Efficiency and envy-freeness in fair division of indivisible goods: logical representation and complexity

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Introduction and context

Fair division of indivisible goods among agents: compact representation and complexity issues.

This subject is motivated by a common work between ONERA, IRIT and CNES about fairness and efficiency in ressource allocation problems.

Several studies about its application to Earth Observation Satellite have been carried out (see Michel's presentation).

Introduction – the two keypoints

Problem studied

fair division of indivisible goods among agents

Fair division problems, two (antagonistic ?) keypoints:

- fairness $\longrightarrow$ envy-freeness
- efficiency $\longrightarrow$ Pareto-efficiency

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- fairness → envy-freeness
- efficiency → Pareto-efficiency

An allocation is *envy-free* iff every agent likes her share at least as much as the share of any other one.

Example: 2 agents, 2 items / Agent 1 wants $item_1$ with utility 10 and $item_2$ with utility 5. / Agent 2 wants $item_2$ with utility 2.
Agent 1$\leftarrow$ $item_1$ / Agent 2$\leftarrow$ $item_2$ is envy-free.
Agent 1$\leftarrow$ $item_2$ / Agent 2$\leftarrow$ $item_1$ isn't envy-free.

Introduction – the two keypoints

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fair division of indivisible goods among agents

Fair division problems, two (antagonistic ?) keypoints:

- fairness → envy-freeness
- efficiency → Pareto-efficiency

An allocation is *Pareto-efficient* iff for every other allocation that increases the satisfaction of an agent, there is at least another agent that is strictly less satisfied in this new allocation.

Example: 2 agents, 2 items / Agent 1 wants $item_1$ with utility 10 and $item_2$ with utility 5. / Agent 2 wants $item_2$ with utility 2.
Agent 1← $item_1$ / Agent 2← $item_2$ is Pareto-efficient.
Agent 1← $item_2$ / Agent 2← $item_1$ isn't Pareto-efficient.

Existing work

- Social choice theory
  - Most of the work concerns divisible goods and / or monetary transfers.
  - Some work on indivisible goods without m.t., but it lacks of a compact representation language.
  - Almost nothing about complexity issues.
- Artificial Intelligence
  - Combinatorial auctions and other related utilitarianistic problems.
  - Complexity and compact representation.
  - Not so much about fairness[1].

[1] apart from recent work such as [Lipton *et al.*, 2004]

In the search for efficiency AND envy-freeness

The problem of the existence of an efficient and envy-free allocation isn't trivial (there are some cases where no efficient and envy-free allocation exists)
$\rightarrow$ *Is it computationally hard to determine whether such an allocation exists ?*

In the search for efficiency AND envy-freeness

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→ *Is it computationally hard to determine whether such an allocation exists ?*

Example: 2 agents, 2 items / Agent 1 wants $item_1$ with utility 5 and $item_2$ with utility 10. / Agent 2 wants $item_2$ with utility 2.
The two Pareto-efficient allocations are:
Agent 1← $item_1$ and $item_2$, Agent 2 nothing / Agent 1← $item_1$, Agent 2← $item_2$, but none of them is envy-free.

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The fair division problem

Definition (fair division problem)

A *fair division problem* is a tuple $\mathcal{P} = \langle I, X, \mathcal{R} \rangle$ where

- $I = \{1, \ldots, N\}$ is a set of agents;
- $X = \{x_1, \ldots, x_p\}$ is a set of indivisible goods;
- $\mathcal{R} = \langle R_1, \ldots, R_N \rangle$ is a preference profile (a set of reflexive, transitive and complete relations on 2^X).

Definition (allocation)

An *allocation* is a mapping $\pi : I \rightarrow 2^X$ such that $\forall i \neq j, \pi(i) \cap \pi(j) = \varnothing$.

About dichotomous preferences

A very particular case of fair division problem:
$\longrightarrow$ the preference relations are under their simplest non-trivial form.

Definition (dichotomous preference relation)

R is dichotomous $\Leftrightarrow$ there is a set of "good" bundles *Good* s.t. $A \succeq_R B \Leftrightarrow A \in Good$ or $B \notin Good$.

Example:

$$X = \{a, b, c\} \Rightarrow 2^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$$Good \longrightarrow \{\{a, b\}, \{b, c\}\}$$

$$\overline{Good} \longrightarrow \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b, c\}\}$$

Where the propositional logic can help us

A dichotomous preference is exhaustively represented by its set of good bundles.
A quite obvious way to represent this set uses propositional logic.

Example (cont'd):

$$Good_i = \{\{a, b\}, \{b, c\}\} \Rightarrow \varphi_i = (a \wedge b \wedge \neg c) \vee (\neg a \wedge b \wedge c)$$

The fair division problem with dichotomous preferences (1)

When every preference relations are dichotomous, the fair division problem can be represented as the set of propositional formulae for each agent:

$$\mathcal{P} = \langle \varphi_1, \ldots, \varphi_N \rangle$$

We introduce one truth variable per pair (good, agent): x_i is true iff the good x is allocated to agent i, and rewrite the φ_i with the $x_i \rightarrow \varphi_i^*$.

Example:

$$Good_1 = \{\{a, b\}, \{b, c\}\};\ Good_2 = \{\{b\}\{b, c\}\}$$

$$\varphi_1^* = (a_1 \wedge b_1 \wedge \neg c_1) \vee (\neg a_1 \wedge b_1 \wedge c_1);\ \varphi_2^* = b_2 \wedge \neg a_2$$

The fair division problem with dichotomous preferences (2)

$\Rightarrow$ An allocation "is"[2] a truth assignment of the x_i, satisfying:

$$\Gamma_{\mathcal{P}} = \bigwedge_{x \in X} \bigwedge_{i \neq j} \neg(x_i \wedge x_j)$$

Example (cont'd):

$$\Gamma_{\mathcal{P}} = \neg(a_1 \wedge a_2) \wedge \neg(b_1 \wedge b_2) \wedge \neg(c_1 \wedge c_2)$$

[2] to be precise, can be bijectively mapped to (let F be this bijection)

Envy-freeness and dichotomous preferences

Envy-freeness has a simple expression within the dichotomous framework:

$$\Lambda_{\mathcal{P}} = \bigwedge_{i=1,\dots,N} \left[\varphi_i^* \vee \left(\bigwedge_{j \neq i} \neg \varphi_{j|i}^* \right) \right]$$

where $\varphi_{j|i}^* = \varphi_i^*(x_i \leftarrow x_j)$

Proposition

π is envy-free if and only if $F(\pi) \models \Lambda_{\mathcal{P}}$.

Example (cont'd):

$$\begin{aligned} \Lambda_{\mathcal{P}} &= [[(a_1 \wedge b_1 \wedge \neg c_1) \vee (\neg a_1 \wedge b_1 \wedge c_1)] \vee \neg [(a_2 \wedge b_2 \wedge \neg c_2) \vee (\neg a_2 \wedge b_2 \wedge c_2)]] \\ &\quad \wedge [[b_2 \wedge \neg a_2] \vee \neg [b_1 \wedge \neg a_1]] \end{aligned}$$

Pareto-efficiency and dichotomous preferences

Efficiency requires that allocations satisfy a *maximal* (in the sense of inclusion) set of agents, while being admissible (satisfying $\Gamma_{\mathcal{P}}$). This can be very naturally expressed as a maximal-consistency condition.

Proposition

π is efficient if and only if $\{\varphi_i^* | F(\pi) \models \varphi_i^*\}$ is a maximal $\Gamma_{\mathcal{P}}$-consistent subset of $\{\varphi_1^*, \ldots, \varphi_N^*\}$.

Example (cont'd): The 2 maximal $\Gamma_{\mathcal{P}}$-consistent subsets of $\{\varphi_1^*, \varphi_2^*\}$ are $\{\varphi_1^*\}$ and $\{\varphi_2^*\}$.

Linking efficiency and envy-freeness together

By putting things together, we can find the following condition for the existence of an efficient and envy-free allocation:
$\exists S$ maximal $\Gamma_{\mathcal{P}}$-consistent subset of $\{\varphi_1^*, \ldots, \varphi_N^*\}$ such that $\bigwedge S \wedge \Gamma_{\mathcal{P}} \wedge \Lambda_{\mathcal{P}}$ is consistent.

The link with skeptical inference (1)

Interestingly, this is exactly an instance of a well-known problem in non-monotonic reasoning: default theory and skeptical inference. The aim of default reasoning [Reiter 1980] is to build a framework for general rules with exceptions. The particular case[3] that interests us is based on:

- a fact $\rightarrow$ logical formula β,
- normal defaults without prerequisites $\rightarrow$ a set of logical formulae $\Delta = \{\alpha_1, \ldots, \alpha_m\}$: if nothing prevents α_i from being true, assume it is.

We are looking for the maximal sets of default, consistent with the fact (called *extensions*).

[3] Normal defaults without prerequisites

The link with skeptical inference (2)

Definition (Skeptical consequence)

Δ a set of formulae, β and ψ formulae. ψ is a *skeptical consequence* of $\langle \beta, \Delta \rangle$ (denoted $\langle \beta, \Delta \rangle {\mid\!\sim}^{\forall} \psi$) iff $\forall S \in \mathit{MaxCons}(\Delta, \beta)$ (extension) $\bigwedge S \wedge \beta \models \psi$.

$\Rightarrow$ the existence of an efficient and envy-free allocation can be reduced to:

$$\langle \Gamma_{\mathcal{P}}, \{\varphi_1^*, \ldots, \varphi_N^*\} \rangle \not{\mid\!\sim}^{\forall} \neg \Lambda_{\mathcal{P}}$$

The link with skeptical inference (2)

Consequences:

- The problem can be reduced to the negation of the skeptical inference problem ($\rightarrow$ gives us a good idea of its computational complexity)
- We can use the default reasoning algorithms to find EEF allocations in a single step.

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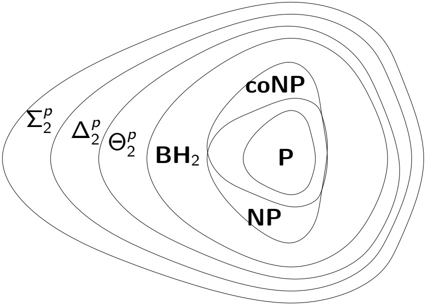
Background about computational complexity results

- $\mathbf{BH}_2 = \{L_1 \cap L_2 | L_1 \in \mathbf{NP} \text{ and } L_2 \in \mathbf{coNP}\}$
- $\Delta_2^p = \mathbf{P}^{\mathbf{NP}}$ (languages recognizable in polynomial time by a deterministic TM using **NP** oracles).
- $\Sigma_2^p = \mathbf{NP}^{\mathbf{NP}}$
- $\Theta_2^p = \Delta_2^p[\mathcal{O}(\log n)]$ (only a logarithmic number of oracles).



Computational complexity of the existence of an Efficient and Envy-Free allocation

Proposition

The problem EEF EXISTENCE for a ressource allocation problem with monotonous[a], dichotomous preferences under logical form is Σ_2^p-complete.

[a] *i.e.*every formulae are positive

Some other results about dichotomous preferences (1)

We also studied some particular cases of the EEF EXISTENCE problem:

- N identical dichotomous, monotonous preferences $\longrightarrow$ **NP**-complete.
- Monotonous, dichotomous preferences, 2 agents $\longrightarrow$ **NP**-complete.
- N identical dichotomous preferences $\longrightarrow$ $\mathbf{coBH}_2$-complete.
- Dichotomous preferences, 2 agents $\longrightarrow$ $\mathbf{coBH}_2$-complete.

Some other results about dichotomous preferences (2)

What about weakening Pareto-efficiency ?

- **Complete** envy-free allocations existence ($N \geq 2$) $\rightarrow$ **NP**-complete (even for monotonous preferences).
- Envy-free allocation satisfying a **maximal number** of agents $\rightarrow$ Θ_2^p-complete (even for monotonous preferences).

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Extension to non-dichotomous preferences

The previous main result can be extended to non-dichotomous preferences under the condition that preferences are represented by a “reasonable” compact representation language.

Proposition

EEF EXISTENCE with monotonous preferences under logical form is Σ_2^p-complete.

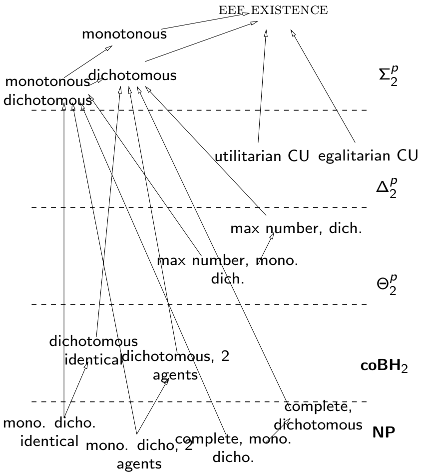
With utility functions

We also studied the case where efficiency is based on social welfare functions (more on this in Michel's talk).

Proposition

- Existence of envy-free allocation maximizing the utilitarian social welfare (*i.e.* sum of utilities) is Δ_2^p-complete.
- Existence of envy-free allocation maximizing the egalitarian social welfare (*i.e.* min of utilities) is Δ_2^p-complete.

Summary of the complexity results



Future work

- How can we approximate the efficiency and envy-freeness ?
- How can we design algorithms to find (almost) efficient and envy-free allocations ?