Smith and Rawls Share a Room: Stability and Medians

Bettina Klaus and Flip Klijn

Maastricht University, The Netherlands and Institute for Economic Analysis (CSIC), Spain

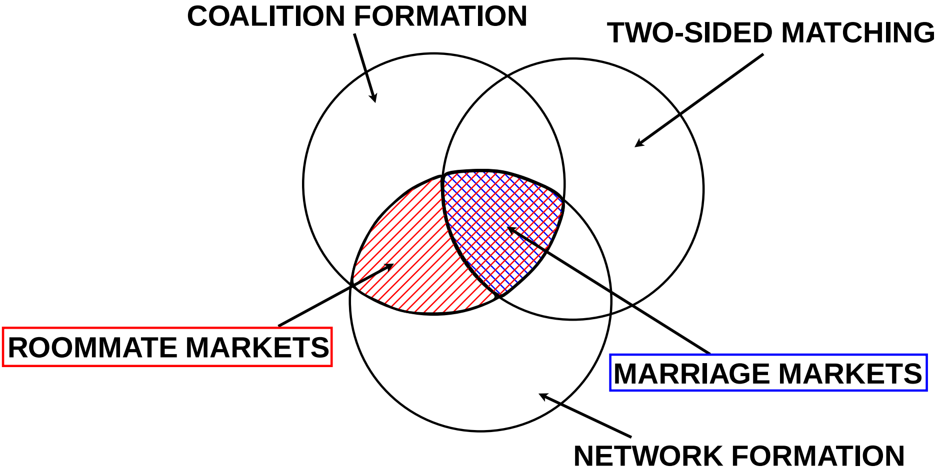
Our Quest

- Selection of a particularly appealing stable matching for matching problems with multiple stable matchings.
- Elementary, graphic proofs.
- Identification of key properties.

Roommate Markets

- In their seminal paper Gale and Shapley (AMM 1962) introduced the very simple (?) and appealing roommate problem as follows:
- "An even number of boys wish to divide up into pairs of roommates."
- A very common extension of this problem is to allow also for odd numbers of agents and to consider the formation of pairs and singletons (rooms can be occupied either by one or by two agents).

- $N = \{1, \ldots, n\}$: ***set of agents***.
- $\succeq_i$: ***agent i's preferences*** over sharing a room with any of the agents in $N \backslash \{i\}$ and having a room for himself (or outside option).
- Assumption: preferences are strict, e.g., $j \succ_i k \succ_i i \succ_i h \succ_i \ldots$
- A *roommate market* consists of a set of agents N and their preferences $\succeq$ and is denoted by $(N, \succeq)$.
- A *marriage market* is a roommate market $(N, \succeq)$ such that N is the union of two disjoint sets M and W, and each agent in M (respectively W) prefers being single to being matched with any other agent in M (respectively W).


COALITION FORMATION
TWO-SIDED MATCHING
ROOMMATE MARKETS
MARRIAGE MARKETS
NETWORK FORMATION

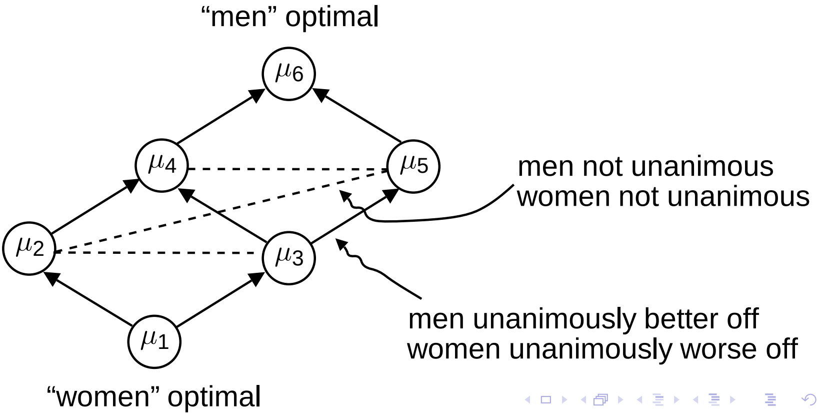
- A *matching* μ for roommate market $(N, \succeq)$ is a function $\mu : N \rightarrow N$ of order two, i.e, for all $i \in N$, $\mu(\mu(i)) = i$.

- For a matching μ, $\{i, j\}$ is a *blocking pair* if $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$.
- Matching μ is *individually rational* if no blocking pair $\{i, i\}$ exists.
- Matching μ is *stable* if no blocking pair $\{i, j\}$ exists.

- The *core* equals the set of stable matchings.

The Core for Marriage Markets

- For marriage markets and college admission markets the core is always non-empty and has the very strong structure of a distributive lattice that reflects the polarization between the two sides of the market.



The Core for Marriage Markets

- In addition, for marriage markets and college admission markets there is an easy and fast algorithm to find the two optimal stable matchings: Gale and Shapley's deferred acceptance algorithm. To compute men optimal matching μ_M:
  - ***Step 1.a.*** Each man proposes to his favorite woman.
  - ***Step 1.b.*** Each woman rejects any unacceptable man, and each woman who receives more than one proposal rejects all but her most preferred of these (this man is kept "engaged")
  - ...
  - ***Step k.a.*** Each man currently not engaged proposes to his favorite woman among those who have not yet rejected him.
  - ***Step k.b.*** Each woman rejects any unacceptable man, and each woman rejects all proposals but her most preferred among the group consisting of the new proposers together with the man she was engaged with (if any).
  - REPEAT until no man is rejected. Final matching: μ_M.

A Roommate Market with an Empty Core

Example

Agent 1: 2 P_1 3 P_1 1,

Agent 2: 3 P_2 1 P_2 2,

Agent 3: 1 P_3 2 P_3 3.

- All agents being single is not a core matching.
- If agents 1 and 2 are matched, then agent 3 will "seduce" agent 2 to block.
- If agents 2 and 3 are matched, then agent 1 will "seduce" agent 3 to block.
- If agents 1 and 3 are matched, then agent 2 will "seduce" agent 1 to block.

A roommate market with a non-empty core is called *solvable*.

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Henceforth, we consider solvable roommate markets. Typically, there are multiple stable matchings.

Selection problem: can we select a particularly appealing stable matching?

- Can selection be based on the number of matched agents?
- Can we choose a stable matching without favoring any agent?

Henceforth, the red matching μ and the blue matching μ' are two *stable* matchings.

We introduce a *bi-choice graph* $G(\mu, \mu') = (V, E)$.

- Vertices: $V = N$.
- Edges: E. Let $i, j \in N$. Then there is an edge

E1. $i \bullet\!\!\longrightarrow\!\!\bullet\, j$ if $j = \mu(i) \succ_i \mu'(i)$;

E2. $i \bullet\!\!\dashrightarrow\!\!\bullet\, j$ if $j = \mu'(i) \succ_i \mu(i)$;

E3. $i \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\, j$ if $j = \mu(i) \sim_i \mu'(i)$ (i.e., a loop $i \circlearrowleft$ if $j = i$).

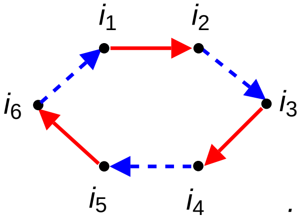
Lemma

Bi-choice graph components

Consider $G(\mu, \mu')$. Let $i \in N$. Then, agent i's component of $G(\mu, \mu')$ either

- **(a)** *equals i •——• j for some agent j (i.e., i •⬭ if $j = i$), or*
- **(b)** *is a directed even cycle (with $\geq$ 4 agents) where continuous and discontinuous edges alternate.*

An example of such a cyclical component is



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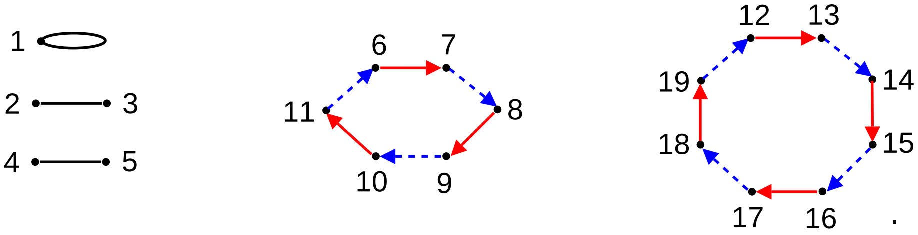

Hence, any two stable matchings μ and μ' decompose the set of agents into a set of even cycles and singletons.

We prove the following basic results for solvable roommate markets with our graphic approach:

- The lonely wolf theorem
- Decomposability
- Smith and Rawls share a room: stability versus justice

Theorem

Lonely wolves

μ and μ' have the same set of single agents, i.e., $\mu(i) = i \Leftrightarrow \mu'(i) = i$.

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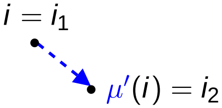

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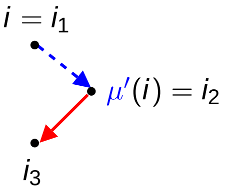

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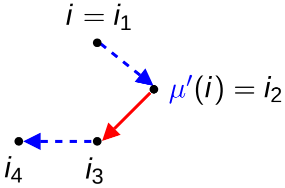

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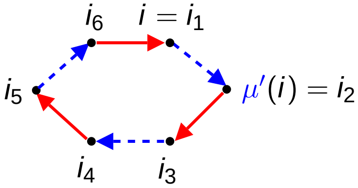

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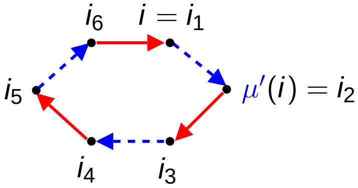

$$\Rightarrow \begin{cases} i_6 \neq i \\ \mu(i_6) = i, \text{ i.e., } \mu(i) = i_6 \end{cases}$$

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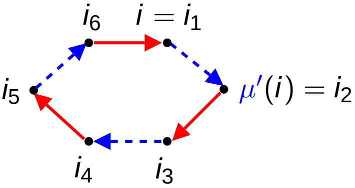

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$$\Downarrow$$

$\mu(i) \neq i \Rightarrow$ contradiction! □

Lemma

Decomposability

Let $\mu(i) = j$. *Then,*

(a) $\mu(i) \succ_i \mu'(i)$ *implies* $\mu'(j) \succ_j \mu(j)$ *and*

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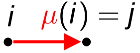

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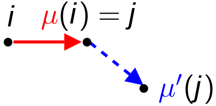

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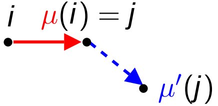

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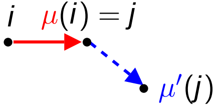

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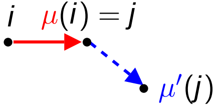

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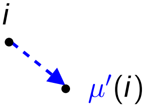

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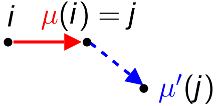

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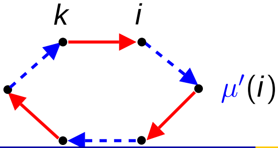

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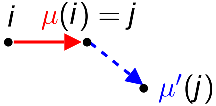

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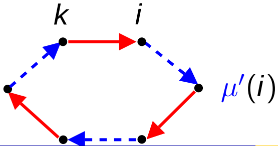

$\Rightarrow$ $i = \mu(k)$, i.e., $j = k$ and $\mu(j) \succ_j \mu'(j)$.

□

Let $\mu_1, \ldots, \mu_{2k+1}$ be an odd number of (possibly non-distinct) stable matchings. Let each agent rank these matchings according to his preferences, e.g.,

$$\mu_1(i) \succ_1 \mu_2(i) \sim_i \mu_3(i) \succ_i \underbrace{\mu_4(i)}_{\text{med}\{\mu_1(i),\ldots,\mu_7(i)\}} \sim_i \mu_5(i) \succ_i \mu_6(i) \succ_i \mu_7(i).$$

We denote agent i's $(k+1)$-st ranked (the median) match by $\mu_{\text{med}}(i) \equiv \text{med}\{\mu_1(i), \ldots, \mu_{2k+1}(i)\}$.

Theorem

Smith and Rawls share a room

Let $\mu_1, \ldots, \mu_{2k+1}$ be an odd number of stable matchings. Then, the median matching μ_{med} is a well-defined stable matching.

W.l.o.g.,

$$\mu_1(i) \succeq_1 \mu_2(i) \succeq_i \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \succeq_i \mu_4(i) \succeq_i \mu_5(i).$$

Then,

$$\underbrace{\mu_3(i)}_{\mu_{med}(i)=j}$$

$$\underbrace{\mu_3(j)}_{=i}$$

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$$\mu_1(i) \succeq_1 \mu_2(i) \succeq_i \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \succeq_i \mu_4(i) \succeq_i \mu_5(i).$$

Then,

$$\mu_1(i) \quad \succeq_i \quad \mu_2(i) \quad \succeq_i \quad \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \quad \succeq_i \quad \mu_4(i)$$

$$\mu_4(j) \quad \succeq_j \quad \underbrace{\mu_3(j)}_{=i} \quad \succeq_j \quad \mu_2(j) \quad , \quad \mu_1(j)$$

W.l.o.g.,

$$\mu_1(i) \succeq_1 \mu_2(i) \succeq_i \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \succeq_i \mu_4(i) \succeq_i \mu_5(i).$$

Then,

$$\mu_1(i) \quad \succeq_i \quad \mu_2(i) \quad \succeq_i \quad \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \quad \succeq_i \quad \mu_4(i) \quad \succeq_i \quad \mu_5(i)$$

$$\mu_4(j) \quad \succeq_j \quad \underbrace{\mu_3(j)}_{=i} \quad \succeq_j \quad \mu_2(j) \quad , \quad \mu_1(j)$$

W.l.o.g.,

$$\mu_1(i) \succeq_1 \mu_2(i) \succeq_i \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \succeq_i \mu_4(i) \succeq_i \mu_5(i).$$

Then,

$$\mu_1(i) \quad \succeq_i \quad \mu_2(i) \quad \succeq_i \quad \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \quad \succeq_i \quad \mu_4(i) \quad \succeq_i \quad \mu_5(i)$$

$$\mu_5(j) \quad , \quad \mu_4(j) \quad \succeq_j \quad \underbrace{\mu_3(j)}_{=i} \quad \succeq_j \quad \mu_2(j) \quad , \quad \mu_1(j)$$

W.l.o.g.,

$$\mu_1(i) \succeq_1 \mu_2(i) \succeq_i \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \succeq_i \mu_4(i) \succeq_i \mu_5(i).$$

Then,

$$\mu_1(i) \quad \succeq_i \quad \mu_2(i) \quad \succeq_i \quad \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \quad \succeq_i \quad \mu_4(i) \quad \succeq_i \quad \mu_5(i)$$

$$\mu_5(j) \quad , \quad \mu_4(j) \quad \succeq_j \quad \underbrace{\mu_3(j)}_{\mu_{med}(j)=i} \quad \succeq_j \quad \mu_2(j) \quad , \quad \mu_1(j)$$

W.l.o.g.,

$$\mu_1(i) \succeq_1 \mu_2(i) \succeq_i \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \succeq_i \mu_4(i) \succeq_i \mu_5(i).$$

Then,

$$\mu_1(i) \quad \succeq_i \quad \mu_2(i) \quad \succeq_i \quad \underbrace{\mu_3(i)}_{\mu_{med}(i)=j} \quad \succeq_i \quad \mu_4(i) \quad \succeq_i \quad \mu_5(i)$$

$$\mu_5(j) \quad , \quad \mu_4(j) \quad \succeq_j \quad \underbrace{\mu_3(j)}_{\mu_{med}(j)=i} \quad \succeq_j \quad \mu_2(j) \quad , \quad \mu_1(j)$$

Hence, μ_{med} is a well-defined matching.

W.l.o.g., $\{i,j\}$ blocking pair for μ_{med}. Then,

$$j \succ_i \overbrace{\mu_{\text{med}}(i) \succeq_i \qquad \cdots \qquad}^{k+1 \text{ stable partners}},$$

$$i \succ_j \underbrace{\mu_{\text{med}}(j) \succeq_j \qquad \cdots \qquad}_{k+1 \text{ stable partners}}.$$

W.l.o.g., $\{i, j\}$ blocking pair for μ_{med}. Then,

$$j \succ_i \overbrace{\mu_{\text{med}}(i) \succeq_i \ldots \mu'(i) \ldots \qquad\qquad\qquad}^{k+1 \text{ stable partners}},$$

$$i \succ_j \underbrace{\mu_{\text{med}}(j) \succeq_j \qquad\qquad\qquad \ldots \mu'(j) \ldots}_{k+1 \text{ stable partners}}.$$

W.l.o.g., $\{i, j\}$ blocking pair for μ_{med}. Then,

$$j \succ_i \overbrace{\mu_{\text{med}}(i) \succeq_i \ldots \mu'(i) \ldots \qquad\qquad}^{k+1 \text{ stable partners}},$$

$$i \succ_j \underbrace{\mu_{\text{med}}(j) \succeq_j \qquad\qquad \ldots \mu'(j) \ldots}_{k+1 \text{ stable partners}}.$$

By "transitivity of blocking," $\{i, j\}$ is a blocking pair for matching μ', which contradicts stability of μ'.
Hence, μ_{med} is a stable matching. □

Corollary

Smith and Rawls (almost) share a room

Let $\mu_1, \ldots, \mu_{2k}$ be an even number of stable matchings. Then, there exists a stable matching at which each agent is assigned a match of rank k or $k+1$.

Key properties in “Smith and Rawls share a room:”

- Decomposability
- Transitivity of blocking

Using these properties and the same proof technique we obtain an even stronger result for marriage markets.

Let $\mu_1, \ldots, \mu_k$ be (possibly non-distinct) stable matchings. Let each agent rank these matchings according to his/her preferences.

For any $l \in \{1, \ldots, k\}$, we define the generalized median matching α_l as the function $\alpha_l : M \cup W \to M \cup W$ such that

$$\alpha_l(i) := \begin{cases} l\text{-th ranked match of } i & \text{if } i \in M; \\ (k - l + 1)\text{-st ranked match of } i & \text{if } i \in W. \end{cases}$$

Theorem

Marriage and compromise – generalized median

Let $\mu_1, \ldots, \mu_k$ be stable matchings. Then, for any $l \in \{1, \ldots, k\}$, α_l is a well-defined stable matching.

In fact, the same proof is essentially valid for its generalization to the college admissions model.

However, the extended proof is no longer elementary in the sense that the key properties identified earlier are based on well-known but non-trivial results for the college admissions model.

- $S = \{s_1, \ldots, s_m\}$: *set of students*.
- $\mathcal{C} = \{C_1, \ldots, C_n\}$: *set of colleges*. College C has quota q_C.
- $\succeq_s$: student s's strict preferences over $\mathcal{C} \cup \{s\}$.
- $\succeq_C$: college C's preferences over feasible sets of students $\mathcal{P}(S, q_C) := \{S' \subseteq S : |S'| \leq q_C\}$.
- Assumption on $\succeq_C$: responsiveness, i.e.,
  - if $s \notin S'$ and $|S'| < q_C$, then $(S' \cup s) \succ_C S'$ if and only if $s \succ_C \emptyset$ and
  - if $s \notin S'$ and $t \in S'$, then $((S' \backslash t) \cup s) \succ_C S'$ if and only if $s \succ_C t$.
- A *college admissions market* is a triple $(S, \mathcal{C}, (\succeq_i)_{i \in S \cup \mathcal{C}})$.

- A *matching* μ for college admissions market $(S, \mathcal{C}, (\succeq_i)_{i \in S \cup \mathcal{C}})$ is a function μ on the set $S \cup \mathcal{C}$ such that
  - for all $s \in S$, either $\mu(s) \in \mathcal{C}$ or $\mu(s) = s$,
  - for all $C \in \mathcal{C}$, $\mu(C) \in \mathcal{P}(S, q_C)$, and
  - for all $s \in S$ and $C \in \mathcal{C}$, $\mu(s) = C$ if and only if $s \in \mu(C)$.

- Matching μ is *individually rational* if $\mu(s) = C$, then $C \succ_s s$ and $\mu(C) \succ_C (\mu(C) \backslash s)$.
- A pair (s, C) *blocks* $(\mu(s), \mu(C))$ if $C \succ_s \mu(s)$ and

  B1. $[\,|\mu(C)| < q_C$ and $s \succ_C \emptyset\,]$ or

  B2. [there exists $t \in \mu(C)$ such that $s \succ_C t$].
- Matching μ is *stable* if it is individually rational and there is no pair (s, C) that blocks $(\mu(s), \mu(C))$.

Lemma

Weak decomposability, *Roth and Sotomayor 1990*
Let μ and μ' be stable matchings.
Let $C \in \mathcal{C}$, $s \in S$, and $s \in \mu(C) \cup \mu'(C)$. Then,

(a) $\mu(C) \succ_C \mu'(C)$ *implies* $\mu'(s) \succeq_s \mu(s)$;

(b) $\mu(s) \succ_s \mu'(s)$ *implies* $\mu'(C) \succeq_C \mu(C)$.

Lemma

Transitivity of blocking for college admissions
Let μ and μ' be matchings, $C \in \mathcal{C}$, and $s \in S$. Suppose (s, C) blocks $(\mu(s), \mu(C))$. Suppose also that C is assigned groups of students $\mu(C)$ and $\mu'(C)$ under some stable matchings.

If $\mu(s) \succeq_s \mu'(s)$ and $\mu(C) \succeq_C \mu'(C)$, then (s, C) blocks $(\mu'(s), \mu'(C))$.

Let $\mu_1, \dots, \mu_k$ be (possibly non-distinct) stable matchings. Let each student/college rank these matchings according to his/its preferences.

For any $l \in \{1, \dots, k\}$, we define the generalized median matching α_l by

$$\alpha_l(i) := \begin{cases} l\text{-th ranked match of } i & \text{if } i \in S; \\ (k - l + 1)\text{-st ranked match of } i & \text{if } i \in \mathcal{C}. \end{cases}$$

Theorem

College admissions and compromise – generalized median

Let $\mu_1, \dots, \mu_k$ be stable matchings. Then, for any $l \in \{1, \dots, k\}$, α_l is a well-defined stable matching.

Example 1

No compromise for q-separable and substitutable preferences

Consider the college admissions market with 4 students s_1, s_2, s_3, s_4, 2 colleges C_1 and C_2 with 2 seats each, and preferences as listed in the table below (Martínez et al., 2000, Example 2). The colleges' preferences are q-separable and substitutable.

$\succ_{C_1}$	$\succ_{C_2}$	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
$\{s_1, s_2\}$	$\{s_3, s_4\}$	C_2	C_2	C_1	C_1
$\{s_1, s_3\}$	$\{s_2, s_4\}$	C_1	C_1	C_2	C_2
$\{s_2, s_4\}$	$\{s_1, s_3\}$				
$\{s_3, s_4\}$	$\{s_1, s_2\}$				
$\{s_1, s_4\}$	$\{s_1, s_4\}$				
$\{s_2, s_3\}$	$\{s_2, s_3\}$				
$\{s_1\}$	$\{s_1\}$				
$\{s_2\}$	$\{s_2\}$				
$\{s_3\}$	$\{s_3\}$				
$\{s_4\}$	$\{s_4\}$				

There are 4 stable matchings:

$$\mu_1 = \{\{C_1, s_1, s_2\}, \{C_2, s_3, s_4\}\}$$
$$\mu_2 = \{\{C_1, s_1, s_3\}, \{C_2, s_2, s_4\}\}$$
$$\mu_3 = \{\{C_1, s_2, s_4\}, \{C_2, s_1, s_3\}\}$$
$$\mu_4 = \{\{C_1, s_3, s_4\}, \{C_2, s_1, s_2\}\}$$

Violation of weak decomposability:

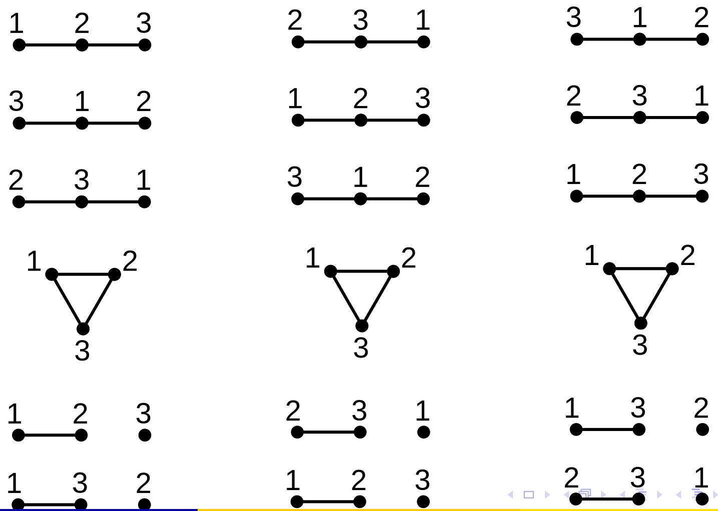
$$s_3 \in \mu_2(C_1),\ \mu_2(s_3) \succ_{s_3} \mu_3(s_3), \text{ and } \mu_2(C_1) \succ_{C_1} \mu_3(C_1).$$

Considering the first three matchings, one straightforwardly checks that matching each agent with its median match is not a matching: C_1 would be matched with $\{s_1, s_3\}$, but at the same time s_3 would be matched with C_2. ◇

Example 2

An unstable compromise for a network formation problem

$\succ_1$	$\succ_2$	$\succ_3$



We extend the notion of blocking for the matching problems considered so far to this network formation problem in a natural way as follows.

Two agents can block a given network by adding a link if and only if this is beneficial for both agents. Furthermore, a single agent can block a given network by destroying a link if that is beneficial for him/her.

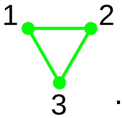
Then, there are 3 stable networks μ_1, μ_2, and μ_3 which are given by

,


, and


,

respectively. Let each agent choose the median of the three sets of links with which he can be associated. Then, each agent chooses to connect with both of the other agents. Hence, the resulting median network is the well-defined but *unstable* complete network:



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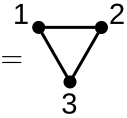
Transitivity of blocking for network formation

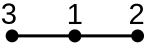
Let μ and μ' be networks and i, j agents such that $\{i, j\}$ (possibly $i = j$) blocks network μ. Suppose also that for all $k = i, j$, agent k is assigned the set of links $\mu(k)$ and $\mu'(k)$ under some *stable* network.[1]
If $\mu \succeq_i \mu'$ and $\mu \succeq_j \mu'$, then $\{i, j\}$ also blocks μ'.

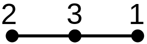
We now show that transitivity of blocking is violated. Consider

$\mu =$ 

and $\mu' =$ 

.

- Note $\{i, j\} = \{1\}$ blocks μ by breaking the link with agent 2 (or 3),
- $\mu_2 =$ 

and $\mu_3 =$ 

are stable with $\mu_2(1) = \mu(1)$ and $\mu_3(1) = \mu'(1)$,
- $\mu \succeq_1 \mu'$, BUT
- in contradiction to transitivity of blocking, $\{1\}$ cannot block μ'.

[1] That is, there are stable $\bar{\mu}$ and $\bar{\mu}'$ with $\bar{\mu}(k) = \mu(k)$ and $\bar{\mu}'(k) = \mu'(k)$.

Note: also a violation of weak decomposability:

$$\mu_2 \succ_1 \mu_3, \text{ agent 1 is linked to agent 3 at } \mu_2, \text{ but } \mu_2 \succ_3 \mu_3.$$

◇

So far, we did not succeed in constructing an example where

- the median outcome is well-defined but unstable,
- (weak) decomposability is satisfied,
- and transitivity of blocking is violated.