Kleene Algebra — Lecture 6

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1 Today's lecture

Previously, we saw how to obtain an expression for the language of a state in a given finite automaton. Along the way, we developed a theory of vectors and matrices over rational terms. In particular, showed that given a Q-matrix M and a Q-vector b, we can construct a Q-vector s such that $b + M \cdot s \leqq s$, and moreover if t is a Q-vector and $e \in \mathbb{E}$ with $b \fatsemi e + M \cdot t \leqq t$, then $s \fatsemi e \leqq t$.

This lecture is focused on further building out the theory of matrices and vectors over rational terms. In particular, we will see that matrices and vectors over rational terms satisfy all of the laws that we know of Kleene algebra. Most importantly, this includes the fact that one can compute the "star" of a matrix, and have that matrix satisfy the familiar laws about the star.

Towards the end, we will use these facts about matrices over rational terms to prove an illustrative, and ultimately useful result, namely that bisimilar states in an automaton yield expressions that are not only language-equivalent, but *provably* equivalent. On the one hand, this tells us that if you are synthesizing an expression from an automaton, the particular structure of the automaton does not matter — as long as you start with a bisimilar state, the resulting expression will be provably equivalent. On the other hand, this result will be one of the cornerstones to the completeness proof, in the next lecture.

2 The star of a matrix

It is not too hard to show that addition and multiplication of matrices satisfy the axioms that we have become used to for rational expressions.

Lemma 6.1 (Semiring laws for matrices)**.** *Let Q be a finite set, and let X, Y and Z be Q-matrices. Let's write $\mathbf{1}$ for the identity Q-matrix, where $\mathbf{1}(q, q') = [q = q']$, and $\mathbf{0}$ for the null Q-matrix, where $\mathbf{0}(q, q') = 0$. The following hold:*

$$X + \mathbf{0} \equiv X \qquad X + X \equiv X \qquad X + Y \equiv Y + X$$

$$X + (Y + Z) \equiv (X + Y) + Z \qquad X \cdot (Y \cdot Z) \equiv (X \cdot Y) \cdot Z$$

$$X \cdot (Y + Z) \equiv X \cdot Y + X \cdot Z \qquad (X + Y) \cdot Z \equiv X \cdot Z + Y \cdot Z$$

$$X \cdot \mathbf{1} \equiv X \equiv X \cdot \mathbf{1} \qquad X \cdot \mathbf{0} \equiv \mathbf{0} \equiv \mathbf{0} \cdot X$$

Proof. Each law can be proved by unfolding the definitions of the operators, and applying on the corresponding law on the level of rational expressions. □

These laws are very nifty, but perhaps not entirely satisfactory — we are missing the laws about the star. Of course, to state those laws at all, we first need to define what it means to take the star of a matrix. To this end, let's reconsider the construction from the last lecture.

Lemma 6.2 (Restated)**.** *Let Q be a finite set, with M a Q-matrix and b a Q-vector. We can construct a Q-vector s, such that $b + M \cdot s \leqq s$, and furthermore if t is a Q-vector and $z \in \mathbb{E}$ with $b \fatsemi z + M \cdot t \leqq t$, then $s \fatsemi z \leqq t$.*

If you zoom in on the constraint $b + M \cdot s \leqq s$, then you might realise that it looks quite familiar to the premise of the left-fixpoint axiom that we know from Kleene Algebra! In fact, if we imagine for a second that b, M and s are all rational expressions, then we could take the "star" of M to find that $s = M^* \cdot b$ fits the constraints on s in Lemma 6.2! After all, by the fixpoint axioms:

$$b + M \cdot M^* \cdot b \equiv (\mathbf{1} + M \cdot M^*) \cdot b \equiv M^* \cdot b \qquad b \cdot z + M \cdot t \leqq t \implies M^* \cdot b \cdot z \leqq t$$

If you have experience with linear algebra, you might realize that the homework from last lecture provides you with a clue about how to derive the star of a matrix. More precisely, the vector-to-vector transformation described in Lemma 6.2 is *linear* (i.e., it preserves scalar multiplication and vector addition). This means that the transformation can be represented by multiplication with a certain matrix, derived by the action of the transformation on the unit vectors. If you do not have experience with linear algebra, don't worry — the lemma below spells out what is meant by the above, in elementary terms.

Lemma 6.3 (Star of a matrix)**.** *Let Q be a finite set, and let M be a Q-matrix. We can construct a matrix M^* such that the following hold:*

(i) if s and b are Q-vectors such that $b + M \cdot s \leqq s$, then $M^* \cdot b \leqq s$; and

(ii) $\mathbf{1} + M \cdot M^* \equiv M^*$, where $\mathbf{1}$ is the Q-matrix given by $\mathbf{1}(q, q') = [q = q']$.

Proof. For each $q \in Q$, we write u_q for the Q-vector where $u_q(q') = [q = q']$. Furthermore, we write s_q for the least Q-vector such that $u_q + M \cdot s_q \leqq s_q$; note that we can construct this Q-vector, per Lemma 6.2.

We now choose the Q-matrix M^* as follows:

$$M^*(q, q') = s_{q'}(q)$$

It remains to show that M^* satisfies the two requirements above.

(i) Suppose that s and b are Q-vectors such that $b + M \cdot s \leqq s$. We then need to show that $M^* \cdot b \leqq s$, or, in other words, show that for all $q \in Q$:

$$\sum_{q' \in Q} M^*(q, q') \cdot b(q') \leqq s(q)$$

It thus suffices to show that, for all $q, q' \in Q$, we have $s_{q'}(q) \cdot b(q') \leqq s(q)$, or, in other words, that $s_{q'} \fatsemi b(q') \leqq s$. By construction of $s_{q'}$, it then suffices to show that $u_{q'} \fatsemi b(q') + M \cdot s \leqq s$. To this end, we derive:

$$(u_{q'} \fatsemi b(q') + M \cdot s)(q) \leqq (b + M \cdot s)(q) \leqq s(q)$$

which completes this part of the proof.

(ii) For the second part of the claim, let $q, q' \in Q$; we derive as follows:

$$\begin{aligned}
(\mathbf{1} + M \cdot M^*)(q, q') &\equiv \mathbf{1}(q, q') + \sum_{q'' \in Q} M(q, q'') \cdot M^*(q'', q') && \text{(by def.)} \\
&\equiv u_{q'}(q) + \sum_{q'' \in Q} M(q, q'') \cdot s_{q'}(q'') && \text{(def. } u_{q'}, M^*\text{)} \\
&\equiv (u_{q'} + M \cdot s_{q'})(q) && \text{(by def.)} \\
&\equiv s_{q'}(q) && \text{(see below)} \\
&\equiv M^*(q, q') && \text{(def. } M^*\text{)}
\end{aligned}$$

where the second to last equivalence follows from the fact that

$$u_{q'} + M \cdot (u_{q'} + M \cdot s_{q'}) \leqq u_{q'} + M \cdot s_{q'}$$

and hence $u_{q'} + M \cdot s_{q'} \leqq s_{q'}$, meaning that $u_{q'} + M \cdot s_{q'} \equiv s_{q'}$. □

So, what's the upshot of this theorem? Why do we care about this operation on matrices? Well, for one thing, you could be in a situation where you want to compute the vector resulting from Lemma 6.2 for a fixed Q-vector M, but many different Q-vectors b_i. In particular, this could happen when you have an automaton that you want to turn into an equivalent rational expression, but you want to experiement with which states are accepting — thus changing the input vector b, but keeping the matrix M fixed. In that case, it might be more efficient to first compute M^*, and then compute $M^* \cdot b_i$ for each of your vectors.

As another bit of linear algebra, we note that the star of a matrix satisfies a version of the least fixpoint axiom for matrices, as well.

Corollary 6.4. *Let Q be a finite set, and let M and B be Q-matrices. If S is a Q-matrix such that $B + M \cdot S \leqq S$, then $M^* \cdot B \leqq S$.*

Proof. For $q \in Q$, let's define the Q-vectors s_q and b_q by setting $s_q(q') = S(q', q)$ and $b_q(q') = B(q', q)$. We claim that, for all $q \in Q$, we have $b_q + M \cdot s_q \leqq s_q$.

To see this, we derive as follows:

$$\begin{aligned}
(b_q + M \cdot s_q)(q') &= b_q(q') + \sum_{q'' \in Q} M(q', q'') \cdot s_q(q'') \\
&\equiv B(q, q') + \sum_{q'' \in Q} M(q', q'') \cdot S(q'', q) \\
&= (B + M \cdot S)(q', q) \\
&\leqq S(q', q) = s_q(q')
\end{aligned}$$

Now, by Lemma 6.3, we know that $M^* \cdot b_q \leqq s_q$, and we can derive

$$\begin{aligned}
(M^* \cdot B)(q, q') &= \sum_{q'' \in Q} M^*(q, q'') \cdot B(q'', q') \\
&= \sum_{q'' \in Q} M^*(q, q'') \cdot b_{q'}(q'') \\
&= (M^* \cdot b_{q'})(q) \\
&\leqq s_{q'}(q) = S(q, q')
\end{aligned}$$

This completes the proof. □

3 Dualization

We can compute the "star" of a Q-matrix, provided Q is finite, and that this operation satisfies two of the rules about the star at the level of rational expressions. More precisely, following hold for rational expressions e, f, g:

$$1 + e \cdot e^* \equiv e \qquad\qquad e + f \cdot g \leqq g \implies f^* \cdot e \leqq g$$

which are replicated by the following results about Q-matrices M, S and B:

$$\mathbf{1} + M \cdot M^* \equiv M \qquad\qquad B + M \cdot S \leqq S \implies M^* \cdot B \leqq S$$

However, we also have the mirrored version of the star axioms:

$$1 + e^* \cdot e \equiv e \qquad\qquad e + f \cdot g \leqq f \implies g \cdot e^* \leqq f$$

The question then arises: can we recover analogous properties of the star of a matrix? One way to tackle this question is to realize that we can mirror Lemma 6.2 and its proof. To make this precise, we need to define the mirrored version of vector-matrix multiplication, as follows.

Definition 6.5 (Left-multiplication of vector by a matrix)**.** Let S be a finite set, with b an S-vector and M an S-matrix. We define the S-vector $b \cdot M$ by

$$(b \cdot M)(s) = \sum_{s' \in S} b(s') \cdot M(s', s)$$

Based on this operation, we can state the mirrored version of Lemma 6.2:

Lemma 6.6 (Reflection of Lemma 6.2)**.** *Let Q be a finite set, with M a Q-matrix and b a Q-vector. We can construct a Q-vector s, such that $b + s \cdot M \leqq s$, and furthermore if t is a Q-vector and $z \in \mathbb{E}$ with $b \fatsemi z + t \cdot M \leqq t$, then $s \fatsemi z \leqq t$.*

Proof. The proof proceeds using exactly the same structure as Lemma 6.2 as laid out in the previous lecture, but using the reflected fixpoint axioms instead. □

It stands to reason that this lemma should give rise to an operation on matrices, in the same way Lemma 6.3 leverages Lemma 6.2.

Lemma 6.7 (Dagger of a matrix)**.** *Let Q be a finite set, and let M be a Q-matrix. We can construct a matrix $M^\dagger$ such that the following hold:*

(i) if s and b are Q-vectors such that $b + s \cdot M \leqq s$, then $b \cdot M^\dagger \leqq s$; and

(ii) $\mathbf{1} + M^\dagger \cdot M \equiv M^\dagger$, where $\mathbf{1}$ is the Q-matrix given by $\mathbf{1}(q, q') = [q = q']$.

Proof. Analogous to how Lemma 6.3 follows from Lemma 6.2. □

Notice that we do not know *a priori* that the star and dagger of a matrix are the same operation — their calculation uses two very different subroutines. However, the dagger of a matrix does satisfy the reflected fixpoint axiom.

Corollary 6.8. *Let Q be a finite set, and let M and B be Q-matrices. If S is a Q-matrix such that $B + S \cdot M \leqq S$, then $B \cdot M^\dagger \leqq S$.*

Proof. Analogous to how Corollary 6.4 follows from Lemma 6.3. □

With all of this in hand, we can now show that the star and dagger of a matrix are really the same operation, up to $\equiv$, as follows.

Lemma 6.9. *Let Q be a finite set, and let M be a Q-matrix. Now $M^* \equiv M^\dagger$.*

Proof. To prove that $M^* \leqq M^\dagger$, it suffices to prove that $M^* \cdot \mathbf{1} \leqq M^\dagger$, where $\mathbf{1}$ is the identity Q-matrix. By Corollary 6.4, this holds if we can show that $\mathbf{1} + M \cdot M^\dagger \leqq M^\dagger$, or equivalently, $\mathbf{1} \leqq M^\dagger$ and $M \cdot M^\dagger \leqq M^\dagger$. The former follows immediately from the fact that $\mathbf{1} + M^\dagger \cdot M \equiv M^\dagger$. As for the latter, by Corollary 6.8, it suffices to show that $M + M^\dagger \cdot M \leqq M^\dagger$, or equivalently $M \leqq M^\dagger$ and $M^\dagger \cdot M \leqq M^\dagger$. The former again follows from the fact that $\mathbf{1} + M^\dagger \cdot M \equiv M^\dagger$. As for $M \leqq M^\dagger$, this follows from $M \leqq \mathbf{1} \cdot M \leqq M^\dagger \cdot M \leqq \mathbf{1} + M^\dagger \cdot M \equiv M^\dagger$.

The proof that $M^\dagger \leqq M^*$ is analogous. □

This means that we can wrap up our consideration of the star-operation on matrices by stating the desired properties, as follows.

Theorem 6.10. *The addition, multiplication and star operations on matrices satisfy all of the laws of Kleene algebra. In particular, in addition to the laws listed in Lemma 6.1, if X, Y and Z are Q-matrices for some finite set Q, then*

$$\mathbf{1} + X \cdot X^* \equiv X^* \qquad X + Y \cdot Z \leqq Z \implies Y^* \cdot Z \leqq Z$$

$$\mathbf{1} + X^* \cdot X \equiv X^* \qquad X + Y \cdot Z \leqq Y \implies X \cdot Z^* \leqq Y$$

Proof. The first law is stated in Lemma 6.3, and the third law is a consequence of Lemma 6.7 and Lemma 6.9. Similarly, the second law is stated in Corollary 6.4, and the fourth law is a consequence of Corollary 6.8 and Lemma 6.9. □

4 Matrices and bisimulations

So far, we saw that we can compute the star of a matrix over rational terms, and that this operator behaves similarly to the star of a plain rational term. We are now going to use this operation to show that if you have bisimilar states in automata yield (provably) equivalent expressions. On the one hand, it's a great validation of our work so far: if you have two *equivalent machines*, in that they can mimic one another in terms of behavior, then converting those machines back into programs gives you *equivalent expressions*. On the other hand, this is an important step on the road to showing completeness.

To this end, we need to first widen our view of matrices and matrix multiplication to include "rectangular" matrices, as follows.

Definition 6.11. Let S_1, S_2 and S_3 be sets. A *S_1-by-S_2 matrix* is a M function from $S_1 \times S_2$ to $\mathbb{E}$. If M is an S_1-by-S_2 matrix, and N is an S_2-by-S_3 matrix, then their multiplication is the S_1-by-S_3 matrix given by

$$(M \cdot N)(s_1, s_3) = \sum_{s_2 \in S_2} M(s_1, s_2) \cdot N(s_2, s_3)$$

Furthermore, if b is an S_2-vector, then $M \cdot b$ is the S_1-vector given by

$$(M \cdot b)(s_1) = \sum_{s_2 \in S_2} M(s_1, s_2) \cdot b(s_2)$$

Clearly, S-matrices and their operations on other S-matrices or S-vectors as used up to this point are a special case of the above. Specifically, what we have referred to as an S-matrix up to this point is an S-by-S matrix, and multiplication of S-matrices coincides with their multiplications as S-by-S-matrices. This broader view of matrices allows us to encode relations as well.

Definition 6.12. Let S_1 and S_2 be sets, and let $R \subseteq S_1 \times S_2$ be a relation between S_1 and S_2. We write M_R for the matrix given by $M(s_1, s_2) = [s_1 \; R \; s_2]$.

Also, recall that when $A = \langle Q, F, \delta \rangle$, we write M_A for the Q-matrix where

$$M(q, q') = \sum_{\delta(q, \mathtt{a}) = q'} \mathtt{a}$$

We can now state two useful properties of our encoding of automata and relations into matrice. The property is a relation between the acceptance vectors.

Lemma 6.13. *Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ be an automaton for $i \in \{0, 1\}$. Furthermore, let R be a bisimulation between A_0 and A_1. Then $M_R \cdot b_{A_1} \leqq b_{A_0}$.*

Proof. First, let's check our types. We know that M_R is a Q_0-by-Q_1 matrix, and b_{A_1} is a Q_1-vector; this makes $M_R \cdot b_{A_1}$ a Q_0-vector, just like b_{A_0}. To show that $M_R \cdot b_{A_1} \leqq b_{A_0}$, we need to show that for all $q \in Q_0$, we have

$$\sum_{q_1 \in Q_1} M_R(q_0, q_1) \cdot b_{A_1}(q_1) \leqq b_{A_0}(q_0)$$

Expanding the definitions of M_R, b_{A_1} and b_{A_0}, the following works out to be

$$\sum_{q_1 \in Q_1} [q_0 \; R \; q_1] \cdot [q_1 \in F_1] \leqq [q_0 \in F_0]$$

We can show that this holds by showing that each of the terms on the left is below $[q_0 \in F_0]$. There are two cases to distinguish. On the one hand, if $q_0 \not R \; q_1$ or $q_1 \not\in F_1$, then $[q_0 \; R \; q_1] \cdot [q_1 \in F] \equiv 0$, and so the claim holds immediately. Otherwise, if $q_0 \; R \; q_1$ and $q_1 \in F_1$, then $q_0 \in F_0$, by virtue of R being a bisimulation. Hence $[q_0 \; R \; q_1] \cdot [q_1 \in F_1] \equiv 1 = [q_0 \in F_0]$. □

The second property relates the transition matrices, as follows.

Lemma 6.14. *Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ be an automaton for $i \in \{0, 1\}$. Furthermore, let R be a bisimulation between A_0 and A_1. Then $M_R \cdot M_{A_1} \leqq M_{A_0} \cdot M_R$.*

Proof. Let's get our types right first. Since R is a relation between Q_0 and Q_1, we know that M_R is a Q_0-by-Q_1 matrix. We also know that M_{A_0} is a Q_0-matrix, and M_{A_1} is a Q_1-matrix. This makes $M_R \cdot M_{A_1}$ a Q_0-by-Q_1 matrix, just like $M_{A_0} \cdot M_R$. So at least the comparison between them is sensible.

To prove our goal, we need to show that for $q_0 \in Q_0$ and $q_1 \in Q_1$, we have $(M_R \cdot M_{A_1})(q_0, q_1) \leqq (M_{A_0} \cdot M_R)(q_0, q_1)$, or, writing out the multiplication:

$$\sum_{q_1' \in Q_1} M_R(q_0, q_1') \cdot M_{A_1}(q_1', q_1) \leqq \sum_{q_0' \in Q_0} M_{A_0}(q_0, q_0') \cdot M_R(q_0', q_1)$$

Expanding out the definitions of these matrices, the above amounts to showing

$$\sum_{q_1' \in Q_1} [q_0 \mathrel{R} q_1'] \cdot \Big(\sum_{\delta(q_1', \mathtt{a}) = q_1} \mathtt{a} \Big) \leqq \sum_{q_0' \in Q_0} \Big(\sum_{\delta(q_0, \mathtt{a}) = q_0'} \mathtt{a} \Big) \cdot [q_0' \mathrel{R} q_1]$$

Applying distributivity to both ends tells us the above is equivalent to

$$\sum_{q_1' \in Q_1} \sum_{\delta(q_1', \mathtt{a}) = q_1} [q_0 \mathrel{R} q_1'] \cdot \mathtt{a} \leqq \sum_{q_0' \in Q_0} \sum_{\delta(q_0, \mathtt{a}) = q_0'} \mathtt{a} \cdot [q_0' \mathrel{R} q_1]$$

To show the above, it suffices to prove that every term in the sum on the left-hand side is below (a term on) the right-hand side, w.r.t. $\leqq$. To this end, let $q_1' \in Q_1$ and $\mathtt{a} \in \Sigma$ be such that $\delta(q_1', \mathtt{a}) = q_1$. If $q_0 \not\mathrel{R} q_1'$, then $\mathtt{a} \cdot [q_0 \mathrel{R} q_1'] \equiv 0$, and so the claim holds immediately. Otherwise, if $q_0 \mathrel{R} q_1'$, then $\delta(q_0, \mathtt{a}) \mathrel{R} \delta(q_1', \mathtt{a}) = q_1$, since R is a bisimulation. If we then choose $q_0' = \delta(q_0, \mathtt{a})$, we find that $[q_0 \mathrel{R} q_1'] \cdot \mathtt{a} = 1 \cdot \mathtt{a} \equiv \mathtt{a} \cdot 1 = \mathtt{a} \cdot [q_0' \mathrel{R} q_1]$. The latter is precisely a term on the right-hand side, and so the claim follows. □

As a matter of fact, we can use our newly derived facts about the star of a matrix to leverage the last fact about transition- and bisimulation matrices.

Corollary 6.15. *Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ be an automaton for $i \in \{0, 1\}$. Furthermore, let R be a bisimulation between A_0 and A_1. Then $M_R \cdot M_{A_1}^* \leqq M_{A_0}^* \cdot M_R$.*

Proof sketch. This follows form Lemma 6.14, Lemma 6.9, Corollary 6.8 and the fact that the laws for Kleene algebra also apply to matrices. More precisely, it is analogous to the proof that if $e, f, g \in \mathbb{E}$ with $e \cdot f \leqq g \cdot e$, then $e \cdot f^* \leqq g^* \cdot e$, which is part of today's homework. □

These two facts now allow us to reach the desired conclusion, namely that expressions obtained for two bisimilar states are provably equivalent.

Theorem 6.16. *Let $A_i = \langle Q_i, F_i, \delta_i \rangle$ be an automaton for $i \in \{0, 1\}$. Furthermore, let R be a bisimulation between A_0 and A_1. If $q_0 \mathrel{R} q_1$, and $e_0, e_1 \in \mathbb{E}$ are the rational expressions satisfying $[\![e_0]\!]_{\mathbb{E}} = L_{A_0}(q_0)$ and $[\![e_1]\!]_{\mathbb{E}} = L_{A_1}(q_1)$ obtained from the automata-to-expressions procedure, then $e_0 \equiv e_1$.*

Proof. From our automata-to-expressions procedure, we know that for both $i \in \{0, 1\}$ that $e_i = (M_{A_i}^* \cdot b_{A_i})(q_i)$. This then allows us to derive as follows:

$$\begin{aligned}
e_1 &= (M_{A_1}^* \cdot b_{A_1})(q_1) \\
&\equiv [q_0 \mathrel{R} q_1] \cdot (M_{A_1}^* \cdot b_{A_1})(q_1) && \text{(since } q_0 \mathrel{R} q_1\text{)} \\
&\leqq \sum_{q_1' \in Q_1} [q_0 \mathrel{R} q_1'] \cdot (M_{A_1}^* \cdot b_{A_1})(q_1') \\
&\leqq \sum_{q_1' \in Q_1} M_R(q_0, q_1') \cdot (M_{A_1}^* \cdot b_{A_1})(q_1') && \text{(def. } M_R\text{)} \\
&= (M_R \cdot M_{A_1}^* \cdot b_{A_1})(q_0) \\
&\leqq (M_{A_0}^* \cdot M_R \cdot b_{A_1})(q_0) && \text{(Corollary 6.15)} \\
&\leqq (M_{A_0}^* \cdot b_{A_0})(q_0) && \text{(Lemma 6.13)} \\
&= e_0
\end{aligned}$$

The converse of R is a bisimulation between A_1 and A_0, which allows us to derive $e_0 \leqq e_1$ analogously, and hence conclude that $e_0 \equiv e_1$. □

5 Homework

1. Let Q be a finite set, and let X, Y and Z be Q-matrices. Prove the left-distributivity claimed in Lemma 6.1, namely that

$$X \cdot (Y + Z) \equiv X \cdot Y + X \cdot Z$$

2. Let Q_0 and Q_1 be finite sets, let M be a Q_0-by-Q_1 matrix, let X be a Q_1-matrix, and let Y be a Q_0-matrix.

   Show that $M \cdot X \leqq Y \cdot M$ implies $M \cdot X^* \leqq Y^* \cdot M$.

   We have not proved that the laws of KA apply to non-square matrices. For this exercise, however, you may assume that this is the case.

   Hint: if you're stuck, suppose $e, f, g \in \mathbb{E}$ *with* $e \cdot f \leqq g \cdot e$, *and prove that* $e \cdot f^* \leqq g^* \cdot e$. *The structure of this proof can be adapted to your needs.*

3. Let R and S be relations. Show that $M_{R \circ S} \equiv M_R \cdot M_S$.

4. Let R be a relation. Show that $M_{R^*} \equiv M_R^*$.

   Hint: show that $M_R^* \leqq M_{R^*}$ *and* $M_{R^*} \leqq M_R^*$. *For both inclusions, you need to use facts about the star of a matrix, but the implication is only necessary for one of them.*

6 Bibliographical notes

The idea of using vector spaces over Kleene algebras can be traced back to Conway [Con71] and Backhouse [Bac75]. The observation that matrices over rational terms obeys the laws of Kleene algebra is due to Kozen [Koz94], although he calculates the star of a matrix without going through Lemma 6.2.

The proof that solutions to automata are invariant w.r.t. bisimilarity is adapted from Jacobs's account [Jac06], who credits the idea to Kozen [Koz01].

References

[Bac75] Roland Backhouse. *Closure algorithms and the star-height problem of regular languages.* PhD thesis, University of London, 1975.

[Con71] John Horton Conway. *Regular Algebra and Finite Machines.* Chapman and Hall, Ltd., London, 1971.

[Jac06] Bart Jacobs. A bialgebraic review of deterministic automata, regular expressions and languages. In *Algebra, Meaning, and Computation, Essays Dedicated to Joseph A. Goguen on the Occasion of His 65th Birthday*, pages 375–404, 2006. `doi:10.1007/11780274_20`.

[Koz94] Dexter Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Inf. Comput.*, 110(2):366–390, 1994. `doi:10.1006/inco.1994.1037`.

[Koz01] Dexter Kozen. Myhill-Nerode relations on automatic systems and the completeness of Kleene algebra. In *STACS*, pages 27–38, 2001. `doi:10.1007/3-540-44693-1_3`.