\mu} - 1} + \frac{J^2}{\mu^2}.$

Density: $\frac{\langle \mathcal{N} \rangle}{V} = \frac{(2M)^{3/2}}{4\pi^2\hbar^3} \int_0^\infty \frac{\mathrm{d}\varepsilon\ \sqrt{\varepsilon}}{\mathrm{e}^{\beta(\varepsilon-\mu)} - 1} + \frac{1}{V}\frac{1}{\mathrm{e}^{-\beta\mu} - 1} + \frac{J^2}{V\mu^2}$

$$\frac{N}{V} = \frac{N_{\mathrm{n}}}{V} + \frac{N_{\mathrm{c}}}{V} = 0.165\ \frac{(M\ \ T)^{3/2}}{\hbar^3} + \left(\frac{T}{V|\mu|} + \frac{J^2}{V\mu^2} \right)$$

N_c dominated by J if $1 \gg \frac{|\mu|}{T} \gg \frac{1}{N}, \quad \frac{|\mu|}{J} = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$

Then $x = \pm\sqrt{\frac{2\mu^2 N_c}{\hbar g^2}}$ follows from measuring N_c and $\langle a_0 \rangle$.

Dynamics of the apparatus in its bath

$\dot{a}_i = -i(\omega_i + \alpha)a_i - \gamma a_i + \sqrt{2\gamma}\; b_i(t) \qquad (i \neq 0),$

with $\omega_i \equiv \frac{\varepsilon_i}{\hbar}$, $\alpha \equiv -\frac{\mu}{\hbar}$,

γ is damping rate, $b_i(t)$ Gaussian white noise from bath.

Thermal occupation: $\langle b_i^\dagger(t) b_k(t')\rangle = \delta_{ik}\delta(t-t')\, n_i^{\mathrm{eq}}$

$n_i^{\mathrm{eq}} = \frac{1}{\mathrm{e}^{\beta(\varepsilon_i - \mu)} - 1}$:

Exact solution:

$n_i(t) \equiv \langle a_i^\dagger(t) a_i(t)\rangle = \mathrm{e}^{-2\gamma(t)} n_i(0) + (1 - \mathrm{e}^{-2\gamma t}) n_i^{\mathrm{eq}}$

Dynamics of order parameter

Let $a \equiv a_0$, $b \equiv b_0$. Then $\omega_0 = 0$ and

$\dot{a} = -i\alpha a - \gamma a + \sqrt{2\gamma}\; b(t) + \frac{i}{\sqrt{2\hbar}}\; g\; x(t).$

Exact solution when $x(t) \approx$ constant:

$n(t) = n^{\mathrm{eq}} + \mathrm{e}^{-2\gamma t}[n(0) - n^{\mathrm{eq}}] + \frac{1 + \mathrm{e}^{-2\gamma t} - 2\mathrm{e}^{-\gamma t}\cos\alpha t}{2\hbar(\gamma^2 + \alpha^2)} g^2 x^2$

Dynamics of the tested particle:

$$\dot{x} = \frac{1}{m}p, \quad \dot{p} = -V'(x) + \int_0^t \mathrm{d}s\chi(t-s)x(s) + \eta(t),$$

with $\chi(t) = g^2\ \mathrm{e}^{-\gamma t}\sin \alpha t$.

Noise $\eta(t) = \eta_0(t)+\eta_1(t)$, from bath and from apparatus:
$$\eta_0(t) = \sqrt{\hbar\gamma}\ g\int_0^t \mathrm{d}s\ \mathrm{e}^{-\gamma s}[b^\dagger(t-s)\mathrm{e}^{i\alpha s} + b(t-s)\mathrm{e}^{-i\alpha s}],$$
$$\eta_1(t) = \sqrt{\frac{\hbar}{2}}\ g\mathrm{e}^{-\gamma t}[a^\dagger(0)\mathrm{e}^{i\alpha t} + a(0)\mathrm{e}^{-i\alpha t}],$$

Noise autocorrelation:

$$K(t,t') = \frac{1}{2}\langle\eta(t)\eta(t') + \eta(t')\eta(t)\rangle = K_0(t,t') + K_1(t,t'),$$
$$K_0(t,t') = \frac{g^2\hbar}{2}\cos[\alpha(t-t')]\ \coth\frac{\hbar\alpha}{2T}\ \left(\mathrm{e}^{-\gamma|t-t'|} - \mathrm{e}^{-\gamma(t+t')}\right),$$
$$K_1(t,t') = \frac{g^2\hbar}{2}\cos[\alpha(t-t')]\ \coth\frac{\hbar\alpha}{2T}\quad \mathrm{e}^{-\gamma(t+t')},$$

$K(t,t')$ is time-translation invariant, but $K_{0,1}$ are not.

Brownian motion

$$m\ddot{x} = -V'(x) - \int_0^t \mathrm{d}s\tilde{\chi}(t-s)\dot{x}(s) + \eta(t) - \tilde{\chi}(t)x(0) + \tilde{\chi}(0)x(t),$$

Oscillating 'damping kernel' $\tilde{\chi}(t) = \mathrm{e}^{-\gamma t}\,\frac{\alpha\cos\alpha t + \gamma\sin\alpha t}{\alpha^2+\gamma^2}$.

Weak coupling to bath $\mapsto \gamma \ll \alpha$.

Measurement time larger than bath relaxation time:
$\theta \gg \frac{1}{\gamma} \mapsto$ many oscillations.

Wigner function of condensate and particle

$a = \frac{1}{\sqrt{2\hbar}}(X + iP), \qquad a^\dagger = \frac{1}{\sqrt{2\hbar}}(X - iP),$

Wigner fion condensate + particle: $\mathcal{W}(X, P, x, p)$;

Wigner fion condensate: $W(X, P) = \int \frac{\mathrm{d}p\mathrm{d}x}{2\pi}\mathcal{W}(X, P, x, p)$

Wigner fion particle: $w(x, p) = \int \mathrm{d}\xi \ \mathrm{e}^{-i\xi p/\hbar} \ \langle x + \frac{\xi}{2}|\rho|x - \frac{\xi}{2}\rangle$

Intermediate Wigner function: particle in coordinate-representation (useful when x discrete, e.g. spin)

$\mathcal{V}(X, P, x, x') = \int \mathrm{d}p \ \mathrm{e}^{ip(x-x')/\hbar}\mathcal{W}(X, P, \frac{x+x'}{2}, p)$

Ideal measurement: Postmeasurement states

Asssume m large, so x not changed during measurement

Apparatus

Probability that particle at x:

$\int \frac{\mathrm{d}p_0}{2\pi\hbar}\ w(x,p_0;0) = \langle x|\rho(0)|x\rangle$

Wigner function is an average over events at fixed x

$W(X,P;\theta) = \int \mathrm{d}x\ \langle x|\rho(0)|x\rangle\ W_x(X,P),$

W_x is a shifted Gibbsian:

$$W_x(X,P) = \frac{\hbar}{\lambda}\exp\left[-\frac{1}{2\lambda}\left(X-\frac{gx}{\alpha}\right)^2 - \frac{1}{2\lambda}P^2\right], \qquad \lambda = -\frac{T\hbar}{\mu}.$$

[$a, a^\dagger$ shifted; $X \propto a + a^\dagger$ shifted; $P \propto i(a - a^\dagger)$ not.]

Amplification and registration

Before measurement the apparatus is close to phase transition.

Small coupling term $H_{\mathrm{I}} = -gxX$ brings apparatus in condensed phase: apparatus amplifies the measured signal.

After the final time θ the coupling between apparatus and system is switched off, and also the exchange of apparatus with the bath. The apparatus will then stay in this state, and can be read off.

Robustness and accuracy of the measurement

What is the probability that aparatus will leave its state spontaneously?

Assume apparatus has density matrix R characterized by

$$\langle X \rangle = \frac{gx}{\alpha}. \tag{1}$$

Transition probability to state R' associated with x'?
If states are pure, $\mathrm{Pr}(x \to x') = \mathrm{tr}(RR')$.
For mixed states we use

$$\begin{aligned} \mathrm{Pr}(x \to x') \propto \mathrm{tr}(RR') &= \int \frac{\mathrm{d}X\ \mathrm{d}P}{2\pi\hbar}\ W_x(X,P)W_{x'}(X,P) \\ &= \exp\left[-\frac{g^2(x-x')^2}{4\lambda\alpha^2}\right]. \end{aligned}$$

- Above the phase transition, it is $\mathcal{O}(1)$, so non-robust.
- Below phase transition: robustness for large N:

$$\mathrm{Pr}(x \to x') \sim \exp\left[-\sqrt{N}(x-x')^2\right]$$

- Different positions of pointer variable constitute *exclusive* events.
- Accuracy of measurement is good:

signal-to-noise-ratio: $\frac{\{\langle X^2\rangle\}_{\mathrm{av}} - \{\langle X\rangle^2\}_{\mathrm{av}}}{\{\langle X\rangle^2\}_{\mathrm{av}}} \sim \frac{1}{\sqrt{N}}$

averages: $\langle ... \rangle$ over apparatus; $\{...\}_{\mathrm{av}}$ over $\langle x|\rho(0)|x\rangle$

Tested particle after measurement

Collaps of wavefunction = Reduction of wavepacket

The density matrix in the x-basis is given by

$$\langle x|\rho(\theta)|x'\rangle = \langle x|\rho(0)|x'\rangle \exp\left[-\frac{\lambda\Delta}{2\hbar^2}(x-x')^2 + \frac{ig^2\theta}{2|\mu|}(x^2-x'^2)\right].$$

$\lambda\Delta$ large $\mapsto$ only x' very close to x survives.

Decoherence time of the off-diagonal elements:

$\tau \sim \frac{\hbar}{TN^{1/4}}$ $\qquad$ $\tau \ll \theta =$ *duration* of measurement.

Einstein-Podolsky-Rosen experiment

Our approach also works when x stands for a discrete variable, such as a spin.

EPR-setup: object with angular momentum 0 decays in two spins in singlet state. Initial density operator :

$$\langle s_1 s_2|\rho(0)|s_1' s_2'\rangle = \frac{1}{2}\delta_{s_1+s_2,0}\ (\delta_{s_1,s_1'}\delta_{s_2,s_2'} - \delta_{s_1,s_2'}\delta_{s_2,s_1'}).$$

Measurement only of z-component of spin 1. For large N reduction $s_1 = s_1'$ after time τ. This automatically implies $s_2 = s_2'$:

$$\langle s_1 s_2|\rho(\tau)|s_1' s_2'\rangle \simeq \frac{1}{2}\ \delta_{s_1+s_2,0}\ \delta_{s_1,s_1'}\ \delta_{s_2,s_2'}.$$

“Speed of quantum signals”

For $T \simeq 1$ K, $N \sim 10^{24}$: $\tau = 10^{-11} N^{-1/4}\,\mathrm{s} \sim 10^{-17}\,\mathrm{s}$.

Distance between spins = 1 m $\mapsto$ speed $\sim 10^{17}$ m/s.

No energy transfered yet from apparatus to spin 1.

Fate of Schrödinger cats & kittens

$t = 0$: superposition of two eigenvectors, $\phi_1|x_1\rangle + \phi_2|x_2\rangle$.

$w(x,p;0) = \sum_{i=1}^{2} \varphi_i^2 \delta(x - x_i) + 2\varphi_1\varphi_2 \delta\left(x - \frac{x_1+x_2}{2}\right) \cos \frac{p(x_1-x_2)}{\hbar}$.

classical probabilities + quantum *interference*

After measurement:

$\mathcal{W}(X,P,x,p;\theta) = \sum_{i=1}^{2} \varphi_i^2\, W_{x_i}(X,P)\, \delta(x - x_i) + \mathcal{W}_{\text{if}}(X,P,x,p;\theta)$

$\mathcal{W}_{\text{if}} \propto 2\varphi_1\varphi_2 \delta\left(x - \frac{x_1+x_2}{2}\right)\ \exp[-\gamma\theta N^{3/2}(x_2 - x_1)^2]$

Discrete spectrum: $x_1 - x_2 = \mathcal{O}(1)$: $\mathcal{W}_{\text{if}} \ll 1$.
Schrödinger cats are automatically suppressed,

Continuous spectrum, $|x_2 - x_1| \ll x_1$: *Schrödinger kitten*.

Can be detected iff $\Pr(x_1 \to x_2) \ll 1$. But then $\mathcal{W}_{\text{if}} \ll 1$:
Schrödinger kittens are also automatically suppressed.

Summary

- apparatus should be macroscopic. Here: Bose gas.

decoherence $\mapsto$ definite result in the measurement; robust and accurate registration of the observable.

- initial state apparatus extremely sensitive to interaction with microscopic system. Here: near transition.

- duration of the measurement larger than relaxation time of apparatus.

- coupling constant g finite: Macroscopic effect on condensate though interaction Hamiltonian not extensive.

- statistical distribution of measured quantity (position of the tested particle) should remain constant during measurement process.

- Two-step process: 1) collaps of wavefunction;
  2) adjustment of apparatus.

1) Decoherence time $\tau \sim \frac{\hbar}{TN^{1/4}}$ much smaller than duration measurement 2).

- EPR setups: "speed of quantum information transfer" can exceed speed of light.

- Schrödinger cats and kittens donot survive robust measurement.