Majority rule in the absence of a majority

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Abstract

What is the meaning of "majoritarianism" as a principle of democratic group decision-making in a judgement aggregation problem, when the propositionwise majority view is logically inconsistent? We argue that the majoritarian ideal is best embodied by the principle of *supermajority efficiency* (SME). SME reflects the idea that smaller supermajorities must yield to larger supermajorities. We show that in a well-demarcated class of judgement spaces, the SME outcome is generically unique. But in most spaces, it is not unique; we must make trade-offs between the different supermajorities. We axiomatically characterize the class of *additive majority rules*, which specify how such trade-offs are made. This requires, in general, a hyperreal-valued representation.

1 Introduction

In social choice with two social alternatives, Majority Rule needs no explanation: Majority rule obtains if the group chooses what the greater number of its members would choose; axiomatically, it has been characterized in this setting in a classical theorem by May (1952).

With more than two social alternatives, the very meaning of "Majority Rule" is no longer obvious; to begin with, "the majority" may no longer exist. There is a very straightforward, if unsatisfactory, way to extend majority rule, plurality rule, according to which the group chooses what would be chosen by the plurality, that is: the largest number of its members. This has the virtue of simplicity, but the vice of oversimplification, as both practice and theory attest.

To make progress, we need to formulate more sharply the problem that "majoritarianism" is meant to answer. We take this to be a kind of group decision which might be

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called a *democratic disagreement problem.* Such a problem is characterized by five background assumptions: *self-governance, pluralism, decision by procedure, political equality,* and *democratic agnosticism.* The first three assumptions define a "disagreement problem" in general, while the last two spell out its distinctly democratic character. *Self-governance* means that the group's decision is to be taken on the basis of the views of the members of the group themselves, rather than, for example, some "benevolent dictator" or "philosopher king". *Pluralism* means there is disagreement among the members on the group decision and its basis. This disagreement has not been resolved at the time of decision. It may well be deep-seated and not resolveable by further deliberation among the group; deliberation has effectively ended. *Decision by procedure* means that, consistent with the principle of self-governance and the fact of pluralism, the disagreement cannot be resolved by trying to determine "who is right". Instead, the disagreement among the group needs to be resolved by some aggregation rule or decision procedure. *Political equality* means that all members of the group are equal qua members; the aggregation rule must therefore equally rely on the views of all members. The last assumption is *democratic agnosticism.* As political equality is based on equal membership, not on equal wisdom, epistemic competence, or preference intensity, the aggregation rule should rely as little as possible on the judgemental performance of any particular group member. This consideration provides a motivation for a distinctly "majoritarian" character of the aggregation rule. Indeed, majoritarian aggregation rules reflect "democratic agnosticism" in a clear-cut manner at least in those simple cases in which a unified majority exists, namely by simply discounting all minority opinions completely.

We do not undertake to *defend* majoritarianism here, based on these or any other principles. Our task is squarely analytic: to define and articulate what majoritarianism can mean, on a sufficiently broad and relevant conception of it, within the overall framework of social choice theory. If this analytic work is done successfully, it should help prepare the ground for substantive normative discussion. The five background assumptions provide a common denominator with important strands in contemporary political theory; see for example Waldron (1999) or Christiano (2006).

As a bridge towards the formal definitions and analysis, we shall adopt the following informal definition. Its first part is intended to capture self-governance under pluralism, the second the distinct majoritarian character.

> *A group acts under majority rule if its choice accords with (one of) the majority views of its members on the matters at issue. A majority view is one that is most representative of the distribution of members' views among feasible views.*

First, by making majoritarianism a matter of determining an appropriate majority *view,* this informal definition places the problem squarely within the emerging field of judgement aggregation theory. Majority Rule is viewed as a "solution criterion" to judgement aggregation problems viewed as disagreement problems; in other words, it is viewed as describing a class of methods to "best" resolve the disagreement among group members.[1]

[1]The word "solution criterion" is meant to be analogous to solution criteria for cooperative bargaining, for example.

In line with the general approach of judgement aggregation theory, we shall adopt a very broad, abstract perspective of what a "view" may be: essentially, any set of binary (yes-no) judgements on an interrelated set of propositions or "issues" . Formally, if $\mathcal{K}$ is a (finite) set of issues, then a ***view*** is an element of some feasible set of views $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$. We will also refer to the set $\mathcal{X}$ as a ***judgement space***.[2]

A classical example is that of ordinal preference aggregation; it will play a leading role in this paper as well. Here, a view —for the group and for each member —is given by a (linear) ranking on a set of alternatives. The different issues are given by the comparisons between different pairs of alternatives. The interpretation is that the members of the group disagree in their assessment of which alternatives are better or worse for the group as a whole.

Another example, which originated in the law-and-economics literature (Kornhauser and Sager, 1986) is the set of factual or normative assessments by a multi-member court that serve as "reasons" for its decision ("conclusion"); this example has been one of the stimulants behind the recent growth of judgement aggregation literature (List and Pettit, 2002). In many cases, the views can be seen as beliefs. Conceptually, these beliefs can be quantitative (as in the aggregation of probabilities), not just qualitative.[3]

The conception of Majority Rule as "disagreement solution" differs sharply from the more usual appeal to majoritarian considerations in the more commonly studied problem of welfare aggregation, including the Arrowian problem of how to derive an ordinal social welfare function from individual ordinal preference (betterness) rankings. It is important to keep this distinction in mind here, since majoritarianism appears to be a lot more plausible in the context of resolving disagreement than in the context of welfare aggregation.[4]

The second important part of the informal definition is to identify the majority view with the "most representative" view: the view that, on the whole, "best represents" or "is best aligned with" the distribution of members' individual views. More specifically, the overall alignment of a feasible group view with the distribution of individual views is measured by the alignment of each proposition that makes up the group view with the distribution of individual views *on that proposition.* The latter, in turn, is measured simply by how many members of the group affirm this proposition, its *numerical support.* This focus on the numerical support of alternative views has a natural democratic rationale in terms of giving an equal voice to each member on each proposition. In view of this feature, Majority Rule can arguably claim a priveleged status as a *democratic* disagreement solution —"democratic" referring to the last two of our background assumptions, political equality and democratic agnosticism. At the same time, we do not claim that majority rule is a

[2]Each issue $k \in \mathcal{K}$ can be identified with a proposition that is to be affirmed or negated; in a more syntactic vein, if p is a proposition, then the ordered pair $(p, \neg p)$ is the issue "whether or not p".

[3]Technically, probability aggregation leads beyond the scope of this paper since it requires the aggregation of judgements on an infinite number of issues. the present paper contains, however, a model which can also be interpreted as a model of discretized probability aggregation, and which has particularly attractive properties from a majoritarian perspective.

[4]Formally, the Arrowian preference aggregation framework admits both the "welfarist" and the group choice/disagreement interpretation. Arrow's 1951 classic, *Social Choice and Individual Welfare*, appeals to both themes. Unfortunately, as indeed already indicated by the title, the two themes are not always clearly kept apart there, nor in a lot of the follow-up literature.

priveleged method for resolving disagreement of views *in general*. For example, much of the literature on probabilistic opinion pooling tries to determine the epistemically best view derivable from a given profile of views of "experts"; such methods can be viewed as aiming for maximization of *epistemic* rather than *numerical* support.

The numerical support for a single proposition measures alignment with that proposition in an ordinal manner. The overall alignment of a view is thus simply measured by adding up the numerical support for each proposition constituting that view, after transformation by a common gain function ϕ. This idea yields the class of "additive majority rules". The main result of this paper, Theorem 4.2, characterizes additive majority rules in terms of two axioms, Supermajority Efficiency and Decomposition. While Decomposition is applicable to disagreement solutions quite broadly, Supermajory Efficiency serves as the hallmark of properly majoritarian disagreement resolution, and we now turn to the motivation of this central analytical concept of the paper.

A first step towards making formally precise our informal "representativeness" conception of Majority Rule has been taken in Nehring et al. (2011). It is based on a minimalist criterion of greater representativeness, according to which a view $\mathbf{x}$ is more representative of the distribution of individual views than some other view $\mathbf{y}$ if $\mathbf{x}$ agrees with the majority on each issue on which $\mathbf{y}$ does, and also on some issue on which $\mathbf{y}$ does not. The maximal elements with respect to this partial ordering are called *Condorcet admissible*.

To illustrate how Condorcet admissibility reflects democratic agnosticism, consider a situation of preference aggregation in which everyone agrees on the ranking of all but one alternative a. Then it is easily checked that there is a unique Condorcet admissible ranking (up to ties), which ranks a at the median ranking (relative to the other alternatives). This is clearly robust against changes of individual views. By contrast, the Borda rule, to take a paradigmatic non-majoritarian preference aggregation rule, ranks a roughly at the mean ranking relative to the others. This is clearly very sensitive to changes in individual views, and may thus be questioned from the point of view of democratic agnosticism as putting too much store into every individual view.

A problem with Condorcet admissibility is its indecisiveness. Ignoring potential ties, Condorcet admissibility yields a unique maximal element if and only if the majority judgement on each issue forms itself a feasible view. Consider, for example, the problem of preference aggregation on three alternatives $\{a, b, c\}$, with 40% of the members holding the ranking abc (a first, b second, and c third), 35% the ranking bca, and 25% the ranking cab. Then a majority ranks a above b, b above c, and c above a. But the combination of these three opinions does not form of well-defined ranking itself, hence it does not amount to a feasible view of the group. By consequence, the Condorcet admissible views are those that agree with the majority on exactly two out of three propositions; these are the three rankings abc, bca, and cba.

But this conclusion is more indecisive than necessary. While each of the three Condorcet admissible rankings departs from the majority position on one issue, the ranking abc departs from the majority on the issue of comparing a and c, overruling a supermajority of 60%; by contrast, the ranking bac overrules a supermajority of 65% (on the comparison of a versus b), while the ranking cba overrules an even higher supermajority of 75% (on the comparison of a versus b). Thus, it stands to reason that there is a unique "most representative" ranking, namely the one overruling the *smallest* majority, which is the ranking *abc*.

This advance does not come entirely for free, as it rests on the assumption that all issues are treated on par; if, somehow, there was an overriding interest in comparing a and c, it might be questionable to identify the majority view with a ranking which takes a minority position on exactly that issue. But, in many situations, it is natural and well-justified to treat all issues on par; for such situations, generalizing the above example, we propose the criterion of *supermajority efficiency* as refinement of Condorcet admissibility.

The general idea behind supermajority efficiency is that smaller supermajorities must yield to larger supermajorities. Yet, in contrast to the above example, in general judgement aggregation problems, it may no longer be enough to compare one supermajority won to one supermajority lost. Instead, the entire vectors of supermajorities won or lost need to be compared to arrive at an appropriate partial ordering of "supermajority dominance"; the comparison is analogous to a comparison of risky prospects in terms of first-order stochastic dominance. A view is *supermajority efficient* (SME) if it is not supermajority dominated by any other view.

We shall address two main questions in the following. One: when does SME single out a determinate (i.e. essentially unique) solution? Two: how can a selection among supermajority efficient views be made otherwise?

To address question one, we ask: which judgement spaces *guarantee* determinacy of the SME solution for all profiles? The answer is given by a pair of results. The first of these provides a general combinatorial characterization (Theorem 6.3); it is amplified by a companion result which renders this characterization geometrically transparent and easily checkable under a regularity constraint (Theorem 6.4). We illustrate these results by both positive and negative examples. Among the positive examples are discrete budget spaces, which can model, for example, the allocation of public goods or (discretized) probability judgments.

However, in many judgement spaces of interest (such as for example the space of linear rankings over four or more alternatives), SME fails to guarantee a determinate solution. Nonetheless, we show that even then there is a "sizeable" set of profiles that are SME determinate, while indeterminate under the weaker criterion of Condorcet admissibility.

The indeterminacy of SME in many spaces results from the fact that SME exploits only ordinal comparisons of supermajority margins. Thus, in general, it may be necessary to make cardinal tradeoffs among supermajority margins of different sizes. The main result of this paper, Theorem 4.2, thus characterizes the systematic ways of making such cardinal trade-offs by means of a representation theorem. It relies on just one axiom besides supermajority efficiency, an axiom called "Decomposition".[5]

The axiom of "Decomposition" requires that if a judgement space can be decomposed into logically unrelated subsets of issues, the group view on the entire space is derived from the group views on each of the subsets considered in isolation. The Decomposition axiom

[5]Theorem 4.2 is based on a more restricted and more elementary version, Theorems 4.1, that is of independent interest.

appears compelling from a majoritarian perspective, but does not rely on this perspective and should be applicable to disagreement solutions quite broadly.

Theorem 4.2 delivers a characterization of the class of "additive majority rules". Each of these is described by a "gain-function" ϕ, which measures how much the numerical support (i.e. net supermajority margin) for any proposition of a view contributes to its overall alignment with the profile of member views. An additive majority rule F_ϕ maximizes the overall alignment among feasible views, computing the overall alignment of a view as the simple sum of alignments ("gains") of all the propositions making up that view.

A privileged place among additive majority rules is occupied by the case in which gains are simply equal to numerical support; in that case, the associated additive majority rule is called the *median rule*, and it simply maximizes the total vote count of a view over all issues. In the context of preference aggregation, the median rule is also known as the *Kemeny rule* (1959), and has been axiomatized by Young and Levenglick (1978). The median rule has been studied quite extensively for other special classes of judgement spaces as well (Barthélémy and Monjardet, 1981, 1988; Barthélémy, 1989; Barthélémy and Janowitz, 1991). We provide a general characterization in the companion paper (Nehring and Pivato, 2012b).

In the context of preference aggregation, two other aggregation rules have been proposed in the literature, which themselves are not quite SME themselves but which admit refinements that are. These are the *Slater rule* (1961), and the *Ranked Pairs* rule proposed by Tideman (1987). These rules and their refinements stand at two extreme ends of the spectrum of additive majority rules. Overall, the mathematical and conceptual unity of the majoritarian disagreement solutions identified by our main result stands in stark contrast to the veritable "zoo" of competing majoritarian (Condorcet-consistent) social choice rules in the standard voting formulation.[6]

A technically interesting feature of the analysis is the need to allow gain-functions to take *hyperreal* values rather than just real values. For example, to ensure SME, the refinements of the Ranked Pairs and Slater rule both require hyperreal-valued representations. We hope that our techniques may be useful in other applications to social choice and decision theory.

The remainder of the paper is organized as follows. Section 2 introduces basic notation and terminology. Section 3 formally defines *supermajority efficiency*, with emphasis on the additive majority rules. Section 4 gives an axiomatic characterization of these rules. Section 5 considers what the totality of all additive majority rules tells us about a single judgement problem. Section 6 concerns judgement spaces which are *supermajority determinate*, meaning that the criterion of supermajority efficiency alone is fully decisive, and determines a unique collective view. Appendix A is a formal introduction to hyperreal numbers. Appendices B-F contain the proofs of all the results in the text.

[6]See Nehring et al. (2011) for a more thorough comparison between the judgement aggregation and voting approaches to majoritarianism.

2 Preliminaries

Let $\mathcal{K}$ be a finite set, representing a collection of propositions, each of which can be either true or false. A *view* on $\mathcal{K}$ is an element $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, where $x_k = 1$ if $\mathbf{x}$ "asserts" proposition k, and $x_k = -1$ if $\mathbf{x}$ "denies" proposition k. A *judgement space* is a subset $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$; typically $\mathcal{X}$ is the set of views which are "admissible" or "logically consistent" according to our interpretation of the elements of $\mathcal{K}$.

For example, let $\mathcal{K} = \{p, q, c\}$, where p and q represent two logically independent "premises", and c is a "conclusion" whose truth value is determined by p and q. Then let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of logically consistent assignments of truth values. (Thus, if $c = (p$ and $q)$ then $\mathcal{X} = \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}};\ x_c = \min(x_p, x_q)\}$.) For another example, let $A \in \mathbb{N}$, and let $\mathcal{A} := [1 \ldots A]$ represent a set of A social alternatives. Let $\mathcal{K} := \{(a,b) \in \mathcal{A}^2;\ a < b\}$. Then any view $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ can be interpreted as a complete, antisymmetric binary relation (i.e. a *tournament*) $\prec$ on $\mathcal{A}$, where $a \prec b$ if and only if either $x_{a,b} = 1$ or $x_{b,a} = -1$. (Recall that exactly one of (a,b) or (b,a) is in $\mathcal{K}$.) Now let $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of views representing *transitive* tournaments (i.e. strict preference orders) on $\mathcal{A}$; this space is sometimes called the *permutahedron*. Other judgement spaces encode social decision problems such as resource allocation, committee selection, and object classification; some of these examples appear later in this paper, while others are discussed by Nehring and Puppe (2007), Nehring et al. (2011), and Nehring and Pivato (2011).

Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be a judgement space, and let $\Delta(\mathcal{X})$ be the set of all functions $\mu : \mathcal{X} \longrightarrow \mathbb{R}_+$ such that $\sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x}) = 1$. An element $\mu \in \Delta(\mathcal{X})$ is called a *profile*, and describes a population of (weighted) voters; for each $\mathbf{x} \in \mathcal{X}$, the value of $\mu(\mathbf{x})$ is the total weight of the voters who hold the view $\mathbf{x}$. We call the pair $(\mathcal{X}, \mu)$ a *judgement problem*.

For example, let $\mathcal{N}$ be a finite set of voters, and let $\omega : \mathcal{N} \longrightarrow \mathbb{R}_+$ be a "weight function" such that $\sum_{n \in \mathcal{N}} \omega(n) = W$ for some $W < \infty$ (reflecting, e.g. the differing expertise or priority of different voters). For all $n \in \mathcal{N}$, let $\mathbf{y}^n \in \mathcal{X}$ describe the opinion of voter n. The profile μ determined by this data is defined:

$$\mu(\mathbf{x}) \quad = \quad \frac{1}{W} \sum \{\omega(n)\ ;\ n \in \mathcal{N} \text{ and } \mathbf{y}^n = \mathbf{x}\}, \qquad \text{for all } \mathbf{x} \in \mathcal{X}. \tag{1}$$

Judgement aggregation is the problem of selecting an element of $\mathcal{X}$ to represent the "collective view" of the voters described by μ. (For example: judgement aggregation over $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is classical Arrovian preference aggregation.) Versions of this problem were studied by Guilbaud (1952), Wilson (1975), Rubinstein and Fishburn (1986), and Barthélémy and Janowitz (1991). Since the work of List and Pettit (2002), there has been much interest in this area. List and Puppe (2009) and List and Polak (2010) provide two recent surveys.

Fix a set $\mathcal{K}$ and a judgement space $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$. An *aggregation rule* on $\mathcal{X}$ is a multifunction $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$; for any $\mu \in \Delta(\mathcal{X})$, it yields a nonempty (usually singleton) subset $F(\mu) \subseteq \mathcal{X}$, which represents the social consensus given the profile $\mathbf{x}$. Sometimes we restrict F to a smaller domain. For example, if $\omega : \mathcal{N} \longrightarrow \mathbb{R}_+$ is a weight function, let $\Delta_\omega(\mathcal{X})$ be the set of all profiles obtained as in equation (1) for some assignment $\{\mathbf{y}^n\}_{n \in \mathcal{N}} \subset \mathcal{X}$ of opinions to the individual voters. We shall sometimes consider a rule $F_\omega : \Delta_\omega(\mathcal{X}) \rightrightarrows \mathcal{X}$.

On the other hand, we sometimes define F over a larger domain. For example, let $\mathfrak{X}$ be a collection of judgement spaces (possibly with varying choices of $\mathcal{K}$). Let $\Delta(\mathfrak{X})$ be the set of all ordered pairs $(\mathcal{X}, \mu)$, where $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta(\mathcal{X})$. Now an *aggregation rule* on $\mathfrak{X}$ is a multifunction $F : \Delta(\mathfrak{X}) \rightrightarrows \bigcup_{\mathcal{X} \in \mathfrak{X}} \mathcal{X}$ such that, for each $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta(\mathcal{X})$, we have $F(\mathcal{X}, \mu) \subseteq \mathcal{X}$. (We may indicate $F(\mathcal{X}, \mu)$ by "$F(\mu)$" if $\mathcal{X}$ is clear from context.)

3 Supermajority efficiency

Treat $\{\pm 1\}^{\mathcal{K}}$ as a subset of the vector space $\mathbb{R}^{\mathcal{K}}$. For any profile $\mu \in \Delta(\mathcal{X})$, we define the *majority vector* $\widetilde{\mu} = (\widetilde{\mu}_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\mathcal{K}}$ by setting

$$\widetilde{\mu}_k \quad := \quad \sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x}) \cdot x_k \qquad \text{for all } k \in \mathcal{K}. \tag{2}$$

Thus, $\widetilde{\mu} \in \operatorname{conv}(\mathcal{X})$ (the convex hull of $\mathcal{X}$ in $\mathbb{R}^{\mathcal{K}}$). The vector $\widetilde{\mu}$ records how much "numerical support" there is for each of the propositions in $\mathcal{K}$. For any $k \in \mathcal{K}$, we have $\widetilde{\mu}_k > 0$ (resp. < 0) if a majority asserts (resp. denies) proposition k, and $\widetilde{\mu}_k = 1$ (resp. -1) if the voters unanimously assert (resp. deny) proposition k.

Given a profile $\mu \in \Delta(\mathcal{X})$, a view $\mathbf{x}$ is *at least as widely shared* as view $\mathbf{y}$ ("$\mathbf{x} \underset{\mu}{\gtrsim} \mathbf{y}$") if, on each issue $k \in \mathcal{K}$, the judgement x_k entailed by the former is at least as widely shared as the judgement y_k entailed by the latter. Formally, $\mathbf{x} \underset{\mu}{\gtrsim} \mathbf{y}$ if, for all $k \in \mathcal{K}$,

$$x_k \, \widetilde{\mu}_k \;\geq\; y_k \, \widetilde{\mu}_k. \tag{3}$$

The view $\mathbf{x}$ is *more widely shared* than $\mathbf{y}$ ("$\mathbf{x} \underset{\mu}{>} \mathbf{y}$") if the inequality in (3) is strict for some $k \in \mathcal{K}$. Note that (3) is equivalent to requiring that the judgement entailed by $\mathbf{x}$ on some issue k is aligned with the majority judgement on that issue whenever the judgement entailed by $\mathbf{y}$ is: for all $k \in \mathcal{K}$, we have

$$x_k \widetilde{\mu}_k \geq 0 \quad \text{if} \quad y_k \widetilde{\mu}_k \geq 0.$$

This paraphrase establishes a clear connection to basic intuitions about "majoritarianism".

A view is a candidate for a best majoritarian disagreement resolution if it is maximal among the feasible (i.e. logically consistent) views with respect to the partial order $>_\mu$; these will be called the *Condorcet admissible* views, and their set will be denoted by $\operatorname{Cond}(\mathcal{X}, \mu)$. Formally,

$$\operatorname{Cond}(\mathcal{X}, \mu) \quad := \quad \{\mathbf{x} \in \mathcal{X} : \mathbf{y} \underset{\mu}{>} \mathbf{x} \text{ for no } \mathbf{y} \in \mathcal{X}\}.$$

Clearly (ignoring majority ties), the Condorcet admissible view is unique if and only if there exists a feasible view that agrees with the majority judgement on each issue. Formally, for any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, we define

$$\mathcal{M}(\mu, \mathbf{x}) \quad := \quad \{k \in \mathcal{K} \,;\, x_k \, \widetilde{\mu}_k \geq 0\}. \tag{4}$$

This is the set of all propositions where $\mathbf{x}$ agrees with the majority view. (Thus, $\mathbf{x} \underset{\mu}{\gtrsim} \mathbf{y}$ if and only if $\mathcal{M}(\mu, \mathbf{x}) \supseteq \mathcal{M}(\mu, \mathbf{y})$.) Let $\operatorname{Maj}(\mu) := \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}};\, \mathcal{M}(\mu, \mathbf{x}) = \mathcal{K}\}$. This set is always nonempty, and is usually a singleton, unless there is a "perfect tie" on some propositions. (If $\mathrm{Maj}(\mu) = \{\mathbf{x}\}$, then we will abuse notation by writing "$\mathrm{Maj}(\mu) = \mathbf{x}$" and defining $\mathrm{Maj}_k(\mathbf{x}) := x_k$ for all $k \in \mathcal{K}$.)

If $\mathrm{Maj}(\mu) \cap \mathcal{X} \neq \emptyset$, then it is possible to comply with the majority opinion on every proposition, while still respecting the logical constraints defining $\mathcal{X}$. Unfortunately, as shown by the "Condorcet paradox" in the context of preference aggregation mentioned in the introduction, in many judgement spaces, we may have $\mathrm{Maj}(\mu) \cap \mathcal{X} = \emptyset$ for some $\mu \in \Delta(\mathcal{X})$. In this case, the Condorcet admissible view is not unique. Indeed, the Condorcet set may easily be large; it may even happen that $\mathrm{Cond}\,(\mathcal{X}, \mu) = \mathcal{X}$ in some cases.

But a majoritarian approach to disagreement resolution does not need to stop here, as it can avail itself of considerations that, while not distinctly majoritarian, are perfectly sound from a majoritarian perspective. One such consideration – simple but powerful – derives from the observation that, in many judgement aggregation problems, it makes sense to treat all issues "on par", to give them equal weight as parts of the group view that is to be chosen. This motivates the central idea of this paper: *supermajority efficiency.* Let $\mu \in \Delta(\mathcal{X})$ and let $q \in [0, 1]$. For any $\mathbf{x} \in \mathcal{X}$, let

$$\gamma_{\mu,\mathbf{x}}(q) \quad := \quad \#\{k \in \mathcal{K}\ ;\ x_k\, \widetilde{\mu}_k \geq q\}. \tag{5}$$

This measures the number of coordinates of $\mathbf{x}$ for which the popular support exceeds the supermajority threshold q. For example, $\gamma_{\mu,\mathbf{x}}(0)$ is the number of coordinates where $\mathbf{x}$ receives at least a bare majority, $\gamma_{\mu,\mathbf{x}}(0.5)$ is the number of coordinates where $\mathbf{x}$ receives at least a 75% supermajority, and $\gamma_{\mu,\mathbf{x}}(1)$ is the number of coordinates where $\mathbf{x}$ receives unanimous support.

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, write "$\mathbf{x} \underset{\mu}{\trianglerighteq} \mathbf{y}$" ("$\mathbf{x}$ weakly supermajority-dominates $\mathbf{y}$") if $\gamma_{\mu,\mathbf{x}}(q) \geq \gamma_{\mu,\mathbf{y}}(q)$ for all $q \in (0, 1]$. This relation is transitive and reflexive, but generally not complete. Write "$\mathbf{x} \underset{\mu}{\equiv} \mathbf{y}$" ("$\mathbf{x}$ is supermajority equivalent to $\mathbf{y}$") if $\mathbf{x} \underset{\mu}{\trianglerighteq} \mathbf{y}$ and $\mathbf{x} \underset{\mu}{\trianglelefteq} \mathbf{y}$ (equivalently, $\gamma_{\mu,\mathbf{x}}(q) = \gamma_{\mu,\mathbf{y}}(q)$ for all $q \in (0, 1]$). Finally, write "$\mathbf{x} \underset{\mu}{\triangleright} \mathbf{y}$" ("$\mathbf{x}$ supermajority-dominates $\mathbf{y}$") if $\mathbf{x} \underset{\mu}{\trianglerighteq} \mathbf{y}$ but $\mathbf{x} \underset{\mu}{\not\equiv} \mathbf{y}$. A view $\mathbf{x}$ is *supermajority efficient* (SME) if it is maximal among the feasible (i.e. logically consistent) views with respect to the relation $\underset{\mu}{\triangleright}$. The set of SME views will be denoted by $\mathrm{SME}\,(\mathcal{X}, \mu)$. Formally,

$$\mathrm{SME}\,(\mathcal{X}, \mu) \quad := \quad \big\{\mathbf{x} \in \mathcal{X}\ ;\ \mathbf{y} \underset{\mu}{\triangleright} \mathbf{x} \text{ for no } \mathbf{y} \in \mathcal{X}\big\}.$$

Supermajority efficiency thus mandates overruling, if necessary, small supermajorities in favor of an equal or larger number of supermajorities of equal or greater size. Formally, supermajority dominance $\underset{\mu}{\triangleright}$ goes beyond majority dominance $\underset{\mu}{>}$, by replacing a coordinatewise comparison by a distributional comparison, paralleling the step from ex-post dominance to first-order stochastic dominance in the theory of decision making under risk.

Example 3.1. Let $\mathcal{A} = \{a, b, c\}$, and let $\mathcal{K} := \{$ "$a \succ b$", "$b \succ c$", "$c \succ a$"$\}$; then $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$ is a subset of $\{\pm 1\}^{\mathcal{K}}$. If $\mu \in \Delta(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}})$ and $\mathrm{Maj}(\mu) \not\in \mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$, then $\mathrm{Maj}(\mu)$ must be a Condorcet cycle —say "$a \succ b \succ c \succ a$". Thus, we have $\widetilde{\mu}_{a\succ b} > 0$, $\widetilde{\mu}_{b\succ c} > 0$, $\widetilde{\mu}_{c\succ a} > 0$. To compute $\mathrm{SME}\,(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}, \mu)$, we break the "weakest link(s)" in this cycle. For instance, if $\widetilde{\mu}_{c\succ a} < \min\{\widetilde{\mu}_{a\succ b},\ \widetilde{\mu}_{b\succ c}\}$, then $\mathrm{SME}\,(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}, \mu) = \{a \succ b \succ c\}$. On the other hand, if $\widetilde{\mu}_{c\succ a} = \widetilde{\mu}_{a\succ b} < \widetilde{\mu}_{b\succ c}$, then $\mathrm{SME}\,(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}, \mu) = \{a \succ b \succ c,\ b \succ c \succ a\}$. And if $\widetilde{\mu}_{c\succ a} = \widetilde{\mu}_{a\succ b} = \widetilde{\mu}_{b\succ c} > 0$, then $\mathrm{SME}\,(\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}, \mu) = \{a \succ b \succ c,\ b \succ c \succ a, c \succ a \succ b\}$.

This illustrates the general pattern. If there is a single "'weakest link"' in the Condorcet cycle, then $\mathrm{SME}\left(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}, \mu\right)$ is a singleton. This case is generic: it holds for a dense open subset of profiles in $\Delta(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}})$. In the exceptional case when there are two (respectively, three) "weakest links", the set $\mathrm{SME}\left(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}, \mu\right)$ contains two (respectively, three) elements. Finally, if $\widetilde{\mu}_{c\succ a} = \widetilde{\mu}_{a\succ b} = \widetilde{\mu}_{b\succ c} = 0$, then clearly $\mathrm{SME}\left(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}, \mu\right) = \mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$. ♢

As Example 3.1 shows, on some judgement spaces, the supermajority efficient set is typically a singleton. A judgement space with this property is called *supermajority determinate*; these spaces are characterized in §6. However, in many judgement spaces $\mathcal{X}$, the size of the set $\mathrm{SME}(\mathcal{X}, \mu)$ will depend on the profile μ, and may be large. For example, if $|\mathcal{A}| \geq 4$, then the analysis in Example 3.1 breaks down, and $\mathrm{SME}\left(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}, \mu\right)$ is no longer generically a singleton (see Proposition 6.7(a)). In this case, it is necessary to select from this set, which means we must make tradeoffs between supermajorities of different sizes. A systematic way of making such tradeoffs determines an *aggregation rule*. We will thus be interested in aggregation rules which satisfy the following axiom:

Axiom 1 (***Supermajority efficiency***) *An aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is supermajority efficient if $F(\mu) \subseteq \mathrm{SME}(\mathcal{X}, \mu)$ for all $\mu \in \Delta(\mathcal{X})$.*

For example, the *median rule* is defined:

$$\begin{aligned} \mathrm{Median}(\mathcal{X}, \mu) &:= \operatorname*{argmax}_{\mathbf{x}\in\mathcal{X}} (\mathbf{x} \bullet \widetilde{\mu}), \qquad \text{for all } \mu \in \Delta(\mathcal{X}), \\ \text{where} \quad \mathbf{x} \bullet \widetilde{\mu} &:= \sum_{k\in\mathcal{K}} x_k\, \widetilde{\mu}_k, \quad \text{for any } \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{6}$$

In the setting of Arrovian preference aggregation (i.e. when $\mathcal{X}$ is a permutahedron), this corresponds to the Kemeny (1959) rule.

A (real-valued) *gain function* is an increasing function $\phi : [-1, 1] \longrightarrow \mathbb{R}$. For any judgement space $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ and any gain function ϕ, we define the *additive majority rule* $F_\phi : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ as follows:

$$\text{for all } \mu \in \Delta(\mathcal{X}), \qquad F_\phi(\mu) \quad := \quad \operatorname*{argmax}_{\mathbf{x}\in\mathcal{X}} \left(\sum_{k\in\mathcal{K}} \phi(x_k\, \widetilde{\mu}_k) \right). \tag{7}$$

In the functional form (7), each view $\mathbf{x}$ is evaluated according to its "overall alignment" with the profile μ of individual views, as measured by $\Phi(\mathbf{x}, \mu) := \sum_{k\in K} \phi(x_k \widetilde{\mu}_k)$. The gain function ϕ translates, for each issue $k \in \mathcal{K}$, the numerical support for the judgement on k entailed by the view $\mathbf{x}$ (given by $x_k \widetilde{\mu}_k$) into the "gain" $\phi(x_k \widetilde{\mu}_k)$, which measures the alignment of $\mathbf{x}$ with μ on issue k. To determine the overall alignment of $\mathbf{x}$ with μ, these gains are then added up over all issues $k \in \mathcal{K}$. Since gain functions are increasing by definition, it is easily seen that additive majority rules are always SME.

Importantly, as shown below in Proposition 3.2, without loss of generality, it can be assumed that the gain function ϕ is *odd*, i.e. that, for all $r \in [-1, 1]$, $\phi(-r) = -\phi(r)$, or, equivalently, $\phi(r) = \mathrm{sign}(r)\, \phi(|r|)$. Call $\phi_{||} : r \mapsto \phi(|r|)$ the *absolute gain function*.[7]

[7] While the gain function is most useful and natural mathematically, the absolute gain function is most useful to discuss specification and shape.

If gain equals numeric support, i.e. if ϕ is the identity function or linear, then F_ϕ is the median rule (5). While this is a natural baseline case, a non-proportional response of gains to numeric support makes a lot of intuitive sense.

To better understand alternative shapes of the gain function, assume that ϕ is odd and, to simplify, that it is differentiable. Consider the impact on the overall alignment of view $\mathbf{x}$ of a change in the view of one among N voters from view $\mathbf{y}$ to view $\mathbf{z}$, when the distribution of others' views is $\nu \in \Delta(\mathcal{X})$. This impact is given by $\Phi\left(\mathbf{x}, \frac{1}{N}\delta_{\mathbf{z}} + \frac{N-1}{N}\nu\right) - \Phi\left(\mathbf{x}, \frac{1}{N}\delta_{\mathbf{y}} + \frac{N-1}{N}\nu\right)$, which is easily seen to be equal to $\sum_{k:y_k \neq z_k} \phi\left(x_k\left(\frac{1}{N}\delta_{z_k} + \frac{N-1}{N}\widetilde{\nu}_k\right)\right) - \phi\left(x_k\left(\frac{1}{N}\delta_{y_k} + \frac{N-1}{N}\widetilde{\nu}_k\right)\right)$. If ϕ is differentiable and N is sufficiently large, this in turn is approximately equal to

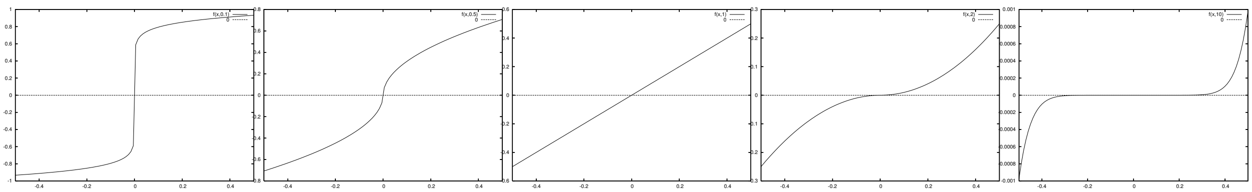
$$\frac{1}{N}\sum_{k:y_k \neq z_k} \phi'\left(\widetilde{\nu}_k\right) x_k\left(z_k - y_k\right).$$

Thus, the rule F_ϕ values the change of support on issue k (namely, $x_k\left(z_k - y_k\right)$) with weight proportional to $\phi'\left(\widetilde{\nu}_k\right)$, which is equal to $\phi'\left(|\widetilde{\nu}_k|\right)$ by oddness. Thus, at the margin, the F_ϕ weighs the "vote" of any member as function of the extent of the agreement on k among the other member $|\widetilde{\nu}_k|$. F_ϕ is the median rule if and only if the weighting of the votes is independent of the other members' views; call this the *agreement-neutral* case. If $\phi'_{||}$ is increasing (in other words, if the absolute gain function $\phi_{||}$ is convex, and the gain function is "inverse-S-shaped"), then votes matter more on issues on which there already is a lot of agreement; call such rules the *agreement-focused*. Agreement focusing seems intuitively rather attractive, especially at the high end. Consider the extreme case of unanimity among all the other members. Then a lone dissent, if it materializes, would naturally carry special weight, since it amounts to a qualitative change from complete agreement to some disagreement overall. The resulting convexity of the absolute gain function implies that large supermajorities carry disproportionate weight, and are unlikely to be overruled.

By contrast, if $\phi'_{||}$ is decreasing (that is, if the absolute gain function $\phi_{||}$ is concave and the gain function ϕ is "S-shaped"), then votes matter more on issues which are highly contested in that there already is a lot of disagreement; call such rules *disagreement-focused*. Disagreement focusing seems somewhat less plausible, but not outlandish, especially near the low end, when an individual change of opinion on an issue can be pivotal for the majority preference on that issue. The resulting concavity of the absolute gain function implies that F_ϕ puts emphasis on the number of issues in which the group judgement agrees with the majority of members. The more concave ϕ is, the more easily a large supermajority, possibly even unanimity, is overruled by a number of smaller supermajorities.

To illustrate with a simple functional form, fix $d \in (0, \infty)$, and define $\phi^d : [-1, 1] \longrightarrow \mathbb{R}$ by $\phi^d(r) := \mathrm{sign}(r) \cdot |r|^d$ for all $r \in [-1, 1]$, as shown in Figure 1. The corresponding additive majority rule $H^d := F_{\phi^d}$ is called the ***homogeneous*** rule of degree d.

If $d = 1$, then H^d is the median rule. If $1 < d < \infty$, then the gain-function ϕ^d is inverse-S-shaped, thus F_{ϕ^d} is agreement-focused. The agreement focus and resulting privileging of large supermajorities becomes more exacerbated as d increases; in the limit, that privileging becomes lexical; the limiting *Leximax* rule can be described as follows. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, write "$\mathbf{x} \underset{\mu}{\approx} \mathbf{y}$" if $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$; otherwise write "$\mathbf{x} \underset{\mu}{\succ} \mathbf{y}$" if there exists some $Q \in (0, 1]$ such that $\gamma_{\mu,\mathbf{x}}(q) = \gamma_{\mu,\mathbf{y}}(q)$ for all $q > Q$, while $\gamma_{\mu,\mathbf{x}}(q) > \gamma_{\mu,\mathbf{y}}(q)$. Then $\underset{\mu}{\succeq}$ is a



$\phi_{1/10}$ $\phi_{1/2}$ ϕ_1 ϕ_2 ϕ_{10}

Figure 1: Gain functions for homogeneous rules

complete, transitive ordering of $\mathcal{X}$. We then define

$$\text{Leximax}\,(\mathcal{X},\mu) \quad := \quad \max(\mathcal{X},\ \underset{\mu}{\succeq}\ ). \tag{8}$$

In other words, Leximax first maximizes the number of coordinates which receive unanimous support (if any); then, for every possible supermajoritarian threshold $q \in (0,1]$, Leximax maximizes the number of coordinates where the support exceeds q, with higher values of q given lexicographical priority over lower ones.[8]

On the other hand, if $0 < d < 1$, then the gain-function ϕ^d is S-shaped, and thus F_{ϕ^d} is disagreement-focused. The disagreement focus and resulting deprivileging of large supermajorities becomes more exacerbated as d decreases; in the limit, F_{ϕ^d} becomes a refinement of the Slater rule.[9] Slightly abusing notation, the Slater rule can be defined as Slater $(\mathcal{X},\mu) := F_{\phi}(\mathcal{X},\mu)$, where $\phi\,(r) := \text{sign}(r)$ for all $r \in [-1,1]$. Note that ϕ is *not* a proper gain-function, since it is merely non-decreasing, not strictly increasing. As a consequence, while majority admissible, the Slater rule is *not* SME. On the other hand, there are refinements of the Slater rule that are SME, since Slater is always consistent with SME, in the sense that Slater $(\mathcal{X},\mu) \cap \text{SME}\,(\mathcal{X},\mu) \neq \varnothing$ for any $\mu \in \Delta(\mathcal{X})$.

At first sight, the Leximax rule and these SME refinements of the Slater rules appear to be natural examples of SME rules that are *not* additive majority rules, indicating a significant limitation of this family. But this limitation is more apparent than real, since it can be overcome by allowing gains to be infinite and/or infinitesimally-valued —technically, by extending the co-domain of the gain function ϕ.

Perhaps the broadest such extension which makes sense is that to what is called a *linearly ordered abelian group* $(\mathbb{L},+,>)$. Here, *abelian group* means that $+$ is an associative, commutative, invertible binary operation on $\mathbb{L}$. Meanwhile, *linearly ordered* means that $>$ is a linear ordering relation compatible with $+$, in that $r > 0$ iff $r + s > 0$ for all $r,s \in \mathbb{L}$. An $\mathbb{L}$*-valued gain function* is now any increasing function $\phi : [-1,1] \longrightarrow \mathbb{L}$. (Note

[8] The Leximax rule is a refinement of the Ranked Pairs proposed by Tideman (1987) in the setting of preference aggregation; see also Zavist and Tideman (1989). The Ranked Pairs rule itself is not SME, but it agrees with the Leximax rule on profiles μ for which $\widetilde{\mu}_k \neq \widetilde{\mu}_\ell$ if $k \neq \ell$. With a finite number of members, it can easily be substantially coarser.

[9] *Proof sketch:* For any $\mathbf{x},\mathbf{y} \in \mathcal{X}$, if $\mathbf{x} \in$ Slater $(\mathcal{X},\mu)$ and $\mathbf{y} \notin$ Slater $(\mathcal{X},\mu)$, then $\sum_{k\in\mathcal{K}} \text{sign}(\widetilde{\mu}_k\, x_k) > \sum_{k\in\mathcal{K}} \text{sign}(\widetilde{\mu}_k\, y_k)$, which implies that $\Phi^d(\mathbf{x},\mu) > \Phi^d(\mathbf{y},\mu)$ for all d close enough to 0. Thus, $\mathbf{y} \notin F_{\phi^d}(\mu)$ if d is close to 0. Since $\mathcal{X}$ is finite, we can repeat this argument for all $\mathbf{y} \notin$ Slater $(\mathcal{X},\mu)$ to conclude that $F_{\phi^d}(\mu) \subseteq$ Slater $(\mathcal{X},\mu)$ for all d close enough to 0.

that a real-valued gain function is a special case, because $\mathbb{R}$ is a linearly ordered abelian group.) Given any $\mathbb{L}$-valued gain function ϕ, we can define the *additive majority rule* F_ϕ as in equation (7). Then F_ϕ is easily seen to be SME.

But this is more general than necessary. Indeed, it turns out that one can always let the co-domain be a *linearly ordered real field* $\mathbb{L}$. As a real field, $\mathbb{L}$ possesses a well-defined multiplication operation and contains the real numbers $\mathbb{R}$ as a subfield. Linear orderedness now means that the ordering relation is compatible with addition and multiplication; the latter requires that $\ell \cdot m > m$ for all $\ell, m \in \mathbb{L}$ with $\ell > 1$ and $m > 0$.

A linearly ordered real field $\mathbb{L}$ can be viewed as $\mathbb{R}$ augmented by infinite and infinitesimal numbers. An element ℓ in $\mathbb{L}$ is *infinite* if $|\ell| > N$ for all $N \in \mathbb{N}$. On the other hand, ℓ is *infinitesimal* if $|\ell| < 1/N$ for any $N \in \mathbb{N}$ (or equivalently, if $1/\ell$ is infinite). If ℓ is neither infinite nor infinitesimal, then it is *finite*. For any finite $\ell \in \mathbb{L}$, there exists a unique $r \in \mathbb{R}$ and infinitesimal $\epsilon \in \mathbb{L}$ such that $\ell = r + \epsilon$. We then write $\mathrm{st}(\ell) = r$; this is called the *standard* part of ℓ.

For example, with linearly ordered real fields containing infinitesimals, we can define refinements of the Slater rule as additive majority rules. Let $\psi : [-1,1] \longrightarrow \mathbb{R}$ be any strictly increasing function, let $\epsilon \in \mathbb{L}$ be an infinitesimal, and define $\phi : [-1,1] \longrightarrow \mathbb{L}$ by $\phi(r) := \mathrm{sign}(r) + \epsilon\, \psi(r)$ for all $r \in [-1,1]$. Then the additive majority rule F_ϕ is a supermajority efficient refinement of the Slater rule. That is, $F_\phi(\mu) \subseteq \mathrm{SME}(\mathcal{X},\mu) \cap \mathrm{Slater}(\mathcal{X},\mu)$ for any $\mu \in \Delta(\mathcal{X})$.[10]

For most purposes, this is an adequate level of generality. To achieve full generality, one may need to ensure that number of infinite and infinitesimal elements is sufficiently "large". This is done by assuming $\mathbb{L}$ to be an appropriate *hyperreal* field $^*\mathbb{R}$. (Formally, $^*\mathbb{R}$ is an ultrapower of $\mathbb{R}$; see Appendix A for details.)

As a bonus, hyperreal fields allow well-defined exponentiation as well. This is useful in particular to make the range of different infinities arithmetically accessible, as, for example, in obtaining a representation of the Leximax rule in terms of a hyperreal valued gain function. In sum: hyperreals are very user-friendly and quite intuitive in describing various additive majority rules in a unified manner. For clarity, we sum up the above discussion as follows.

Definition. Let $\mathcal{K}$ be a finite set of propositions, and let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ be a judgement space. An aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is an *additive majority rule* if there exists a linearly ordered abelian group[11] $\mathbb{L}$ and a strictly increasing *gain function* $\phi : [-1,1] \longrightarrow \mathbb{L}$ such that, for all profiles $\mu \in \Delta(\mathcal{X})$, we have

$$F(\mu) \quad = \quad \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} \left(\sum_{k \in \mathcal{K}} \phi(x_k\, \widetilde{\mu}_k) \right). \tag{9}$$

[10] *Proof:* For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, if $\mathbf{x} \in \mathrm{Slater}(\mathcal{X},\mu)$ and $\mathbf{y} \notin \mathrm{Slater}(\mathcal{X},\mu)$, then $\sum_{k\in\mathcal{K}} \mathrm{sign}(\widetilde{\mu}_k)\, x_k > \sum_{k\in\mathcal{K}} \mathrm{sign}(\widetilde{\mu}_k)\, y_k$, which implies that $\phi(\widetilde{\mu}) \bullet \mathbf{x} > \phi(\widetilde{\mu}) \bullet \mathbf{y}$. Thus, $\mathbf{y} \notin F_\phi(\mu)$. Thus, $F_\phi(\mu) \subseteq \mathrm{Slater}(\mathcal{X},\mu)$. The supermajority efficiency of F_ϕ follows from Theorem 4.2(a) below.

[11] This includes the special case when $\mathbb{L}$ is the additive structure of a linearly ordered field; in particular, it includes the case when $\mathbb{L}$ is the field of real numbers $\mathbb{R}$ or a hyperreal field $^*\mathbb{R}$.

More generally, let $\mathfrak{X}$ be a collection of judgement spaces. An aggregation rule F on $\mathfrak{X}$ is an *additive majority rule* if there exist $\mathbb{L}$ and ϕ as above such that, for all $\mathcal{X} \in \mathfrak{X}$ and all $\mu \in \Delta(\mathcal{X})$, the outcome $F(\mathcal{X}, \mu)$ is defined by formula (9).

For the sake of generality, we allow $\mathbb{L}$ to have the general structure of linearly ordered abelian group in this definition. However, Proposition 4.4 (in Section 4) will establish that, without loss of generality, $\mathbb{L}$ can always be assumed to be a hyperreal field. This representation is appealing because of the similarity between hyperreal arithmetic and ordinary real-valued arithmetic, which may sit more comfortably with some readers than an abstract abelian group. Thus, in the remainder of the paper, we will focus on hyperreal-valued gain functions.

It is a notable feature of the functional form (9) that one can assume without loss of generality that the gain function ϕ is *odd*, i.e. that $\phi(-r) = -\phi(r)$ for all $r \in [-1, 1]$. Heuristically, this means that ϕ gives exactly the same weight towards a majority opinion that some proposition k is true as it does towards an equal-sized majority opinion that k is false; hence F_ϕ is not biased towards either truth or falsehood. In terms of the function's graph, oddness of ϕ is equivalent to symmetry of the graph around the point $(0, 0)$, as illustrated by the homogeneous rules depicted in Figure 1.

Odd functions are completely determined by their behavior on the positive unit interval $[0, 1]$ (or, symmetrically, the negative unit interval $[-1, 0]$). For example, an odd ϕ function is inverse-S shaped if and only if it is convex on the positive unit interval. In view of the next result, we will henceforth take gain-functions to be increasing and odd functions from $[-1, 1]$ to ${}^*\mathbb{R}$.

Proposition 3.2 *Let $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$ be any increasing gain function. Define the function $\widehat{\phi}$ by $\widehat{\phi}(r) := \phi(r) - \phi(-r)$ for $r \in [-1, 1]$. Then $\widehat{\phi}$ is odd and increasing, and yields the same aggregation rule, i.e. $F_{\widehat{\phi}} = F_\phi$.*

Proposition 3.2 is due to the fact that at a given profile μ, an aggregation rule can only choose between affirming or overriding the majority of size $|\widetilde{\mu}_k|$ on a particular proposition k. Accordingly, $\widehat{\phi}(r)$ describes the gain from affirming rather than overriding a majority of the same size r on any particular proposition; it follows that the gains associated with negative majority margins are the mirror images of the gains associated with positive ones. If ${}^*r \in {}^*\mathbb{R}$ and $x \in \{\pm 1\}$, then $x \cdot {}^*r$ is also an element of ${}^*\mathbb{R}$. Thus, if ${}^*\mathbf{r} \in {}^*\mathbb{R}^{\mathcal{K}}$ and $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, then we can define $\mathbf{x} \bullet {}^*\mathbf{r} := \sum_{k \in \mathcal{K}} x_k \cdot {}^*r_k$, an element of ${}^*\mathbb{R}$. In particular, for any $\mu \in \Delta(\mathcal{X})$, we define $\phi(\widetilde{\mu}) := [\phi(\widetilde{\mu}_k)]_{k \in \mathcal{K}} \in {}^*\mathbb{R}^{\mathcal{K}}$; then $\mathbf{x} \bullet \phi(\widetilde{\mu}) = \sum_{k \in \mathcal{K}} x_k \cdot \phi(\widetilde{\mu}_k)$.

Corollary 3.3 *Let $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$ be an odd gain function, and let $(\mathcal{X}, \mu)$ be a judgement problem.*

(a) $F_\phi(\mathcal{X}, \mu) = \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} \sum_{k \in \mathcal{K}} \phi\left[(x_k\, \widetilde{\mu}_k)_+\right] = \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} \sum_{k \in \mathcal{M}(\mathbf{x}, \mu)} \phi\, |\widetilde{\mu}_k|.$[12]

(b) *Also,* $F_\phi(\mathcal{X}, \mu) = \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} \ \mathbf{x} \bullet \phi(\widetilde{\mu})$.

[12] For any $r \in \mathbb{R}$, recall that $r_+ := \min\{r, 0\}$. Meanwhile, $\mathcal{M}(\mu, \mathbf{x})$ is as defined in equation (4).

In view of Corollary 3.3(b), all additive majority rules are alike. This is particular clear in the real-valued case. In this case, a view is maximal under the rule F_ϕ at the profile μ iff it is maximal under the median rule F_{med} at a fictitious profile with the vector of majority margins $\phi(\widetilde{\mu})$.[13]

Additive majority rules have two other attractive properties: they are monotone and generically single-valued. For any $\mu \in \Delta(\mathcal{X})$, let $\mathcal{X}(\mu) := \{\mathbf{x} \in \mathcal{X};\ \ \mu(\mathbf{x}) > 0\}$. Let $\mu' \in \Delta(\mathcal{X})$ and let $\mathbf{y} \in \mathcal{X}$. We say that μ' is *more supportive than μ of* $\mathbf{y}$ if $\mu'(\mathbf{y}) > \mu(\mathbf{y})$, while $\mu'(\mathbf{x}) < \mu(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}(\mu) \setminus \{\mathbf{y}\}$, and $\mu'(\mathbf{x}) = \mu(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{X} \setminus \mathcal{X}(\mu)$. (For example: if $\delta_\mathbf{y} \in \Delta(\mathcal{X})$ is the unanimous profile at $\mathbf{y}$, then for any $\mu \in \Delta(\mathcal{X})$ and any $r \in (0, 1]$, the convex combination $r\,\delta_\mathbf{y} + (1 - r)\,\mu$ is more supportive than μ of $\mathbf{y}$.) A judgement aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is *monotone* if, for any $\mu, \mu' \in \Delta(\mathcal{X})$, and $\mathbf{y} \in F(\mu)$, if μ' is more supportive than μ of $\mathbf{y}$, then $F(\mu') = \{\mathbf{y}\}$. In other words: if $\mathbf{y}$ is already one of the winning alternatives, then any increase in the support for $\mathbf{y}$ at the expense of support for other elements of $\mathcal{X}$ will make $\mathbf{y}$ the *unique* winning alternative. The rule F is *generically single-valued* if there is an open dense subset $\mathcal{O} \subset \Delta(\mathcal{X})$ such that $|F(\mu)| = 1$ for all $\mu \in \mathcal{O}$.

Proposition 3.4 *Let $^*\mathbb{R}$ be any hyperreal field, and let $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$ be any gain function. Then for any judgement space $\mathcal{X}$, the additive majority rule F_ϕ is* (a) *monotone, and* (b) *generically single-valued on $\Delta(\mathcal{X})$.*

Technically, genericity is defined only for a continuum of individuals. Nevertheless, if the number of voters is finite but "large", then Proposition 3.4(b) can be interpreted heuristically as saying that ties are uncommon. However, if the number of voters is finite but "small", then genericity has no bearing.

A judgement aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is *upper hemicontinuous* (uhc) if, for every $\mu \in \Delta(\mathcal{X})$, each of the following two equivalent statements is true:

(UHC1) There exists some $\bar{\epsilon} > 0$ such that, for any $\epsilon \in (0, \bar{\epsilon})$ any other $\nu \in \Delta(\mathcal{X})$, we have $F(\epsilon\,\nu + (1 - \epsilon)\,\mu) \subseteq F(\mu)$.

(UHC2) For every sequence $\{\mu_n\}_{n=1}^\infty \subset \Delta(\mathcal{X})$, and every $\mathbf{x} \in \mathcal{X}$, if $\lim_{n\to\infty} \mu_n = \mu$, and $\mathbf{x} \in F(\mu_n)$ for all $n \in \mathbb{N}$, then $\mathbf{x} \in F(\mu)$.

Statement (UHC1) is sometimes described as the "overwhelming majority" property. Heuristically, the profile $\epsilon\,\nu + (1-\epsilon)\,\mu$ represents a mixture of two populations: a small "minority" described by the profile ν, and a large "majority" represented by the profile μ. Statement (UHC1) says that, if the majority is large enough, then its view determines the group view (except that the minority can perhaps act as a "tie-breaker" in some cases). Statement (UHC2) means that the outcome of judgement aggregation is robust under small

[13] In general, the vector $\phi(\widetilde{\mu})$ need not realizable in terms of an actual profile $\nu \in \Delta(\mathcal{X})$. However, in a significant class of interesting judgement spaces, —the *McGarvey* spaces —the vector $\phi(\widetilde{\mu})$ is always equal to the vector of margins $\widetilde{\nu}$ associated with some profile $\nu \in \Delta(\mathcal{X})$, possibly after multiplication by a positive scalar (Nehring and Pivato, 2011).

measurement errors or perturbations of public opinion.[14] The proof of the next result is straightforward.

Proposition 3.5 *Let $\phi : [-1,1] \longrightarrow \mathbb{R}$ be any continuous, real-valued gain function. Then for every judgement space $\mathcal{X}$, the rule F_ϕ is upper hemicontinuous on $\Delta(\mathcal{X})$.*

4 Characterization of additive majority rules

The Decomposition Axiom. In order to provide a normative foundation for the class of additive majority rules, a second normative axiom besides SME is needed. It applies to situations in which judgement spaces can be decomposed into independent subspaces. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be disjoint sets, and let $\mathcal{K} := \mathcal{K}_1 \sqcup \mathcal{K}_2$. Let $\mathcal{X}_1 \subseteq \{\pm 1\}^{\mathcal{K}_1}$, let $\mathcal{X}_2 \subseteq \{\pm 1\}^{\mathcal{K}_2}$ and let $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \subseteq \{\pm 1\}^{\mathcal{K}}$. If $\mathbf{x} \in \mathcal{X}$, then we write $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ where $\mathbf{x}^n \in \mathcal{X}_n$ for $n = 1, 2$. For any $\mu \in \Delta(\mathcal{X})$, let $\mu^{(1)} \in \Delta(\mathcal{X}_1)$ be the ***marginal profile*** of μ on $\mathcal{X}_1$. That is:

$$\text{for all } \mathbf{x}^1 \in \mathcal{X}_1, \qquad \mu^{(1)}(\mathbf{x}^1) \quad := \quad \sum_{\mathbf{x}^2 \in \mathcal{X}_2} \mu(\mathbf{x}^1, \mathbf{x}^2). \tag{10}$$

Likewise, define $\mu^{(2)} \in \Delta(\mathcal{X}_2)$. (Observe that $\widetilde{\mu} = (\widetilde{\mu}^{(1)}, \widetilde{\mu}^{(2)})$, because $\widetilde{\mu}^{(1)}_k = \widetilde{\mu}_k$ for all $k \in \mathcal{K}_1$ and $\widetilde{\mu}^{(2)}_k = \widetilde{\mu}_k$ for all $k \in \mathcal{K}_2$.)

Let $\mathfrak{X}$ be a collection of judgement spaces. We say $\mathfrak{X}$ is ***closed under Cartesian products*** if, for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}$, we also have $\mathcal{X} \times \mathcal{Y} \in \mathfrak{X}$. (For an example, let $\mathfrak{X}$ be the set of *all* judgement spaces.) Let F be a judgement aggregation rule on $\Delta(\mathfrak{X})$ (so $F(\mathcal{X}, \mu) \subseteq \mathcal{X}$ for any $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta(\mathcal{X})$). Consider the following axiom.

Axiom 2 (*Decomposition*) *F is **decomposable** if, for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}$ and all $\mu \in \Delta(\mathcal{X}_1 \times \mathcal{X}_2)$, we have $F(\mathcal{X}_1 \times \mathcal{X}_2, \mu) \ = \ F(\mathcal{X}_1, \mu^{(1)}) \times F(\mathcal{X}_2, \mu^{(2)})$.*

Decomposition appears compelling as a requirement on majoritarian aggregation, especially if majoritarianism is is interpreted as looking for the "most representative" view. Indeed, it seems eminently sensible to say that a view on a set of logically independent issues (reflected in a Cartesian product) is most representative overall if and only if it is made up views on the independent constituent issues that themselves are most representative of voters' views on these subissues.

To take the simplest example, consider a hypercube $\mathcal{X} = \{\pm 1\}^{\mathcal{K}}$, reflecting a set of $\mathcal{K}$ logically independent yes-no issues. If F coincides with majority voting over binary issues $\{\pm 1\}^k$, then Decomposition implies that F coincides with issue-wise majority voting over the entire hypercube. Note, though, that this is simple, normatively appealing conclusion is entailed here by SME already.

To illustrate the additional force of the Decomposition axiom beyond SME, consider two distinct judgement spaces $\mathcal{X}$ and $\mathcal{Y}$, and suppose that the aggregation in $\mathcal{X}$ and $\mathcal{Y}$ is based on additive majority rules with distinct gain functions ϕ and ψ, respectively. If F is

[14] Note that a nontrivial judgement aggregation rule can never be *lower* hemicontinuous (because it is a nonconstant function from $\Delta(\mathcal{X})$ into a discrete set).

decomposable, then F may need to violate SME on $\mathcal{X} \times \mathcal{Y}$, essentially because F would trade off supermajorities of different sizes on $\mathcal{X}$ and on $\mathcal{Y}$ in an inconsistent manner.

Fixed Population, Fixed Judgment Space. We shall first present a basic version of the main result of the paper, an (almost-)characterization of the class of additive majority rules for the case of a fixed finite population of voters and a fixed judgement space $\mathcal{X}$. To apply the Decomposition axiom, we need to consider the aggregation on $\mathcal{X}$ itself together with its finite Cartesian powers representing, for example, repetitions of the same type of aggregation problem with different profiles.

For any judgement space $\mathcal{X}$, and any $M \in \mathbb{N}$, let $\mathcal{X}^M := \mathcal{X} \times \mathcal{X} \times \cdots \times \mathcal{X}$ be the M-fold Cartesian product of $\mathcal{X}$. Let $\langle \mathcal{X} \rangle := \{\mathcal{X}^M;\ M \in \mathbb{N}\}$. (Thus, $\langle \mathcal{X} \rangle$ is closed under Cartesian products.) Let $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be a judgement aggregation rule. For all $M \in \mathbb{N}$, define the rule $F^M : \Delta(\mathcal{X}^M) \rightrightarrows \mathcal{X}^M$ as follows:

$$F^M(\mu) \quad := \quad F(\mu^{(1)}) \times F(\mu^{(2)}) \times \cdots \times F(\mu^{(M)}), \qquad \text{for all } \mu \in \Delta(\mathcal{X}^M). \tag{11}$$

Here, $\mu^{(1)}, \ldots, \mu^{(M)} \in \Delta(\mathcal{X})$ are the marginal profiles of μ onto the M copies of $\mathcal{X}$ which comprise $\mathcal{X}^M$. If we define $\Delta\langle \mathcal{X} \rangle := \bigcup_{M=1}^{\infty} \Delta(\mathcal{X}^M)$, then we obtain a decomposable aggregation rule $F^* : \Delta\langle \mathcal{X} \rangle \rightrightarrows \bigcup_{M=1}^{\infty} \mathcal{X}^M$. Indeed, F^* is the *unique* extension of F to a decomposable rule on $\Delta\langle \mathcal{X} \rangle$.

Now, fix $N \in \mathbb{N}$. Let $\Delta_N\langle \mathcal{X} \rangle$ be the set of all profiles on $\langle \mathcal{X} \rangle$ involving exactly N equally weighted voters. If $\mu \in \Delta_N(\mathcal{X}^M)$, then $\mu^{(1)}, \ldots, \mu^{(M)} \in \Delta_N(\mathcal{X})$. Thus, for any judgement aggregation rule $F : \Delta_N(\mathcal{X}) \rightrightarrows \mathcal{X}$, we can extend F to a decomposable rule F^* on $\Delta_N\langle \mathcal{X} \rangle$ by applying equation (11).

We say that F is *extended supermajority efficient* (or ESME) if F^M is supermajority efficient on $\Delta_N(\mathcal{X}^M)$ for every $M \in \mathbb{N}$. Let $G : \Delta_N(\mathcal{X}) \rightrightarrows \mathcal{X}$ be another aggregation rule; we say G *covers* F if $F^M(\mu) \subseteq G^M(\mu)$ for all $M \in \mathbb{N}$ and $\mu \in \Delta_N(\mathcal{X}^M)$.

Theorem 4.1 *Let $N \in \mathbb{N}$ be a finite number of voters, let $\mathcal{X}$ be a judgement space, and let $F : \Delta_N(\mathcal{X}) \rightrightarrows \mathcal{X}$ be a judgement aggregation rule. Then the rule F is ESME if and only if F is covered by an additive majority rule $G : \Delta_N(\mathcal{X}) \rightrightarrows \mathcal{X}$*

Furthermore, we can choose G to be "minimal" in the following sense: if H is any other additive majority rule which covers F, then H also covers G. This minimal covering rule G is unique.

Finally, any additive majority rule G on $\Delta_N\langle \mathcal{X} \rangle$ has a real-valued representation.

Theorem 4.1 associates with any ESME rule a unique additive majority rule G that minimally covers F. This minimal cover can be seen as the closest approximation to F which can be justified by a "systematic" way of trading off between conflicting majorities (as described by the gain function ϕ). Thus, additive majority rules can be viewed as the systematic part of ESME rules.

The key steps in the proof of Theorem 4.1 are roughly as follows. For any $M \in \mathbb{N}$, $\mu \in \Delta_N(\mathcal{X}^M)$, and $\mathbf{x} \in \mathcal{X}$, the function $\gamma_{\mu,\mathbf{x}}$ can be represented as a vector $\mathbf{g}_{\mu,\mathbf{x}} \in \mathbb{R}^N$. Let $\mathcal{P}$ be the closure of the set $\{(\mathbf{g}_{\mathbf{x},\mu} - \mathbf{g}_{\mathbf{y},\mu})/M;\ M \in \mathbb{N},\ \mu \in \Delta_N(\mathcal{X}^M),\ \mathbf{x} \in F^M(\mathcal{X}^M, \mu)$, and

$\mathbf{y} \in \mathcal{X}^M\}$. First, using definition (11), we show that $\mathcal{P}$ is a compact, convex polyhedron in $\mathbb{R}^N$. Second, using ESME, we show that $\mathcal{P}$ is disjoint from nonnegative orthant $\mathbb{R}^N_{\not -}$. Third, using a slightly enhanced version of the Separating Hyperplane Theorem for finite dimensional Euclidean spaces, we obtain a *strictly positive* vector $\mathbf{v} \in \mathbb{R}^N_+$ which separates $\mathcal{P}$ from $\mathbb{R}^N_{\not -}$. Finally, let $\mathcal{Q}_N := \{1 - \frac{2k}{N};\ k \in [0 \ldots N]\}$ (equivalently: $\mathcal{Q}_N = \{\widetilde{\mu}_k;\ k \in \mathcal{K}$ and $\mu \in \Delta_N\langle\mathcal{X}\rangle\}$). We use $\mathbf{v}$ to define an odd, increasing function $\phi : \mathcal{Q}_N \longrightarrow \mathbb{R}$ such that F_ϕ covers F. Now let $G = F_\phi$. A bit of fine-tuning of ϕ ensures that G is the unique minimal cover.[15] The existence of a real-valued representation in the absence of further assumptions is obtained as a consequence of Hahn's embedding theorem for linearly ordered abelian groups.

Variable Population, Variable Judgment Space. If N is fixed in advance, and all voters have equal weight, then Theorem 4.1 is adequate for most purposes. Proposition C.1 (in Appendix C) extends Theorem 4.1 to a fixed population of voters having fixed, distinct weights (reflecting, for example, varying levels of expertise); in this case, F need only be defined on the subset $\Delta_\omega(\mathcal{X}) \subset \Delta(\mathcal{X})$ of profiles which could be produced by these weights. However, if the number N of voters or the weights of individual voters are allowed to vary, then F must be well-defined on all of $\Delta(\mathcal{X})$. Also, Theorem 4.1 only states that F is *covered* by G; if we want to ensure that $F = G$, we must also require upper hemicontinuity, which only makes sense if F is defined on a dense subset of $\Delta(\mathcal{X})$. Finally, we may wish to consider rules defined on a larger collection judgement spaces —not just the set of Cartesian powers of one space $\mathcal{X}$.

Let $\mathfrak{X}$ be a collection of judgement spaces. Given two aggregation rules F and G on $\Delta(\mathfrak{X})$, we say that G *covers* F if $F(\mathcal{X}, \mu) \subseteq G(\mathcal{X}, \mu)$ for any $\mathcal{X} \in \mathfrak{X}$ and any $\mu \in \Delta(\mathcal{X})$. We now come to our second main result.

Theorem 4.2 *Let $\mathfrak{X}$ be any set of judgement spaces which is closed under Cartesian products.*

> **(a)** *Any additive majority rule is supermajority efficient and decomposable on $\mathfrak{X}$.*

Now let F be an aggregation rule satisfying Decomposition on $\mathfrak{X}$.

> **(b)** *F is SME on $\Delta(\mathfrak{X})$ if and only if F is covered by an additive majority rule G* (with a hyperreal gain function). *Furthermore, for all $\mathcal{X} \in \mathfrak{X}$, there is a dense open subset $\mathcal{O} \subseteq \Delta(\mathcal{X})$ such that $F(\mathcal{X}, \mu) = G(\mathcal{X}, \mu)$ and is single-valued for all $\mu \in \mathcal{O}$.*
>
> **(c)** *G can be chosen (uniquely) to be "minimal" in the following sense: if H is any other additive majority rule which covers F, then H also covers G. This minimal covering rule G is unique.*

Finally, let F be a upper hemicontinuous aggregation rule on $\mathfrak{X}$. Then:

> **(d)** *F is an additive majority rule if and only if F is SME and decomposable.*

[15]*Note:* although G is unique, the gain function ϕ which defines G is not unique. Also, ϕ is only defined on the set $\mathcal{Q}_N$ —not on all of $[-1, 1]$.

Theorem 4.2 offers two significant enhancements to Theorem 4.1. First, we now have $F(\mathcal{X},\mu) = G(\mathcal{X},\mu)$ (rather than merely $F(\mathcal{X},\mu) \subseteq G(\mathcal{X},\mu)$) for generic $\mu \in \Delta(\mathcal{X})$. Second, with the auxiliary condition of upper hemicontinuity, Theorem 4.2 provides an exact characterization of the class of additive majority rules.

To illustrate the meaning of Theorem 4.2 and to explain how it is derived from Theorem 4.1, it will be helpful to state a simpler, intermediate version of the same result in a setting with a variable population but a fixed judgment space. The proof sketch of this intermediate result shows how the hyperreal-valued representation in a variable-population setting emerges constructively from the real-valued representation in the fixed-population setting of Theorem 4.1. It also shows that Theorem 4.2 does not depend on the use of irrational profiles in $\Delta(\mathfrak{X})$, or use any richness assumptions about the collection $\mathfrak{X}$. Fix a judgement space $\mathcal{X}$. Define

$$\Delta_{\mathbb{Q}}(\mathcal{X}) \;:=\; \bigcup_{N=1}^{\infty} \Delta_N(\mathcal{X}) \;\subset\; \Delta(\mathcal{X}) \qquad \text{and} \qquad \mathcal{Q} \;:=\; \bigcup_{N=1}^{\infty} \mathcal{Q}_N \;=\; \mathbb{Q}\cap[-1,1].$$

Proposition 4.3 *Let $F : \Delta_{\mathbb{Q}}(\mathcal{X}) \rightrightarrows \mathcal{X}$ be a judgement aggregation rule.*

(a) *F is ESME on $\Delta_{\mathbb{Q}}\langle\mathcal{X}\rangle$ if and only if there is gain function $\phi : \mathcal{Q}\longrightarrow{}^*\mathbb{R}$ such that $F^n(\mu) \subseteq F_\phi(\mathcal{X}^n,\mu)$ for all $n \in \mathbb{N}$ and $\mu \in \Delta_{\mathbb{Q}}(\mathcal{X}^n)$.*

In this case, for all $n \in \mathbb{N}$, there is a dense, relatively open subset $\mathcal{O}_n \subseteq \Delta_{\mathbb{Q}}(\mathcal{X}^n)$ such that $F^n(\mu) = F_\phi(\mathcal{X}^n,\mu)$ and is single-valued for all $\mu \in \mathcal{O}_n$.

(b) *Let F and ϕ be as in part* (a)*, and let $n \in \mathbb{N}$. If F^n is upper hemicontinuous on $\Delta_{\mathbb{Q}}(\mathcal{X}^n)$, then $F^n(\mu) = F_\phi(\mathcal{X}^n,\mu)$ for all $\mu \in \Delta_{\mathbb{Q}}(\mathcal{X}^n)$.*

Proof sketch. Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real-valued, $\mathbb{N}$-indexed sequences. A hyperreal number corresponds to an equivalence class of elements of $\mathbb{R}^{\mathbb{N}}$, where two sequences are deemed equivalent if they agree "almost everywhere" (see Appendix A). For every $N \in \mathbb{N}$, Theorem 4.1 yields an increasing function $\phi_N : \mathcal{Q}_N\longrightarrow\mathbb{R}$ such that $F^n(\mu) \subseteq F_{\phi_N}(\mathcal{X}^n,\mu)$ for all $n \in \mathbb{N}$ and $\mu \in \Delta_N(\mathcal{X}^n)$. For each N, we extend ϕ_N to an odd, increasing function $\phi_N : \mathcal{Q}\longrightarrow\mathbb{R}$ in some arbitrary way. We now have an $\mathbb{N}$-indexed sequence $(\phi_N)_{N\in\mathbb{N}}$ of real-valued functions on $\mathcal{Q}$. Equivalently, it may be viewed as a single, $\mathbb{R}^{\mathbb{N}}$-valued function $\widehat{\phi}$. For every $q \in \mathcal{Q}$, let $\phi(q) \in {}^*\mathbb{R}$ be the almost-everywhere equivalence class of the sequence $\widehat{\phi}(q) \in \mathbb{R}^{\mathbb{N}}$; this yields a function $\phi : \mathcal{Q}\longrightarrow{}^*\mathbb{R}$. It can be checked that ϕ is odd and increasing (because each ϕ_N was odd and increasing), and $F^n(\mu) \subseteq F_\phi(\mathcal{X}^n,\mu)$ for all $\mu \in \Delta_{\mathbb{Q}}(\mathcal{X}^n)$ (because $F^n(\mu) \subseteq F_{\phi_N}(\mathcal{X}^n,\mu)$ for all $\mu \in \Delta_N(\mathcal{X}^n)$). Proposition 3.4(b) yields a dense, relatively open domain $\mathcal{O}_n \subseteq \Delta_{\mathbb{Q}}(\mathcal{X}^n)$ where F_ϕ is single-valued. Any $\mu \in \Delta_{\mathbb{Q}}(\mathcal{X}^n)$ can be approximated with elements from $\mathcal{O}_n$; by invoking upper hemicontinuity, we can then establish $F^n(\mu) = F_\phi(\mathcal{X}^n,\mu)$.

To illustrate the construction of the hyperreal representation in a specific example, suppose F was the Leximax rule (8). Then for each $N \in \mathbb{N}$, it suffices to define $\phi_N(q) := \mathrm{sign}(q)\cdot|q|^{d_N}$ for some sufficiently large $d_N \in \mathbb{R}_+$, to obtain an additive majority rule which covers F on $\Delta_N\langle\mathcal{X}\rangle$. Let $\phi(q) := \mathrm{sign}(q)\cdot|q|^{{}^*d}$, where ${}^*d \in {}^*\mathbb{R}$ is the almost-everywhere equivalence class of the sequence $(d_N)_{N\in\mathbb{N}}$. It is easy to see that $\lim_{N\to\infty} d_N = \infty$; thus, ${}^*d$ is an infinite hyperreal. It follows that F_ϕ is the Leximax rule.[16]

Properties of the Gain-Function: Uniqueness, Real-Valuedness and Continuity. The preceeding results raise three issues. The proof sketch of Proposition 4.3 makes clear how hyperreal gain functions enter the picture. But a real-valued gain function would certainly be preferable. By analogy with Myerson (1995), we might expect upper hemicontinuity to be sufficient to guarantee that ϕ is real-valued. But this turns out to be false; there are uhc additive majority rules which do not admit real-valued gain functions. This also explains why the two-stage proof strategy is necessary in Theorem 4.2; it is not possible to use an infinite-dimensional "separating hyperplane" strategy to directly obtain an additive majority representation of F on all of $\Delta(\mathfrak{X})$.[17]

A second issue raised by Theorem 4.2 is the *uniqueness* of the gain function ϕ. This turns out to be quite subtle; in general, ϕ can be made unique up to positive scalar multiplication, but only on a *subset* of $[-1,1]$, and only if ϕ is real-valued. Further, even this degree of uniqueness is obtainable only for certain kinds of judgment spaces, not all, as it hinges on a sufficient degree of multiplicity in the SME views at some profiles. But, in an important class of spaces studied in section 6, SME efficiency alone determines the group view essentially uniquely at all profiles. Hence, in those spaces, all additive majority rules end up selecting exactly the same views; see Proposition 6.2 below.

A third issue is to find conditions on ϕ which are necessary for F_ϕ to be upper hemicontinuous. Proposition 3.5 provides one sufficient condition. But the converse of Proposition 3.5 turns out to be false; the question of upper hemicontinuity is also quite subtle. The issues of upper hemicontinuity, uniqueness, and real-valuedness are all addressed the companion paper Nehring and Pivato (2012a).

Generality of the Hyperreal Representation. While Theorem 4.2 provides strong grounds for focusing on gain-functions with a hyperreal representation, it does not show their full generality, since it does not provide an exact characterization in the general case, only a unique minimal cover. It thus leaves room for additive majority rules with yet more general representations based on more general co-domains; the most general such representation would be that of a linearly ordered abelain group. However, the next result shows that this level of generality is unnecessary. Hyperreal-valued gain function are sufficient to represent any additive majority rule.

[16] *Proof.* For any $r, s \in [0,1]$, if $r > s$, then the ratio $r^{{}^*\omega}/s^{{}^*\omega}$ is infinite (because it must be larger than $(r/s)^N$ for any $N \in \mathbb{N}$). Thus, if $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ and $\mathbf{x} \underset{\mu}{\succ} \mathbf{y}$, then $\phi(\widetilde{\mu}) \bullet \mathbf{x} > \phi(\widetilde{\mu}) \bullet \mathbf{y}$; hence $\mathbf{y} \notin F_\phi(\mu)$. This shows that $F_\phi(\mu) \subseteq \text{Leximax}(\mathcal{X}, \mu)$. On the other hand, for any $\mathbf{x}, \mathbf{y} \in \text{Leximax}(\mathcal{X}, \mu)$, we must have $\mathbf{x} \underset{\mu}{\approx} \mathbf{y}$, which means $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$, which means $\mathbf{x} \bullet \phi(\widetilde{\mu}) = \mathbf{y} \bullet \phi(\widetilde{\mu})$. Thus $F_\phi(\mu) = \text{Leximax}(\mathcal{X}, \mu)$.

[17] With an infinite-dimensional separating hyerplane argument analogous to the one used in the proof of Theorem 4.1, we *can* construct a function $\phi : [-1,1] \longrightarrow \mathbb{R}$ such that F_ϕ covers F. However, in general, this ϕ will *not* be strictly increasing, so F_ϕ will not, in general, be SME itself. Indeed, in extreme cases (e.g. $F = $Leximax), ϕ may be constant on almost all of $[-1,1]$, which means that generally $F_\phi(\mathcal{X}, \mu) = \mathcal{X}$, so that the "covering" of F by F_ϕ is trivial and useless.

Proposition 4.4 *Let $\mathbb{L}$ be any linearly ordered abelian group, and let $\psi : [-1,1] \longrightarrow \mathbb{L}$ be any gain function. Then there exists a hyperreal field ${}^*\mathbb{R}$ and a gain function $\phi : [-1,1] \longrightarrow {}^*\mathbb{R}$ such that, for any judgement problem $(\mathcal{X}, \mu)$, we have $F_\phi(\mathcal{X}, \mu) = F_\psi(\mathcal{X}, \mu)$.*

The proof of Proposition 4.4 uses the same argument for the existence of a real-valued representation in the finite population setting as was used in the proof of Theorem 4.1. However, now we must "glue together" many such real-valued representations; the outcome is a hyperreal-valued representation.

5 Implications for Individual Profiles

What are the implications of the totality of additive majority rules for judgement aggregation at a particular profile? In other words, for a given space $\mathcal{X}$ and profile μ, which views in $\mathcal{X}$ would be chosen by *some* additive majority rule? One may be inclined to conjecture that these must be exactly the supermajority efficient views, but this need not be correct as it misses the potential indirect restrictions imposed by Decomposition.

To characterize these restrictions, let $(\mathcal{X}, \mu)$ be a judgement aggregation problem, and let $\mathbf{x} \in \mathcal{X}$. For any $M \in \mathbb{N}$, let $\mathbf{x}^M := (\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) \in \mathcal{X}^M$, and let $\nu_M \in \Delta(\mathcal{X}^M)$ denote some profile such that $\nu_M^{(1)} = \cdots = \nu_M^{(M)} = \mu$. We say that $\mathbf{x}$ is *strongly supermajority efficient* (SSME) in $(\mathcal{X}, \mu)$ if $\mathbf{x}^M \in \mathrm{SME}\left(\mathcal{X}^M, \nu_M\right)$ for all $M \in \mathbb{N}$. In other words, $\mathbf{x}$ remains supermajority efficient even under arbitrary "replication" of the original judgement aggregation problem. This replication can be interpreted in two ways.

First, we might regard $[1 \ldots M]$ as a sequence of times, and suppose we encounter the same judgement aggregation problem M times in a row. Then $\mathbf{x}$ is SSME if $\mathbf{x}$ is optimal not only as a solution for one of these problems in isolation, but as a solution for the whole sequence; it is not possible to surpass $\mathbf{x}$ through some strategy which alternates between two or more elements of $\mathcal{X}$.

Second, we might regard $[1 \ldots M]$ as a set of possible states of nature. In this case, an element $(\mathbf{y}_1, \ldots, \mathbf{y}_M) \in \mathcal{X}^M$ represents a "randomized" view, which obtains the value $\mathbf{y}_m$ if state m occurs. Then $\mathbf{x}$ is SSME if it is not possible to surpass $\mathbf{x}$ through some randomized view.

Let $\mathrm{SSME}(\mathcal{X}, \mu)$ be the set of SSME elements in $\mathcal{X}$. The following result collects a few basic facts about SSME as a refinement of SME.

Proposition 5.1 *For any judgement space $\mathcal{X}$ and $\mu \in \Delta(\mathcal{X})$, we have $\emptyset \neq \mathrm{SSME}(\mathcal{X}, \mu) \subseteq \mathrm{SME}(\mathcal{X}, \mu) \subseteq \mathrm{Cond}(\mathcal{X}, \mu)$. If $\mathrm{Maj}(\mu) \cap \mathcal{X} \neq \emptyset$, then $\mathrm{SSME}(\mathcal{X}, \mu) = \mathrm{SME}(\mathcal{X}, \mu) = \mathrm{Cond}(\mathcal{X}, \mu) = \mathrm{Maj}(\mu) \cap \mathcal{X}$.*

Recall that Theorem 4.2 says that any SME rule satisfying decomposition is essentially an additive majority rule. The next result is a similar statement at the level of individual views; it says that any element of $\mathrm{SSME}(\mathcal{X}, \mu)$ can be obtained as the output of some additive majority rule

Proposition 5.2 *Let $(\mathcal{X}, \mu)$ be a judgement aggregation problem, and let $\mathbf{x} \in \mathcal{X}$.*

(a) *The following are equivalent.*

[i] $\mathbf{x} \in \mathrm{SSME}(\mathcal{X}, \mu)$.

[ii] $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$ *for some gain function* ϕ.

[iii] $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$ *for some real-valued, continuous gain function* ϕ.

(b) *Furthermore, for any* $\mathbf{x}, \mathbf{y} \in \mathrm{SSME}(\mathcal{X}, \mu)$*, if* $\gamma_{\mathbf{x},\mu} \neq \gamma_{\mathbf{y},\mu}$*, then* ϕ *can be chosen in part* (a)[iii] *such that* $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$ *but* $\mathbf{y} \notin F_\phi(\mathcal{X}, \mu)$.

We say that a judgement space $\mathcal{X}$ is ***neat*** if the SME and SSME views always coincide; in view of Proposition 5.2, a space is neat if supermajority efficient views can always be obtained as the output of some additive majority rule. In view of Proposition 5.1, one class of neat spaces are the *median spaces*, which are exactly those spaces in which the propositionwise majority view $\mathrm{Maj}(\mu)$ is always consistent; see Nehring and Puppe (2007, 2010).[18] Since SSME is a refinement of SME, it is also clear that spaces in which SME is essentially unique will also be neat; such "supermajority-determinate" spaces are characterized in Section 6 below.

On the other hand, some spaces are not neat. For example, spaces of equivalence relations are not neat, provided that they are sufficiently large. To be precise, let $\mathcal{A}$ be a finite set, and let $\mathcal{K}$ be the set of all cardinality-2 subsets of $\mathcal{A}$. For any equivalence relation "$\sim$" on $\mathcal{A}$, we define the element $\mathbf{x}^\sim \in \{\pm 1\}^\mathcal{K}$ by setting $x^\sim_{\{i,j\}} := 1$ if $i \sim j$, while $x^\sim_{\{i,j\}} := -1$ if $i \not\sim j$, for all $i, j \in \mathcal{A}$. Let $\mathcal{X}^{\mathrm{eq}}_\mathcal{A} := \{\mathbf{x}^\sim;\ \text{"}\sim\text{" is any equivalence relation on } \mathcal{A}\}$.

Proposition 5.3 *For any set* $\mathcal{A}$*, if* $|\mathcal{A}| \geq 8$*, then* $\mathcal{X}^{\mathrm{eq}}_\mathcal{A}$ *is not neat.*

6 Supermajority determinacy

We say that a judgement aggregation problem $(\mathcal{X}, \mu)$ is ***majority determinate*** if, for any $\mathbf{x}, \mathbf{y} \in \mathrm{Cond}(\mathcal{X}, \mu)$, we have $\mathbf{x} \underset{\mu}{\equiv} \mathbf{y}$. Clearly, this occurs if and only if $\mathrm{Maj}(\mu) \cap \mathcal{X} \neq \emptyset$ —i.e. if the profile μ is "Condorcet consistent". Condorcet consistency and majority determinacy of profiles have been studied by Dietrich and List (2010), Pivato (2009), and Nehring et al. (2011). A judgement space $\mathcal{X}$ is ***majority determinate*** if every $\mu \in \Delta(\mathcal{X})$ is majority determinate. The majority determinate spaces have been characterized by Nehring et al. (2011), adapting Nehring and Puppe (2007), as coinciding with the median spaces. Examples of median spaces are views ordered as points on a line, and single-peaked preference relations over alternatives ordered as points on a line (Black, 1948; Arrow, 1963).

We say that a judgement aggregation problem $(\mathcal{X}, \mu)$ is ***supermajority determinate*** if, for any $\mathbf{x}, \mathbf{y} \in \mathrm{SME}(\mathcal{X}, \mu)$, we have $\mathbf{x} \underset{\mu}{\equiv} \mathbf{y}$. Typically, this means that $\mathrm{SME}(\mathcal{X}, \mu)$ is a singleton (because typically, $\mathbf{x} \underset{\mu}{\not\equiv} \mathbf{y}$ whenever $\mathbf{x} \neq \mathbf{y}$). A judgement space $\mathcal{X}$ is ***supermajority determinate*** if every $\mu \in \Delta(\mathcal{X})$ is supermajority determinate. Since supermajority

[18] In the present setting, a median space can be defined as follows. For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{\pm 1\}^\mathcal{K}$, let $\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ denote the unique view $\mathbf{w}$ in $\{\pm 1\}^\mathcal{K}$ such that, for all $k \in \mathcal{K}$, we have $w_k = 1$ iff $x_k + y_k + z_k \geq 1$. Then $\mathcal{X}$ is a ***median space*** if, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$, we have $\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{X}$.

dominance (unlike majority dominance) exploits the different strengths of supermajorities on different propositions, supermajority determinacy is a weaker, more broadly applicable requirement than majority determinacy.

For example, call a profile μ *barely Condorcet inconsistent* if it is Condorcet inconsistent and, for some $\ell \in \mathcal{K}$, there exists $\mathbf{x} \in \mathcal{X}$ such that (i) $\widetilde{\mu}_k \cdot x_k \geq 0$ for all $k \neq \ell$, and (ii) $|\widetilde{\mu}_\ell| \leq |\widetilde{\mu}_k|$ for all $k \neq \ell$. Since, at any such profile, any view must override the majority on some proposition, and since that majority is smallest for the view $\mathbf{x}$, one sees that $\mathbf{x} \underset{\mu}{\trianglerighteq} \mathbf{y}$ for all $\mathbf{y} \in \mathcal{X}$; thus any barely Condorcet inconsistent profile is supermajority determinate. At the same time, barely Condorcet inconsistent profiles are not just majority indeterminate by definition, but may be majority indeterminate in a rather dramatic fashion: the Condorcet set may leave the resolution of *every* issue indeterminate, in that, for all $k \in \mathcal{K}$, we have $\{x_k;\ \mathbf{x} \in \text{Cond}(\mathcal{X}, \mu)\} = \{\pm 1\}$.[19]

According to the following result, the barely Condorcet inconsistent profiles form a kind of belt of supermajority determinate profiles around the majority determinate ones. Generic profiles need to be sufficiently "far" from majority determinacy to be supermajority indeterminate.

Proposition 6.1 *Let μ and ν be generic profiles in $\Delta(\mathcal{X})$ such that μ is majority determinate and ν is supermajority indeterminate. Then there exists an open interval $\mathcal{T} \subset (0,1)$ such that, for any $t \in \mathcal{T}$, the profile $t\,\nu + (1-t)\,\mu$ is barely Condorcet inconsistent, hence supermajority determinate yet majority indeterminate.*

Let $MD(\mathcal{X}) \subseteq \Delta(\mathcal{X})$ be the set of majority determinate profiles, and let $SD(\mathcal{X}) \subseteq \Delta(\mathcal{X})$ be the set of supermajority determinate profiles. If $\mathcal{X}$ itself is not majority determinate, then Proposition 6.1 implies that $MD(\mathcal{X}) \subsetneq SD(\mathcal{X})$ —indeed, the set $SD(\mathcal{X}) \setminus MD(\mathcal{X})$ has nonempty interior in $\Delta(\mathcal{X})$.

In a supermajority determinate space, *all* profiles are supermajority determinate. In such a space, majority rule under issue parity is canonically given by the SME criterion itself. To establish this claim formally, we must invoke the following axiom, which is a natural complement to supermajoritarian efficiency.

Axiom 3 (*Supermajority Equivalence*) *An aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ satisfies Supermajority Equivalence (SMEQ) if, for all $\mu \in \Delta(\mathcal{X})$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $\mathbf{x} \underset{\mu}{\equiv} \mathbf{y}$ we have $\mathbf{x} \in F(\mathcal{X}, \mu)$ if and only if $\mathbf{y} \in F(\mathcal{X}, \mu)$.*

Thus, if the SME axiom says that the rule F cares primarily about the $\underset{\mu}{\trianglerighteq}$-ranking of the collective view, then SMEQ says that F cares *only* about the $\underset{\mu}{\trianglerighteq}$-ranking. Note that any additive majority rule satisfies SMEQ in any judgement aggregation problem. Normatively, we could have included SMEQ in the characterizations of additive majority rules, but this would not have added much. Here it adds just what is needed.

Proposition 6.2 *Suppose $\mathcal{X}$ is a supermajority determinate judgement space, and $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is an aggregation rule. Then the following four conditions are equivalent:*

[19] This property can be viewed as a discrete counterpart in judgement aggregation to the celebrated "chaotic" nature of majority voting identified by McKelvey (1976, 1979). It is studied in more detail in Nehring et al. (2011).

(a) *F is supermajority efficient and supermajority equivalent;*

(b) *F is supermajority efficient and upper hemicontinuous;*

(c) $F(\mu) = \mathrm{SME}\,(\mathcal{X}, \mu)$ *for all* $\mu \in \Delta(\mathcal{X})$*;*

(d) $F = F_\phi$ *for any gain function* $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$.

Which spaces are supermajority determinate besides median spaces? We have seen one example already in section 3: the 3-permutahedron, capturing preference aggregation over three alternatives. (Indeed, in this space, *every* Condorcet inconsistent profile is barely Condorcet inconsistent.) However, our general characterization of supermajority-spaces relies on the following condition of "friendliness" . We say that $\mathcal{X}$ is ***friendly*** if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ and $\mu \in \Delta(\mathcal{X})$, if $\mathrm{Median}\,(\mathcal{X}, \mu) = \{\mathbf{x}, \mathbf{y}\}$, then $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$. Here is the first main result of this section.

Theorem 6.3 *A judgement space is supermajority determinate if and only if it is friendly.*

While simple to state, friendliness is not easy to interpret and may be difficult to check. As we shall show momentarily, in typical cases it is equivalent to a more intuitive and often easily checkable condition of "proximality".

Recall that $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}} \subset \mathbb{R}^{\mathcal{K}}$. Let $\mathcal{C} := \mathrm{conv}(\mathcal{X})$. Note that $\mathcal{C} = \{\widetilde{\mu};\ \ \mu \in \Delta(\mathcal{X})\}$ (by defining formula (2)). We say that $\mathcal{X}$ is ***thick*** if $\mathcal{C}$ has non-empty interior in $\mathbb{R}^{\mathcal{K}}$. This means that no coordinate of $\mathcal{X}$ can be expressed as an affine function of other coordinates. Thickness is a mild nondegeneracy condition which is satisfied by most interesting judgement spaces.

Note that $\mathcal{C}$ is a compact, convex polytope in $\mathbb{R}^{\mathcal{K}}$, whose vertices are the elements of $\mathcal{X}$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, the line segment from $\mathbf{x}$ to $\mathbf{y}$ is an ***edge*** of $\mathcal{C}$ if and only if there exists some "supporting" vector $\mathbf{r} \in \mathbb{R}^{\mathcal{K}}$ such that $\mathbf{r} \bullet \mathbf{x} = \mathbf{r} \bullet \mathbf{y} > \mathbf{r} \bullet \mathbf{z}$ for all $\mathbf{z} \in \mathcal{X} \setminus \{\mathbf{x}, \mathbf{y}\}$. We say $\{\mathbf{x}, \mathbf{y}\}$ is an ***internal edge*** if it has a supporting vector $\mathbf{r} \in \mathcal{C}$; in this case, we write "$\mathbf{x}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{y}$". It is easy to see that $\mathbf{x}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{y}$ if and only if there exists some $\mu \in \Delta(\mathcal{X})$ such that $\mathrm{Median}\,(\mathcal{X}, \mu) = \{\mathbf{x}, \mathbf{y}\}$ (simply choose μ such that $\widetilde{\mu} \in \mathcal{C}$ is a supporting vector for the edge between $\mathbf{x}$ and $\mathbf{y}$).

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, let $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) := \{k \in \mathcal{K}\ ;\ x_k \neq y_k\}$. Let $d(\mathbf{x}, \mathbf{y}) := |\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y})|$ be the Hamming distance from $\mathbf{x}$ to $\mathbf{y}$. We say $\mathcal{X}$ is ***proximal*** if $d(\mathbf{x}, \mathbf{y}) \leq 2$ for all $\mathbf{x}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{y} \in \mathcal{X}$. We now come to the second main result of this section.

Theorem 6.4 **(a)** *If $\mathcal{X}$ is proximal, then $\mathcal{X}$ is friendly; hence $\mathcal{X}$ is supermajority determinate.*

(b) *If $\mathcal{X}$ is thick and friendly, then $\mathcal{X}$ is proximal.*

(c) *Thus, if $\mathcal{X}$ is thick, then $\mathcal{X}$ is supermajority determinate if and only if it is proximal.*

Examples of Proximality. To illustrate the content of Theorem 6.4, we will give a few examples of supermajority determinacy due to proximality. First, if $|\mathcal{K}| = 3$, then *any* space $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$ is supermajority determinate. To see this, first note that any judgment space with two elements is majority determinate hence trivially supermajority-determinate. On the other hand, if $|\mathcal{X}| \geq 3$, then $\mathcal{X}$ is proximal, and thus, supermajority determinate by Theorem 6.4(a). (To see this, observe that $d(\mathbf{x}, \mathbf{y}) \leq 3$ for all $\mathbf{x}, \mathbf{y} \in \{\pm 1\}^{\mathcal{K}}$, and $d(\mathbf{x}, \mathbf{y}) = 3$ if and only if $\mathbf{x} = -\mathbf{y}$, in which case clearly not $\mathbf{x} \, \mathrm{I}_{\mathcal{X}} \, \mathbf{y}$.)

For another example of proximality, let $\mathcal{K}$ represent a set of "candidates". Any $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$ represents a "committee" (where $x_k = 1$ if and only if candidate k is on the committee). Let $|\mathbf{x}| := \#\{k \in \mathcal{K};\ \ x_k = 1\}$ be the size of this committee. Let $0 \leq I \leq J \leq K$, and let $\mathcal{X}^{\mathrm{com}}_{I,J} := \{\mathbf{x} \in \{\pm 1\}^{\mathcal{K}};\ \ I \leq |\mathbf{x}| \leq J\}$. Thus, $\mathcal{X}^{\mathrm{com}}_{I,J}$ represents the set of committees drawn from $\mathcal{K}$, containing at least I members and at most J members.

Proposition 6.5 *$\mathcal{X}^{\mathrm{com}}_{I,J}$ is proximal, and thus, supermajority determinate.*

In the committee problem $\mathcal{X}^{\mathrm{com}}_{I,J}$, the content of SME is very transparent. If the issue-wise majority $\mathrm{Maj}(\mu)$ constitutes a feasible committee, then elect it. If not —for example, if less than I candidates are approved by a majority of votes —then elect the I candidates with the largest number of votes (if necessary, one can arbitrarily break ties among candidates with equal votes).

For yet another example of proximality, fix $M, D \in \mathbb{N}$, and consider the D-dimensional "discrete simplex" $\triangle^D_M := \{\mathbf{x} \in [0...M]^D;\ \sum_{d=1}^{D} x_d = M\}$. Any element $\mathbf{x} \in \triangle^D_M$ can be represented by a unique point $\widetilde{\mathbf{x}} \in \{\pm 1\}^{D \times M}$ defined as follows:

$$\text{for all } (d, m) \in [1...D] \times [1...M], \qquad \widetilde{x}_{(d,m)} \quad := \quad \begin{cases} 1 & \text{if} \quad x_d \geq m; \\ -1 & \text{if} \quad x_d < m. \end{cases} \tag{12}$$

Let $\mathcal{X}^{\triangle}_{D,M} := \{\widetilde{\mathbf{x}};\ \mathbf{x} \in \triangle^D_M\} \subset \{\pm 1\}^{D \times M}$. Judgement aggregation over $\mathcal{X}^{\triangle}_{D,M}$ represents the allocation of a budget of M dollars towards D claimants by voting "yes" or "no" to propositions of the form "x_d should be at least m dollars" for each $d \in [1...D]$ and $m \in [1...M]$; see Lindner et al. (2010).

Proposition 6.6 *$\mathcal{X}^{\triangle}_{D,M}$ is proximal, and thus, supermajority determinate.*

Roughly speaking, the proof strategy for Theorem 6.4(a) is this: if $\mathbf{x} \, \mathrm{I}_{\mathcal{X}} \, \mathbf{y}$, and $d(\mathbf{x}, \mathbf{y}) \leq 2$, then $\mathbf{x}$ and $\mathbf{y}$ are always $\underset{\mu}{\trianglerighteq}$-comparable. Thus, if $\mathcal{X}$ is proximal, then it is possible to compute $\mathrm{SME}\,(\mathcal{X}, \mu)$ through a process of gradient ascent, which converges to the set of globally $\underset{\mu}{\trianglerighteq}$-maximal elements. For example, in the case of $\mathcal{X}^{\mathrm{com}}_{I,J}$ and $\mathcal{X}^{\triangle}_{D,M}$, we can always move from one committee (or allocation) to a $\underset{\mu}{\trianglerighteq}$-superior committee (allocation) by exchanging a single candidate (claimant-dollar) favoured by a smaller majority for a candidate (claimant-dollar) favoured by a larger majority. Iterating this process yields a supermajority efficient committee (allocation). Moreover, any two committees (allocations) reached by this process must be $\trianglerighteq_{\mu}$-comparable. Hence $\mathcal{X}$ must be supermajority-determinate.

Counterexamples to Proximality. Proximality is clearly a rather restrictive condition, which many thick judgement spaces fail to satisfy. Hence, supermajority efficiency is often not a decisive criterion in judgement aggregation, and it will matter which additive majority rule is adopted.

For example, we say $\mathcal{X}$ is a ***McGarvey*** space if $\mathbf{0} \in \text{int}(\mathcal{C})$.[20] In this case, *every* edge of the polytope $\mathcal{C}$ is internal. Any McGarvey space is thick (because $\text{int}(\mathcal{C}) \neq \emptyset$ by definition), so Theorem 6.4(b) says that $\mathcal{X}$ is supermajority determinate if and only if *all* edges of the polytope $\mathcal{C}$ connect vertices of distance 1 or 2 —a condition met by few polytopes, hence by few McGarvey spaces. Among the rare examples are the hypercubes $\{\pm 1\}^{\mathcal{K}}$ among median spaces, and the committee spaces $\mathcal{X}_{I,J}^{\text{com}}$ with $I < \frac{K}{2} < J$. In contrast, important examples of *non*-proximal McGarvey spaces are the spaces of linear orders and equivalence relations, according to our next result.

Proposition 6.7 *If* $|\mathcal{A}| \geq 4$*, then*

- **(a)** $\mathcal{X}_{\mathcal{A}}^{\text{pr}}$ *is not proximal, hence not supermajority determinate.*
- **(b)** $\mathcal{X}_{\mathcal{A}}^{\text{eq}}$ *is not proximal, hence not supermajority determinate.*

Part (b) can be verified by applying the following simple combinatorial criterion for non-proximality. Two elements $\mathbf{x}, \mathbf{z} \in \mathcal{X}$ are ***adjacent*** if no other element of $\mathcal{X}$ is between them.[21] Evidently, two adjacent elements must form an internal edge (although the converse need not hold).[22] Thus, Theorem 6.4(b) has the following consequence.

Corollary 6.8 *If* $\mathcal{X}$ *is thick, and there exist any adjacent vertices* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ *with* $d(\mathbf{x}, \mathbf{y}) \geq 3$*, then* $\mathcal{X}$ *is not proximal, hence not supermajority determinate.*

To apply this corollary in the case of equivalence relations, let $\mathbf{x}=(1, 1, ..., 1)$ representing the relation where all objects are equivalent. Then $d(\mathbf{x}, \mathbf{y}) \geq |\mathcal{A}| - 1$ for all $\mathbf{y} \neq \mathbf{x}$. Thus $\mathcal{X}_{\mathcal{A}}^{\text{eq}}$ is non-proximal whenever $|\mathcal{A}| \geq 4$; this proves Proposition 6.7(b).

By contrast, Corollary 6.8 is not sufficient to establish Proposition 6.7(a). All adjacent vertices in $\mathcal{X}_{\mathcal{A}}^{\text{pr}}$ have distance 1. Instead, we prove Proposition 6.7(a) in Appendix E by constructing a (non-adjacent) internal edge of length 3.

Supermajority Determinacy without Proximality. Finally, to explain why the "thick" hypothesis is required in Theorem 6.4(b), we will now construct an example of a supermajority determinate space which is *not* proximal. Let $\mathcal{S}$ be an abstract finite set with cardinality $S := |\mathcal{S}| \geq 4$. Let m be a natural number with $1 \leq m \leq S - 1$, and let $\mathcal{K} := \{\mathcal{T} \subset \mathcal{S};\ |\mathcal{T}| = m\}$; thus, $|\mathcal{K}| = \binom{S}{m}$. For any $s \in \mathcal{S}$, define $\mathbf{x}^s \in \{\pm 1\}^{\mathcal{K}}$ as follows: for all $k \in \mathcal{K}$, set $x_k^s := 1$ if $s \in k$, while $x_k^s := -1$ if $s \notin k$ (recall that k is a subset of $\mathcal{S}$).

[20] In other words, $\mathcal{X}$ is McGarvey space if, for any every view $\mathbf{x} \in \{\pm 1\}^{\mathcal{K}}$, we have $\mathbf{x} = \text{Maj}(\mu)$ for some profile $\mu \in \Delta(\mathcal{X})$. Nehring and Pivato (2011) provides many examples of such spaces.

[21] Another element $\mathbf{y} \in \mathcal{X}$ is ***between*** $\mathbf{x}$ and $\mathbf{z}$ if, for all $k \in \mathcal{K}$, we have $(x_k = z_k) \Longrightarrow (x_k = y_k = z_k)$.

[22] *Proof.* If $\mathbf{c} := (\mathbf{x}+\mathbf{z})/2$, then $\mathbf{c} \in \mathcal{C}$, and $\text{Median}(\mathcal{X}, \mathbf{c}) = \{\text{all elements of } \mathcal{X} \text{ between } \mathbf{x} \text{ and } \mathbf{z}\}$. Thus, if $\mathbf{x}$ is adjacent to $\mathbf{z}$, then $\text{Median}(\mathcal{X}, \mathbf{c}) = \{\mathbf{x}, \mathbf{z}\}$, so $\mathbf{c}$ supports the edge between $\mathbf{x}$ and $\mathbf{z}$.

Let $\mathcal{X} := \{\mathbf{x}^s;\ s \in \mathcal{S}\}$; then $\mathcal{X}$ is supermajority determinate. Indeed, for any $\mu \in \Delta(\mathcal{X})$, the set $\mathrm{SME}(\mathcal{X}, \mu)$ corresponds to the outcome of the *plurality rule*, namely $\underset{\mathbf{x}\in\mathcal{X}}{\operatorname{argmax}}\, \mu(\mathbf{x})$. (See Proposition E.2 in Appendix E.)

On the other hand, $\mathcal{X}$ is thick if and only if $m = 1$ or $m = S - 1$. In these cases, $\mathcal{X}$ is easily verified to be proximal. However, if $S \geq 4$ and $1 < m < S - 1$, then $\mathcal{X}$ is *not* proximal. Indeed, for any distinct $s, t \in \mathcal{S}$, we have

$$d(\mathbf{x}^s, \mathbf{x}^t) \;=\; \#\{k \in \mathcal{K}\ ;\ s \in k\ \&\ t \notin k\} + \#\{k \in \mathcal{K}\ ;\ s \notin k\ \&\ t \in k\} \;=\; 2 \cdot \binom{S-2}{m-1} \;>\; 2,$$

Extending this example, let $\mathcal{K} := 2^{\mathcal{S}}$, to be interpreted as the "universal aggregation space". Then define $\mathbf{x}^s$ as above, for any $s \in \mathcal{S}$, and let $\mathcal{X}^{\mathrm{univ}} := \{\mathbf{x}^s : s \in \mathcal{S}\}$. Then again $\mathrm{SME}\left(\mathcal{X}^{\mathrm{univ}}, \mu\right)$ is the outcome of the plurality rule, so $\mathcal{X}^{\mathrm{univ}}$ is supermajority determinate. It is not proximal, since, for any distinct $s, t \in \mathcal{S}$, $d(\mathbf{x}^s, \mathbf{x}^t) = 2^{S-1}$. Conceptually, the elements of $\mathcal{X}^{\mathrm{univ}}$ can be viewed as (atomic) "worlds", and the elements of $\mathcal{K}$ as corresponding to *all* logically conceivable "propositions" (equated here with *sets* of possible worlds). $\mathcal{X}^{\mathrm{univ}}$ can be viewed as a universal judgement space in that no conceivable proposition is ruled out. If that lack of distinction is viewed as a limiting/default case, then, by Proposition E.2, plurality rule is supported as the "default" majority rule.[23] Of course, it is very special (and degenerate) to treat all conceivable propositions on par.

Conclusion

In this paper, we have introduced supermajority efficiency as a fundamental criterion of majoritarianism in judgement aggregation. In an interesting class of judgement spaces, supermajority efficiency alone determines the outcome. In other spaces, there exist profiles for which this criterion is indeterminate. This indeterminacy can be resolved by application of additive majority rules . We have identified the additive majority rules as the only rules that are supermajority efficient and "decomposable".

Are Supermajority Efficiency and Decomposition exhaustive, or are there additional normative considerations to further pin down the meaning of majoritarianism as equal representation? If so, how do such considerations constrain the selection of the particular additive majority rule F_ϕ?

We believe that there are such criteria, but they do not all pull in one direction; substantial work remains to be done. A central place in these investigations will surely be played by the simplest such rule, the median rule, where the support of a view is measured simply as the overall sum of votes across all issues. A normative, axiomatic foundation of this rule is provided in the companion paper (Nehring and Pivato, 2012b). A broader look is undertaken in further work in progress (Nehring and Pivato, 2012a).

[23] "Universality" of $\mathcal{X}^{\mathrm{univ}}$ holds also in the sense that any clone-free judgement space can be viewed as a coarsening of $\mathcal{X}^{\mathrm{univ}}$ as follows. A space $\mathcal{X}$ is *clone-free* if, for all $k, \ell \in \mathcal{K}$, $x_k \neq x_\ell$ for some $\mathbf{x}\in\mathcal{X}$. For any clone-free judgement space $\mathcal{X}$, let $\mathcal{X}^{\mathrm{univ}}$ be the universal space associated with $\mathcal{S} = \mathcal{X}$. Then there is a subset $\mathcal{L} \subseteq \mathcal{K} := 2^{\mathcal{X}}$ such that $\mathcal{X}$ is isomorphic to $\{y_{\mathcal{L}} : \mathbf{y}\in\mathcal{X}^{\mathrm{univ}}\}$.

Appendices

A Hyperreal fields

In classical mathematics, infinity is treated as a "nonarithmetic" object, and the word "infinitesimal" is merely a figure of speech. However, it is possible to construct a well-defined and well-behaved arithmetic of infinite and infinitesimal quantities, using a *hyperreal field.* Roughly speaking, this is is an arithmetic structure ${}^*\mathbb{R}$ which is obtained by adding a large collection of "infinite" and "infinitesimal" quantities to the set of real numbers. The important properties of ${}^*\mathbb{R}$ are as follows:

1. ${}^*\mathbb{R}$ is a *field*. This means that ${}^*\mathbb{R}$ has binary operations "+" and "·", and distinguished "identity" elements 0 and 1 such that:

   (a) $({}^*\mathbb{R}, +, 0)$ is an abelian group, and $({}^*\mathbb{R} \setminus \{0\}, \cdot, 1)$ is an abelian group.

   (b) For all $r, s, t \in {}^*\mathbb{R}$, we have $r \cdot (s + t) = r \cdot s + r \cdot t$.

2. There is an "exponentiation" operation, which behaves as one would expect. In particular, for all $r, s, t \in {}^*\mathbb{R}$, we have $r^0 = 1$, $r^1 = r$, $r^{-1} = 1/r$, $r^{s+t} = r^s \cdot r^t$, and $(r \cdot s)^t = r^t \cdot s^t$.

3. ${}^*\mathbb{R}$ has a linear order relation $>$ (i.e. transitive, complete, and antisymmetric).

4. For any $r, s \in {}^*\mathbb{R}$, if $r > 0$, then $r + s > s$. If $r > 1$, and $s > 0$, then $r \cdot s > s$.

5. $\mathbb{R} \subset {}^*\mathbb{R}$, and the arithmetic operations and order relation on ${}^*\mathbb{R}$ extend the standard arithmetic and ordering of $\mathbb{R}$.

The field ${}^*\mathbb{R}$ inherits many of the properties of $\mathbb{R}$, but not all. For example, because it contains infinite quantities, ${}^*\mathbb{R}$ violates the *Archimedean* property of the real numbers. Likewise, because ${}^*\mathbb{R}$ contains infinitesimals, it is not *order-complete*: in general, subsets of ${}^*\mathbb{R}$ do not have well-defined suprema and infima. Thus, much of the machinery of classical real analysis breaks down in ${}^*\mathbb{R}$. The order topology of ${}^*\mathbb{R}$ is not well-behaved; there are no nontrivial continuous functions from $\mathbb{R}$ into ${}^*\mathbb{R}$. The study of hyperreal fields is the starting point of *nonstandard analysis*; see Anderson (1991) or Goldblatt (1998) for more information.

Ultrapower construction. The properties listed above are sufficient for a casual user of ${}^*\mathbb{R}$. However, we will now also provide a formal construction of ${}^*\mathbb{R}$, that is central to the proof of the main result of the paper, Theorem 4.2. Let $\mathcal{I}$ be any infinite indexing set. A *free filter* on $\mathcal{I}$ is a collection $\mathfrak{F}$ of subsets of $\mathcal{I}$ satisfying the following axioms:

(F0) No finite subset of $\mathcal{I}$ is an element of $\mathfrak{F}$. (In particular, $\emptyset \notin \mathfrak{F}$.)

(F1) If $\mathcal{E}, \mathcal{F} \in \mathfrak{F}$, then $\mathcal{E} \cap \mathcal{F} \in \mathfrak{F}$.

(F2) For any $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{E} \subseteq \mathcal{I}$, if $\mathcal{F} \subseteq \mathcal{E}$, then $\mathcal{E} \in \mathfrak{F}$ also.

For any $\mathcal{E} \subseteq \mathcal{I}$, axioms (F0) and (F1) together imply that at most one of $\mathcal{E}$ or $\mathcal{E}^{\complement} := \mathcal{I} \setminus \mathcal{E}$ can be in $\mathfrak{F}$. A *free ultrafilter* is filter $\mathfrak{F}$ which also satisfies:

(UF) For any $\mathcal{E} \subseteq \mathcal{I}$, either $\mathcal{E} \in \mathfrak{F}$ or $\mathcal{E}^{\complement} \in \mathfrak{F}$.

Equivalently, a free ultrafilter is a "maximal" free filter: it is not a proper subset of any other free filter. Heuristically, elements of $\mathfrak{F}$ are "large" subsets of $\mathcal{I}$: if $\mathcal{F} \in \mathfrak{F}$ and a certain statement holds for all $i \in \mathcal{F}$, then this statement holds for "almost all" element of $\mathcal{I}$. In particular, axioms (F0) and (UF) imply that $\mathcal{I} \in \mathfrak{F}$.

Ultrafilter lemma. *Every free filter $\mathfrak{F}$ is contained in some free ultrafilter.*

Proof sketch. Consider the set of all free filters containing $\mathfrak{F}$; apply Zorn's Lemma to get a maximal element of this set. □

Let $\mathfrak{F}$ be a free ultrafilter on $\mathcal{I}$, and let $\mathbb{R}^{\mathcal{I}}$ be the space of all functions $r : \mathcal{I} \longrightarrow \mathbb{R}$. For all $r, s \in \mathbb{R}^{\mathcal{I}}$, define $r \underset{\mathfrak{F}}{\sim} s$ if the set $\{i \in \mathcal{I};\, r(i) = s(i)\}$ is an element of $\mathfrak{F}$. Let ${}^*\mathbb{R} := \mathbb{R}^{\mathcal{I}}/(\underset{\mathfrak{F}}{\sim})$. For any $r \in \mathbb{R}^{\mathcal{I}}$, let ${}^*r$ denote the equivalence class of r in ${}^*\mathbb{R}$. For any $r, s \in \mathbb{R}^{\mathcal{I}}$, we set

$$\left({}^*r > {}^*s\right) \iff \left(\text{the set } \{i \in \mathcal{I};\, r(i) > s(i)\} \text{ is an element of } \mathfrak{F}\right). \tag{A1}$$

This defines a linear order ">" on ${}^*\mathbb{R}$. Define the elements $r + s$, $r \cdot s$, r/s, and r^s in $\mathbb{R}^{\mathcal{I}}$ by: $(r+s)_i := r_i + s_i$, $(r \cdot s)_i := r_i \cdot s_i$, $(r/s)_i := r_i/s_i$, and $(r^s)_i := r_i^{s_i}$, for all $i \in \mathcal{I}$. Then, define ${}^*r + {}^*s := {}^*(r+s)$, ${}^*r \cdot {}^*s := {}^*(r \cdot s)$, ${}^*r/{}^*s := {}^*(r/s)$, and ${}^*r^{{}^*s} := {}^*(r^s)$. We can embed $\mathbb{R}$ into ${}^*\mathbb{R}$ by mapping any $r \in \mathbb{R}$ to the element ${}^*\overline{r}$ in ${}^*\mathbb{R}$, where $\overline{r} := (r, r, r, \ldots) \in \mathbb{R}^{\mathcal{I}}$. Then $({}^*\mathbb{R}, +, \cdot, >)$ is called an *ultrapower* of $\mathbb{R}$; it is an ultrareal field in the sense defined above.

B Proofs from Section 3

Proof of Proposition 3.2. (a) Define $\widehat{\phi}(r) := \phi(r) - \phi(-r)$ for all $r \in [-1, 1]$. Then $\widehat{\phi}$ is odd. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Recall that $\mathcal{K}_\pm(\mathbf{x}, \mathbf{y}) := \{k \in \mathcal{K} \,;\, x_k \neq y_k\}$. Then

$$\begin{aligned}
\sum_{k\in\mathcal{K}} \phi(x_k\widetilde{\mu}_k) - \sum_{k\in\mathcal{K}} \phi(y_k\widetilde{\mu}_k) &= \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} \Big(\phi(x_k\widetilde{\mu}_k) - \phi(y_k\widetilde{\mu}_k)\Big) \\
\underset{(*)}{=} \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} \Big(\phi(x_k\widetilde{\mu}_k) - \phi(-x_k\widetilde{\mu}_k)\Big) &\underset{(\dagger)}{=} \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} \widehat{\phi}(x_k\widetilde{\mu}_k) \\
\underset{(\ddagger)}{=} \frac{1}{2} \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} \Big(\widehat{\phi}(x_k\widetilde{\mu}_k) - \widehat{\phi}(-x_k\widetilde{\mu}_k)\Big) &\underset{(*)}{=} \frac{1}{2} \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} \Big(\widehat{\phi}(x_k\widetilde{\mu}_k) - \widehat{\phi}(y_k\widetilde{\mu}_k)\Big) \\
= \frac{1}{2} \left(\sum_{k\in\mathcal{K}} \widehat{\phi}(x_k\widetilde{\mu}_k) - \sum_{k\in\mathcal{K}} \widehat{\phi}(y_k\widetilde{\mu}_k)\right). &
\end{aligned} \tag{B1}$$

Here $(*)$ is because $y_k = -x_k$ for all $k \in \mathcal{K}_\pm(\mathbf{x}, \mathbf{y})$. Meanwhile (†) is simply the definition of $\widehat{\phi}$, and (‡) is because $\widehat{\phi}$ is odd, so $\widehat{\phi}(r) - \widehat{\phi}(-r) = 2\widehat{\phi}(r)$.

Equation (B1) implies that

$$\left(\sum_{k\in\mathcal{K}} \phi(x_k\widetilde{\mu}_k) \geq \sum_{k\in\mathcal{K}} \phi(y_k\widetilde{\mu}_k)\right) \iff \left(\sum_{k\in\mathcal{K}} \widehat{\phi}(x_k\widetilde{\mu}_k) \geq \sum_{k\in\mathcal{K}} \widehat{\phi}(y_k\widetilde{\mu}_k)\right).$$

Thus, $F_\phi(\mathcal{X}, \mu) = F_{\widehat{\phi}}(\mathcal{X}, \mu)$. □

Proof of Corollary 3.3. (a) The second equality follows immediately from the first, because $x_k\widetilde{\mu}_k > 0$ if and only if $k \in \mathcal{M}(\mathbf{x}, \mu)$. To see the first equality, define $\psi(r) := \phi(r_+)$ for all $r \in [-1, 1]$. Then $\phi(r) = \psi(r) - \psi(-r)$ for all $r \in [-1, 1]$ (because ϕ is odd). Thus, in the notation of Proposition 3.2, we have $\phi = \widetilde{\psi}$. Thus, $F_\phi = F_\psi$. (Strictly speaking, ψ is only non-decreasing, rather than strictly increasing. But the proof of Proposition 3.2 still works.)

(b) If ϕ is odd, then for any $\mu \in \Delta(\mathcal{X})$, $\mathbf{x} \in \mathcal{X}$, and $k \in \mathcal{K}$, we have $\phi(x_k\widetilde{\mu}_k) = x_k\phi(\widetilde{\mu}_k)$ (because $x_k = \pm 1$). Thus, $\sum_{k\in\mathcal{K}} \phi(x_k \cdot \widetilde{\mu}_k) = \sum_{k\in\mathcal{K}} x_k\phi(\widetilde{\mu}_k) = \mathbf{x} \bullet \phi(\widetilde{\mu})$. Thus, $\operatorname{argmax}_{\mathbf{x}\in\mathcal{X}} \left(\sum_{k\in\mathcal{K}} \phi(x_k\, \widetilde{\mu}_k)\right) = \operatorname{argmax}_{\mathbf{x}\in\mathcal{X}} \mathbf{x} \bullet \phi(\widetilde{\mu})$. □

Lemma B.1 *Let $\mathcal{X}$ be a judgement space, and let $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be a judgement aggregation rule. If F is monotone, then F is generically single-valued on $\Delta(\mathcal{X})$.*

Proof: Define $\Delta^*(\mathcal{X}) := \{\mu \in \Delta(\mathcal{X});\ \mu(\mathbf{x}) > 0 \text{ for all } \mathbf{x} \in \mathcal{X}\}$ (a dense subset of $\Delta(\mathcal{X})$). For any $\mu \in \Delta^*(\mathcal{X})$ and any $\mathbf{y} \in F(\mu)$, define $\mathcal{O}_\mathbf{y}(\mu) := \{\mu' \in \Delta(\mathcal{X});\ \mu'(\mathbf{y}) > \mu(\mathbf{y}) \text{ while } \mu'(\mathbf{x}) < \mu(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathcal{X} \setminus \{\mathbf{y}\}\}$. Then $\mathcal{O}_\mathbf{y}(\mu) \neq \emptyset$ (because $\mu \in \Delta^*(\mathcal{X})$), and $\mathcal{O}_\mathbf{y}(\mu)$ is an open subset of $\Delta(\mathcal{X})$, and μ is a cluster point of $\mathcal{O}_\mathbf{x}(\mu)$. Furthermore, $F(\mu') = \{\mathbf{y}\}$ for all $\mu' \in \mathcal{O}_\mathbf{y}(\mu)$ (because F is monotone). Thus, F is single-valued on $\mathcal{O}_\mathbf{y}(\mu)$.

Next, for any $\mu \in \Delta^*(\mathcal{X})$, define $\mathcal{O}(\mu) := \bigcup_{\mathbf{x}\in F(\mu)} \mathcal{O}_\mathbf{x}(\mu)$. Then $\mathcal{O}(\mu)$ is a nonempty open subset of $\Delta(\mathcal{X})$, and μ is a cluster point of $\mathcal{O}(\mu)$ (because it is a union of one or more nonempty open subsets of $\Delta(\mathcal{X})$ clustering at μ). Finally, let $\mathcal{O} := \bigcup_{\mu\in\Delta^*(\mathcal{X})} \mathcal{O}(\mu)$. Then $\mathcal{O}$ is an open subset of $\Delta(\mathcal{X})$, and F is single-valued on $\mathcal{O}$. Every element of $\Delta^*(\mathcal{X})$ is a cluster point of $\mathcal{O}$, and $\Delta^*(\mathcal{X})$ is dense in $\Delta(\mathcal{X})$; thus, $\mathcal{O}$ is dense in $\Delta(\mathcal{X})$. □

Proof of Proposition 3.4 (a) Let $\mu \in \Delta(\mathcal{X})$ and $\mathbf{y} \in F_\phi(\mu)$. Let $\mu' \in \Delta(\mathcal{X})$ be more supportive than μ of $\mathbf{y}$; we must show that $F_\phi(\mu') = \{\mathbf{y}\}$. By negating certain coordinates of $\mathcal{X}$ if necessary, we can assume without loss of generality that $\mathbf{y} = \mathbf{1}$. Recall that $\mathcal{X}(\mu) := \{\mathbf{x} \in \mathcal{X};\ \mu(\mathbf{x}) > 0\}$. Define $\mathcal{K}_1 := \{k \in \mathcal{K};\ \widetilde{\mu}_k = 1\} = \{k \in \mathcal{K};\ x_k = 1 \text{ for all } \mathbf{x} \in \mathcal{X}(\mu)\}$. Let $\mathcal{K}_0 := \mathcal{K} \setminus \mathcal{K}_1$.

Claim 1: (a) *For all $k \in \mathcal{K}$, we have $\widetilde{\mu}'_k \geq \widetilde{\mu}_k$.* (b) *If $k \in \mathcal{K}_0$ then $\widetilde{\mu}'_k > \widetilde{\mu}_k$.*

Proof: If μ' is more supportive than μ of $\mathbf{y}$, then $\mathcal{X}(\mu') \subseteq \mathcal{X}(\mu)$. Let $\mathcal{X}_- := \{\mathbf{x} \in \mathcal{X}(\mu);\ x_k = -1\}$ and $\mathcal{X}_+ := \{\mathbf{x} \in \mathcal{X}(\mu);\ \mathbf{x} \neq \mathbf{y} \text{ but } x_k = 1\}$. Then

$$
\begin{aligned}
(\widetilde{\mu}'_k - \widetilde{\mu}_k) &\underset{(\diamond)}{=} \sum_{\mathbf{x}\in\mathcal{X}(\mu)} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big)\, x_k \\
&= \Big(\mu'(\mathbf{y}) - \mu(\mathbf{y})\Big)\, y_k + \sum_{\mathbf{x}\in\mathcal{X}_-} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big)\, x_k + \sum_{\mathbf{x}\in\mathcal{X}_+} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big)\, x_k. \\
&\underset{(\dagger)}{=} \Big(\mu'(\mathbf{y}) - \mu(\mathbf{y})\Big) - \sum_{\mathbf{x}\in\mathcal{X}_-} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big) + \sum_{\mathbf{x}\in\mathcal{X}_+} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big) \\
&\underset{(*)}{\geq} \Big(\mu'(\mathbf{y}) - \mu(\mathbf{y})\Big) + \sum_{\mathbf{x}\in\mathcal{X}_-} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big) + \sum_{\mathbf{x}\in\mathcal{X}_+} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big) \\
&= \sum_{\mathbf{x}\in\mathcal{X}(\mu)} \Big(\mu'(\mathbf{x}) - \mu(\mathbf{x})\Big) \quad = \quad 1 - 1 \quad = \quad 0,
\end{aligned}
$$

and thus, $\widetilde{\mu}'_k \geq \widetilde{\mu}_k$. Here, $(\diamond)$ is by defining equation (2), $(\dagger)$ is by definition of $\mathcal{X}_-$ and $\mathcal{X}_+$, and $(*)$ is because $\mu'(\mathbf{x}) - \mu(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathcal{X}_-$ by the definition of μ'.

If $k \in \mathcal{K}_0$, then $\mathcal{X}_- \neq \emptyset$, so $(*)$ becomes a strict inequality, so $\widetilde{\mu}'_k > \widetilde{\mu}_k$. $\diamond$ Claim 1

Now, let $\mathbf{x} \in \mathcal{X} \setminus \{\mathbf{y}\}$; we will show that $\mathbf{x} \notin F(\mu')$. There are two cases.

Case 1. Suppose $\mathcal{K}_\pm(\mathbf{x},\mathbf{y}) \subseteq \mathcal{K}_1$. Then for all $k \in \mathcal{K}_\pm(\mathbf{x},\mathbf{y})$, we have $\widetilde{\mu}_k = y_k = 1$, while $x_k = -1$. Thus,

$$
(\mathbf{y} - \mathbf{x}) \bullet \phi(\widetilde{\mu}_k) \quad = \quad \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} (y_k - x_k)\, \phi(\widetilde{\mu}_k) \quad = \quad 2\, d(\mathbf{x},\mathbf{y}) \cdot \phi(1) \quad > \quad 0.
$$

Thus, $\mathbf{y} \bullet \phi(\widetilde{\mu}) > \mathbf{x} \bullet \phi(\widetilde{\mu})$, so $\mathbf{x} \notin F_\phi(\mu)$.

Case 2. Suppose $\mathcal{K}_\pm(\mathbf{x},\mathbf{y}) \not\subseteq \mathcal{K}_1$. For all $k \in \mathcal{K}_\pm(\mathbf{x},\mathbf{y})$, we have $y_k = 1$ and $x_k = -1$, while Claim 1(a) says $\widetilde{\mu}'_k \geq \widetilde{\mu}_k$. Furthermore, $\mathcal{K}_0 \cap \mathcal{K}_\pm(\mathbf{x},\mathbf{y}) \neq \emptyset$, and for any $k \in \mathcal{K}_0 \cap \mathcal{K}_\pm(\mathbf{x},\mathbf{y})$, Claim 1(b) says $\widetilde{\mu}'_k > \widetilde{\mu}_k$. Thus,

$$
(\mathbf{y}-\mathbf{x})\bullet\phi(\widetilde{\mu}') \quad = \quad \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} 2\phi(\widetilde{\mu}'_k) \quad \underset{(\diamond)}{>} \quad \sum_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} 2\phi(\widetilde{\mu}_k) \quad = \quad (\mathbf{y}-\mathbf{x})\bullet\phi(\widetilde{\mu}) \quad \underset{(*)}{\geq} \quad 0,
$$

and thus, $\mathbf{x} \notin F_\phi(\mu')$. Here, $(\diamond)$ is because ϕ is increasing, and $(*)$ is because $\mathbf{y} \in F_\phi(\mu)$.

We conclude that $\mathbf{x} \notin F_\phi(\mu')$ for all $\mathbf{x} \in \mathcal{X} \setminus \{\mathbf{y}\}$; thus, $F(\widetilde{\mu}') = \{\mathbf{y}\}$, as desired.

(b) follows immediately from part (a) and Lemma B.1. □

C Proofs from Section 4

Theorem 4.1 is actually a special case of a more general result, that is also needed for the proof of Theorem 4.2, and which we will state and prove in this appendix. A *finitely generated judgement monoid* is a collection $\mathfrak{X} = \{\mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2} \times \cdots \times \mathcal{X}_J^{m_J};\ m_1, \ldots, m_J \in \mathbb{N}\}$, where $J \in \mathbb{N}$, and $\mathcal{X}_1, \ldots, \mathcal{X}_J$ are some fixed judgement spaces. For example $\langle \mathcal{X} \rangle := \{\mathcal{X}^n;\ n \in \mathbb{N}\}$ is a finitely generated judgement monoid.

A *finitary weight function* is a function $\omega : \mathbb{N} \longrightarrow \mathbb{R}_+$ such that the set $\mathrm{supp}(\omega) := \{n \in \mathbb{N};\ \omega(n) > 0\}$ is finite. (If $|\mathrm{supp}(\omega)| = N$, then ω represents a weight function for N voters.) Let Ω be the set of all finitary weight functions. For any $\omega \in \Omega$, let $\Delta_\omega(\mathfrak{X})$ be the set of all profiles in $\Delta(\mathfrak{X})$ generated using ω, in the sense of eqn.(1). Let $\mathcal{Q}_\omega := \{\widetilde{\mu}_k;\ \mu \in \Delta_\omega(\mathfrak{X})$ and $k \in \mathcal{K}\}$. (For example, suppose $N \in \mathbb{N}$, and let $\omega(n) := 1$ for all $n \in [1 \ldots N]$ while $\omega(0) := 1$ for all $n > N$; then $\Delta_\omega\langle \mathcal{X} \rangle = \Delta_N \langle \mathcal{X} \rangle$, as defined prior to Theorem 4.1.)

Proposition C.1 *Let $\omega \in \Omega$, let $\mathfrak{X}$ be a finitely generated judgement monoid, and let F be a judgement aggregation rule on $\Delta_\omega(\mathfrak{X})$ which satisfies the Decomposition axiom.*

Then F is SME on $\Delta_\omega(\mathfrak{X})$ if and only if there is a gain function $\phi_\omega : \mathcal{Q}_\omega \longrightarrow \mathbb{R}$ such that $F(\mathcal{X}, \mu) \subseteq F_{\phi_\omega}(\mathcal{X}, \mu)$ for all $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta_\omega(\mathcal{X})$.

Furthermore, we can choose F_{ϕ_ω} to be "minimal" in the following sense: if H is any other additive majority rule which covers F on $\Delta_\omega(\mathfrak{X})$, then H also covers F_{ϕ_ω}. This minimal covering rule F_{ϕ_ω} is unique.[24]

Finally, any additive majority rule G on $\Delta_\omega(\mathfrak{X})$ has a real-valued representation.

The proof of Proposition C.1 requires seven preliminary lemmas. The first of these is actually one half of Theorem 4.2(a).

Lemma C.2 *Any additive majority rule is decomposable.*

Proof: Let ϕ be any gain function, and let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two judgement spaces. Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, let $\mu \in \Delta(\mathcal{X})$, and let μ_1 and μ_2 be the marginal projections of μ onto $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively.

We must show that $F_\phi(\mathcal{X}, \mu) = F_\phi(\mathcal{X}_1, \mu_1) \times F_\phi(\mathcal{X}_2, \mu_2)$. If $\mathbf{x} \in \mathcal{X}$, then $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ for some $\mathbf{x}_1 \in \mathcal{X}_1$ and $\mathbf{x}_2 \in \mathcal{X}_2$. Meanwhile, $\phi(\widetilde{\mu}) = (\phi(\widetilde{\mu}_1), \phi(\widetilde{\mu}_2))$. Thus, $\phi(\widetilde{\mu}) \bullet \mathbf{x} = \phi(\widetilde{\mu}_1) \bullet \mathbf{x}_1 + \phi(\widetilde{\mu}_2) \bullet \mathbf{x}_2$. Thus,

$$\Big(\mathbf{x} \in F_\phi(\mathcal{X}, \mu)\Big) \iff \Big(\phi(\widetilde{\mu}) \bullet \mathbf{x} \geq \phi(\widetilde{\mu}) \bullet \mathbf{y} \text{ for all } \mathbf{y} \in \mathcal{X}\Big)$$
$$\iff \Big(\phi(\widetilde{\mu}_1) \bullet \mathbf{x}_1 \geq \phi(\widetilde{\mu}_1) \bullet \mathbf{y}_1,\ \forall\ \mathbf{y}_1 \in \mathcal{X}_1 \text{ and } \phi(\widetilde{\mu}_2) \bullet \mathbf{x}_2 \geq \phi(\widetilde{\mu}_2) \bullet \mathbf{y}_2,\ \forall\ \mathbf{y}_2 \in \mathcal{X}_2\Big)$$
$$\iff \Big(\mathbf{x}_1 \in F_\phi(\mathcal{X}_1, \mu_1) \text{ and } \mathbf{x}_2 \in F_\phi(\mathcal{X}_2, \mu_2)\Big) \iff \Big(\mathbf{x} \in F_\phi(\mathcal{X}_1, \mu_1) \times F_\phi(\mathcal{X}_2, \mu_2)\Big),$$

as desired. □

The next lemma is needed to establish the last statement of Proposition C.1. The construction of the proof is also used in the proof of Proposition 4.4.

[24] However, the gain function ϕ_ω which defines F_{ϕ_ω} is not unique.

Lemma C.3 *Let* $(\mathbb{L}, +, >)$ *be a linearly ordered abelian group, and let* $\psi : [-1,1] \longrightarrow \mathbb{L}$ *be a gain function. Let* $\mathfrak{X}$ *be any finitely generated judgement monoid. For any* $\omega \in \Omega$*, there exists a real-valued gain function* $\phi_\omega : \mathcal{Q}_\omega \longrightarrow \mathbb{R}$ *such that* $F_{\phi_\omega}(\mathcal{X}, \mu) = F_\psi(\mathcal{X}, \mu)$ *for all* $\mathcal{X} \in \mathfrak{X}$ *and all* $\mu \in \Delta_\omega(\mathcal{X})$*.*

Proof: Suppose $\mathfrak{X}$ is generated by $\mathcal{X}_1, \ldots, \mathcal{X}_N$. Thus, any $\mathcal{X} \in \mathfrak{X}$ has the form $\mathcal{X} = \mathcal{X}_{n_1} \times \cdots \times \mathcal{X}_{n_J}$ for some $n_1, \ldots, n_J \in \{1, \ldots, N\}$. For any $\mu \in \Delta_\omega(\mathcal{X})$ and any $j \in [1 \ldots J]$, let $\mu^{(j)} \in \Delta_\omega(\mathcal{X}_{n_j})$ be the jth marginal of μ. Then for any gain function $\phi_\omega : \mathcal{Q}_\omega \longrightarrow \mathbb{R}$, the Decomposition property of the rules F_ψ and F_{ϕ_ω} (from Lemma C.2) implies that

$$F_\psi(\mathcal{X}, \mu) \;=\; \prod_{j=1}^{J} F_\psi(\mathcal{X}_{n_j}, \mu^{(j)}) \quad \text{and} \quad F_{\phi_\omega}(\mathcal{X}, \mu) \;=\; \prod_{j=1}^{J} F_{\phi_\omega}(\mathcal{X}_{n_j}, \mu^{(j)}) \tag{C1}$$

Thus, it suffices to construct ϕ_ω such that $F_{\phi_\omega}(\mathcal{X}_n, \mu) = F_\psi(\mathcal{X}_n, \mu)$ for all $\mu \in \Delta_\omega(\mathcal{X}_n)$ and all $n \in [1 \ldots N]$.

Hahn's Embedding Theorem says there is an order-preserving group isomorphism $\alpha : \mathbb{L} \longrightarrow \mathcal{L} \subset \mathbb{R}^{\mathcal{T}}$, where $\mathcal{T}$ is a (possibly infinite) linearly ordered set, and the group $\mathcal{L}$ is endowed with the $\mathcal{T}$-lexicographical order, with smaller t-coordinates given lexicographical priority over larger t coordinates. (In particular, this means that, for any distinct $\ell, \ell' \in \mathcal{L}$, the set $\{t \in \mathcal{T};\ \ell_t \neq \ell'_t\}$ has a minimal element.)[25]

Now, for any $n \in [1 \ldots N]$, any $\mu \in \Delta_\omega(\mathcal{X}_n)$, and any $\mathbf{x}, \mathbf{y} \in \mathcal{X}_n$, if $\mathbf{x} \bullet \psi(\mu) \neq \mathbf{y} \bullet \psi(\mu)$, then $\alpha[\mathbf{x} \bullet \psi(\mu)] \neq \alpha[\mathbf{y} \bullet \psi(\mu)]$ (because α is injective), and thus, there exists some $t^\mu_{\mathbf{x},\mathbf{y}} \in \mathcal{T}$ such that $\alpha[\mathbf{x} \bullet \psi(\mu)]_{t^\mu_{\mathbf{x},\mathbf{y}}} \neq \alpha[\mathbf{y} \bullet \psi(\mu)]_{t^\mu_{\mathbf{x},\mathbf{y}}}$, while $\alpha[\mathbf{x} \bullet \psi(\mu)]_t = \alpha[\mathbf{y} \bullet \psi(\mu)]_t$ for all $t < t^\mu_{\mathbf{x},\mathbf{y}}$. Since α is order-preserving and $\mathcal{L}$ is lexicographically ordered, this implies that

$$\Big(\mathbf{x} \bullet \psi(\mu) > \mathbf{y} \bullet \psi(\mu)\Big) \quad \Longleftrightarrow \quad \Big(\alpha[\mathbf{x} \bullet \psi(\mu)]_{t^\mu_{\mathbf{x},\mathbf{y}}} > \alpha[\mathbf{y} \bullet \psi(\mu)]_{t^\mu_{\mathbf{x},\mathbf{y}}}\Big). \tag{C2}$$

Let $\mathcal{T}' := \{t^\mu_{\mathbf{x},\mathbf{y}};\ n \in [1 \ldots N],\ \mu \in \Delta_\omega(\mathcal{X}_n), \text{ and } \mathbf{x}, \mathbf{y} \in \mathcal{X}_n\}$. Then $\mathcal{T}'$ is finite, because $\mathcal{X}_n$ and $\Delta_\omega(\mathcal{X}_n)$ are finite for all $n \in [1 \ldots N]$. Let $<'$ denote the linear order which $\mathcal{T}'$ inherits from $\mathcal{T}$. For any $t \in \mathcal{T}'$, let $|t| := \#\{t' \in \mathcal{T}';\ t' <' t\}$; thus, $|t| \in \mathbb{N}$, because $\mathcal{T}'$ is finite. For any $t_1, t_2 \in \mathcal{T}'$, clearly $(t_1 < t_2) \Leftrightarrow (|t_1| < |t_2|)$.

Since $\mathcal{L} \subset \mathbb{R}^{\mathcal{T}}$, any $\ell \in \mathcal{L}$ has the form $\ell = (\ell_t)_{t \in \mathcal{T}}$ where $\ell_t \in \mathbb{R}$ for all $t \in \mathcal{T}$. Thus, for any $\epsilon > 0$, we can define a group homomorphism $\beta_\epsilon : \mathcal{L} \longrightarrow \mathbb{R}$ by setting

$$\beta_\epsilon(\ell) \quad := \quad \sum_{t \in \mathcal{T}'} \ell_t\, \epsilon^{|t|}, \qquad \text{for all } \ell \in \mathcal{L}. \tag{C3}$$

Then define $\phi_\epsilon : \mathcal{Q}_\omega \longrightarrow \mathbb{R}$ by setting

$$\phi_\epsilon(q) \quad := \quad \beta_\epsilon(\alpha[\psi(q)]) \quad = \quad \sum_{t \in \mathcal{T}'} \alpha[\psi(q)]_t\, \epsilon^{|t|}, \qquad \text{for all } q \in \mathcal{Q}_\omega. \tag{C4}$$

[25]This technical remark is necessary because while $\mathcal{T}$ is linearly ordered, it may not be *well*-ordered, so the "lexicographical order" is not necessarily well-defined on all of $\mathbb{R}^{\mathcal{T}}$. But Hahn's Embedding Theorem ensures that it *is* well-defined on the sugroup $\mathcal{L}$. See e.g. Hausner and Wendel (1952), Clifford (1954), or Gravett (1956) for details.

For any $n \in [1 \ldots N]$, we have $\mathcal{X}_n \subseteq \{\pm 1\}^{\mathcal{K}_n}$ for some finite set $\mathcal{K}_n$. Thus, for any $\mu \in \Delta_\omega(\mathcal{X}_n)$, and $\mathbf{x}, \mathbf{y} \in \mathcal{X}_n$, we have

$$\begin{aligned}
\mathbf{x} \bullet \phi_\epsilon(\mu) - \mathbf{y} \bullet \phi_\epsilon(\mu) \; &\underset{(*)}{=} \; \sum_{k \in \mathcal{K}_n} x_k \, \beta_\epsilon\left(\alpha\left[\psi(\widetilde{\mu}_k)\right]\right) - \sum_{k \in \mathcal{K}_n} y_k \, \beta_\epsilon\left(\alpha\left[\psi(\widetilde{\mu}_k)\right]\right) \\
&\underset{(\dagger)}{=} \; \beta_\epsilon\left(\alpha\left[\sum_{k \in \mathcal{K}_n} x_k \, \psi(\widetilde{\mu}_k)\right]\right) - \beta_\epsilon\left(\alpha\left[\sum_{k \in \mathcal{K}_n} y_k \, \psi(\widetilde{\mu}_k)\right]\right) \\
&= \; \beta_\epsilon\left(\alpha\left[\mathbf{x} \bullet \psi(\mu)\right]\right) - \beta_\epsilon\left(\alpha\left[\mathbf{y} \bullet \psi(\mu)\right]\right) \\
&\underset{(\diamond)}{=} \; \left(\alpha\left[\mathbf{x} \bullet \psi(\mu)\right]_{t^\mu_{\mathbf{x},\mathbf{y}}} - \alpha\left[\mathbf{y} \bullet \psi(\mu)\right]_{t^\mu_{\mathbf{x},\mathbf{y}}}\right) \epsilon^{|t^\mu_{\mathbf{x},\mathbf{y}}|} + \begin{pmatrix}\text{a finite linear combination} \\ \text{of higher powers of } \epsilon\end{pmatrix}.
\end{aligned}$$

(Here, $(*)$ is by equation (C4) and the definition of $\mathbf{x} \bullet \phi_\epsilon(\mu)$. Next, $(\dagger)$ is because β_ϵ and α are both group homomorphisms, and $x_k, y_k \in \{\pm 1\}$ for all $k \in \mathcal{K}_n$. Finally, $(\diamond)$ is by equation (C3) and the definition of $t^\mu_{\mathbf{x},\mathbf{y}}$.) Thus, there exists some $\epsilon^\mu_{\mathbf{x},\mathbf{y}} > 0$ such that, for any $\epsilon \in (0, \epsilon^\mu_{\mathbf{x},\mathbf{y}}]$, statement (C2) entails

$$\Big(\mathbf{x} \bullet \psi(\mu) > \mathbf{y} \bullet \psi(\mu)\Big) \iff \Big(\mathbf{x} \bullet \phi_\epsilon(\mu) > \mathbf{y} \bullet \phi_\epsilon(\mu)\Big). \tag{C5}$$

Now define $\epsilon := \min\{\epsilon^\mu_{\mathbf{x},\mathbf{y}};\;\; \mathbf{x}, \mathbf{y} \in \mathcal{X}_n,\ \mu \in \Delta_\omega(\mathcal{X}_n), \text{ and } n \in [1 \ldots N]\}$. Then $\epsilon > 0$ because $\mathcal{X}_n$ and $\Delta_\omega(\mathcal{X}_n)$ are finite for all $n \in [1 \ldots N]$. Furthermore, for any $n \in [1 \ldots N]$, and any $\mu \in \Delta_\omega(\mathcal{X}_n)$, statement (C5) now holds for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}_n$. Thus, Corollary 3.3(b) implies that

$$F_{\phi_\epsilon}(\mu) = F_\psi(\mu), \quad \text{for all } n \in [1 \ldots N] \text{ and } \mu \in \Delta_\omega(\mathcal{X}_n). \tag{C6}$$

Now define $\phi_\omega := \phi_\epsilon$. Then combining statements (C1) and (C6), we deduce that $F_{\phi_\omega}(\mathcal{X}, \mu) = F_\psi(\mathcal{X}, \mu)$ for all $\mathcal{X} \in \mathfrak{X}$ and all $\mu \in \Delta_\omega(\mathcal{X})$, as desired. □

The next lemma is used in the proof of Proposition 5.2 as well as that of Proposition C.1; it is useful for separation arguments involving finite-dimensional polytopes. It strengthens a well-known fact about Pareto optimality; finite-dimensionality of the underlying vector space $\mathbb{R}^Q$ is crucial for its validity. Fix $Q \in \mathbb{N}$, and let $\mathbb{R}^Q_+ := \{\mathbf{r} \in \mathbb{R}^Q;\;\; r_q \geq 0, \text{ for all } q \in [1...Q]\}$ be the non-negative orthant in $\mathbb{R}^Q$.

Lemma C.4 *Let $Q \in \mathbb{N}$. Let $\mathcal{P} \subset \mathbb{R}^Q$ be a closed, convex polytope in which $\mathbf{0}$ is Pareto optimal* (i.e. $\mathcal{P} \cap \mathbb{R}^Q_+ = \{\mathbf{0}\}$). *Then there exists an all-positive vector $\mathbf{v} \in \mathbb{R}^Q_+$ such that $\mathbf{v} \bullet \mathbf{p} \leq 0$ for all $\mathbf{p} \in \mathcal{P}$.*

Proof: Let $\mathcal{T} := \mathcal{P} - \mathbb{R}^Q_+$; then $\mathcal{T}$ is also a closed, convex polytope in $\mathbb{R}^Q$.

Claim 1: $\mathcal{T} \cap \mathbb{R}^Q_+ = \{\mathbf{0}\}$.

Proof: (by contradiction) If $\mathbf{t} \in \mathcal{T}$, then $\mathbf{t} = \mathbf{p} - \mathbf{r}$ for some $\mathbf{p} \in \mathcal{P}$ and $\mathbf{r} \in \mathbb{R}^Q_+$. Thus $\mathbf{p} = \mathbf{t} + \mathbf{r}$. Thus, if $\mathbf{t} \in \mathbb{R}^Q_+ \setminus \{\mathbf{0}\}$, then $\mathbf{p} \in \mathbb{R}^Q_+ \setminus \{\mathbf{0}\}$ also, contradicting the Pareto optimality of $\mathbf{0}$. ◇ Claim 1

Thus, $\mathbf{0}$ lies on the boundary of the polytope $\mathcal{T}$, so it is contained in some face. Let $\mathcal{F}$ be the minimal-dimension face of $\mathcal{T}$ containing $\mathbf{0}$. Thus, if $\mathcal{S} \subseteq \mathbb{R}^Q$ is the linear subspace spanned by $\mathcal{F}$, then $\mathcal{F} = \mathcal{S} \cap \mathcal{T}$, and there exists some $\epsilon > 0$ such that $\mathcal{B}_\epsilon \cap \mathcal{S} \subset \mathcal{F}$, where $\mathcal{B}_\epsilon \subset \mathbb{R}^Q$ is the ϵ-ball around $\mathbf{0}$.

$\mathcal{F}$ is a face of $\mathcal{T}$, so there exists some $\mathbf{v} \in \mathbb{R}^{\mathcal{K}}$ such that $\mathcal{F} = \text{argmax}_{\mathbf{t}\in\mathcal{T}} (\mathbf{v} \bullet \mathbf{t})$. Clearly $\mathbf{v} \bullet \mathbf{0} = 0$; hence $\mathbf{v} \bullet \mathbf{f} = 0$ for all $\mathbf{f} \in \mathcal{F}$, while $\mathbf{v} \bullet \mathbf{t} < 0$ for all $\mathbf{t} \in \mathcal{T} \setminus \mathcal{F}$. Thus, if $\mathcal{V} \subset \mathbb{R}^Q$ is the hyperplane orthogonal to $\mathbf{v}$, then $\mathcal{F} = \mathcal{T} \cap \mathcal{V}$.

Claim 2: $\mathbf{v} \bullet \mathbf{r} > 0$ *for all* $\mathbf{r} \in \mathbb{R}^Q_+ \setminus \{\mathbf{0}\}$.

Proof: For any $\mathbf{r} \in \mathbb{R}^Q_+ \setminus \{\mathbf{0}\}$, we have $-\mathbf{r} \in \mathcal{T}$, so $\mathbf{v} \bullet (-\mathbf{r}) \leq 0$, so $\mathbf{v} \bullet \mathbf{r} \geq 0$. It remains to show that $\mathbf{v} \bullet \mathbf{r} \neq 0$. By contradiction, suppose $\mathbf{v} \bullet \mathbf{r} = 0$. Then $\mathbf{r} \in \mathcal{V}$. Thus, $-\mathbf{r} \in \mathcal{V}$. But $-\mathbf{r} \in \mathcal{T}$ also, so $-\mathbf{r} \in \mathcal{T} \cap \mathcal{V} = \mathcal{F} = \mathcal{T} \cap \mathcal{S}$. But $\mathcal{F}$ contains a relative neighbourhood around $\mathbf{0}$ in the subspace $\mathcal{S}$, so if $-\mathbf{r} \in \mathcal{F}$, then there exists some $\epsilon > 0$ such that $\epsilon\,\mathbf{r} \in \mathcal{F}$; hence $\epsilon\,\mathbf{r} \in \mathcal{T}$. But $\epsilon\,\mathbf{r} \in \mathbb{R}^Q_+ \setminus \{\mathbf{0}\}$, so this contradicts Claim 1. ◇ `Claim 2`

For any $q \in [1...Q]$, let $\mathbf{e}_q = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ be the qth unit vector. Then $\mathbf{e}_q \in \mathbb{R}^Q_+$. Thus, Claim 2 says that $\mathbf{v} \bullet \mathbf{e}_q > 0$. But this means that $v_q > 0$, as desired. □

Now, fix $\omega \in \Omega$. Let $\mathcal{Q} := \mathcal{Q}_\omega \cap [0, 1]$; then $\mathbb{R}^{\mathcal{Q}}$ is a finite-dimensional vector space (because $\mathcal{Q}_\omega \subset [-1, 1]$ is a finite set). Let $\mathcal{X}$ be a judgement space, and let $\mu \in \Delta(\mathcal{X})$. For any $\mathbf{x} \in \mathcal{X}$, we define $\mathbf{g}(\mathbf{x}, \mu) \in \mathbb{R}^{\mathcal{Q}}$ by setting

$$g_q(\mathbf{x}, \mu) \quad := \quad \frac{\gamma_{\mu,\mathbf{x}}(q)}{|\mathcal{K}|} \quad = \quad \frac{\#\{k \in \mathcal{K}\ ;\ x_k\, \widetilde{\mu}_k \geq q\}}{|\mathcal{K}|}, \qquad \text{for all } q \in \mathcal{Q}. \tag{C7}$$

We then define $\mathcal{D}(\mathbf{x}; \mathcal{X}, \mu) := \{\mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu);\ \ \mathbf{y} \in \mathcal{X}\}$, a subset of $\mathbb{R}^{\mathcal{Q}}$. If $F : \Delta_\omega(\mathcal{X}) \rightrightarrows \mathcal{X}$ is a judgement aggregation rule, then define

$$\mathcal{D}_\omega(F, \mathcal{X}) \quad := \quad \bigcup_{\mu\in\Delta_\omega(\mathcal{X})} \ \bigcup_{\mathbf{x}\in F(\mathcal{X},\mu)} \mathcal{D}(\mathbf{x}; \mathcal{X}, \mu) \quad \subseteq \quad \mathbb{R}^{\mathcal{Q}}.$$

Finally, let $\mathfrak{X}$ be a judgement monoid. If F is a judgement aggregation rule on $\Delta_\omega(\mathfrak{X})$, then define $\mathcal{D}_\omega(F, \mathfrak{X}) := \bigcup_{\mathcal{X}\in\mathfrak{X}} \mathcal{D}_\omega(F, \mathcal{X}) \ \subseteq \ \mathbb{R}^{\mathcal{Q}}$.

Lemma C.5 *F is SME on $\Delta_\omega(\mathfrak{X})$ if and only if $\mathcal{D}_\omega(F, \mathfrak{X}) \cap \mathbb{R}^{\mathcal{Q}}_+ = \{\mathbf{0}\}$.*

Proof: For any $\mathcal{X} \in \mathfrak{X}$, $\mu \in \Delta_\omega(\mathcal{X})$, and $\mathbf{x} \in \mathcal{X}$, we have

$$\begin{aligned}
&\Big(\mathbf{x} \in \text{SME}\,(\mathcal{X}, \mu)\Big) \\
\iff \quad &\Big(\nexists\ \mathbf{y} \in \mathcal{X} \text{ with } g_q(\mathbf{y}, \mu) \geq g_q(\mathbf{x}, \mu) \text{ for all } q \in \mathcal{Q}, \text{ and } \mathbf{g}(\mathbf{y}, \mu) \neq \mathbf{g}(\mathbf{x}, \mu)\Big) \\
\iff \quad &\Big((\mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu)) \notin \mathbb{R}^{\mathcal{Q}}_+ \setminus \{\mathbf{0}\} \text{ for all } \mathbf{y} \in \mathcal{X}\Big) \\
\iff \quad &\Big(\mathcal{D}(\mathbf{x}; \mathcal{X}, \mu) \cap \mathbb{R}^{\mathcal{Q}}_+ = \{\mathbf{0}\}\Big).
\end{aligned}$$

Thus, F is supermajority efficient on $\Delta_\omega(\mathfrak{X})$ if and only if $\mathcal{D}(\mathbf{x};\mathcal{X},\mu)\cap\mathbb{R}^{\mathcal{Q}}_+ = \{\mathbf{0}\}$ whenever $\mathbf{x}\in F(\mathcal{X},\mu)$ for some $(\mathcal{X},\mu)\in\Delta_\omega(\mathfrak{X})$. The claim follows. □

Let $\mathcal{P}_\omega(F,\mathfrak{X})$ be the closure of $\mathcal{D}_\omega(F,\mathfrak{X})$ in $\mathbb{R}^{\mathcal{Q}}$.

Lemma C.6 *Let $\mathfrak{X}$ be a judgement monoid. Let $\omega\in\Omega$.*

(a) *If F is a judgement aggregation rule on $\Delta_\omega(\mathfrak{X})$ which satisfies the axiom of Decomposition, then $\mathcal{P}_\omega(F,\mathfrak{X})$ is a closed, convex subset of $\mathbb{R}^{\mathcal{Q}}$.*

(b) *Also, if $\mathfrak{X}$ is finitely generated, then $\mathcal{P}_\omega(F,\mathfrak{X})$ is a convex polytope in $\mathbb{R}^{\mathcal{Q}}$.*

Proof: (a) For any $(\mathcal{X},\mu)\in\Delta_\omega(\mathfrak{X})$, define $\mathcal{D}(F,\mathcal{X},\mu) := \bigcup_{\mathbf{x}\in F(\mathcal{X},\mu)} \mathcal{D}(\mathbf{x};\mathcal{X},\mu)$. Thus,

$$\mathcal{D}_\omega(F,\mathcal{X}) \quad = \quad \bigcup_{\mu\in\Delta_\omega(\mathcal{X})} \mathcal{D}_\omega(F,\mathcal{X},\mu). \tag{C8}$$

Claim 1: *Let $\mathcal{X}_1,\mathcal{X}_2\in\mathfrak{X}$, and suppose $\mathcal{X}_1\subseteq\{\pm1\}^{\mathcal{K}_1}$ and $\mathcal{X}_2\subseteq\{\pm1\}^{\mathcal{K}_2}$ for some finite sets $\mathcal{K}_1,\mathcal{K}_2$. If $\mathcal{X}:=\mathcal{X}_1\times\mathcal{X}_2$, then $\mathcal{X}\subseteq\{\pm1\}^{\mathcal{K}}$, where $\mathcal{K}:=\mathcal{K}_1\sqcup\mathcal{K}_2$. Let $s_1:=|\mathcal{K}_1|/|\mathcal{K}|$ and $s_2:=|\mathcal{K}_2|/|\mathcal{K}|$* (so $s_1+s_2=1$).

(a) *For any $\mu_1\in\Delta_\omega(\mathcal{X}_1)$ and $\mu_2\in\Delta_\omega(\mathcal{X}_2)$, there exists a profile $\mu\in\Delta_\omega(\mathcal{X})$ such that $\mu^{(1)}=\mu_1$ and $\mu^{(2)}=\mu_2$.*

(b) *For any $\mathbf{x}=(\mathbf{x}^1,\mathbf{x}^2)\in\mathcal{X}$, we have $\mathbf{g}(\mathbf{x},\mu) \;=\; s_1\,\mathbf{g}(\mathbf{x}^1,\mu^{(1)})+s_2\,\mathbf{g}(\mathbf{x}^2,\mu^{(2)})$.*

(c) *$\mathcal{D}(F,\mathcal{X},\mu) \;=\; \{s_1\,\mathbf{d}^1+s_2\,\mathbf{d}^2;\ \mathbf{d}^1\in\mathcal{D}(F,\mathcal{X}_1,\mu^{(1)})$ and $\mathbf{d}^2\in\mathcal{D}(F,\mathcal{X}_2,\mu^{(2)})\}$.*

Proof: (a) Suppose $\mathrm{supp}(\omega):=[1\ldots N]$ for some $N\in\mathbb{N}$. Let $\mathbf{y}^1_1,\mathbf{y}^1_2,\ldots,\mathbf{y}^1_N\in\mathcal{X}_1$ be such that μ_1 is defined by applying equation (1) to this collection. Likewise, let $\mathbf{y}^2_1,\mathbf{y}^2_2,\ldots,\mathbf{y}^2_N\in\mathcal{X}_2$ be such that μ_2 is defined by applying equation (1) to this collection. Now, for all $n\in[1\ldots N]$, define $\mathbf{y}_n:=(\mathbf{y}^1_n,\mathbf{y}^2_n)\in\mathcal{X}$. Define $\mu\in\Delta(\mathcal{X})$ by applying equation (1) to the collection $\mathbf{y}_1,\ldots,\mathbf{y}_N\in\mathcal{X}$. For any $\mathbf{x}^1\in\mathcal{X}_1$, we have

$$\mu^{(1)}(\mathbf{x}^1) \underset{(*)}{=} \sum_{\mathbf{x}^2\in\mathcal{X}_2}\mu(\mathbf{x}^1,\mathbf{x}^2) \underset{(\dagger)}{=} \sum_{\mathbf{x}^2\in\mathcal{X}_2}\sum\left\{\omega(n)\ ;\ n\in[1\ldots N] \text{ and } (\mathbf{y}^1_n,\mathbf{y}^2_n)=(\mathbf{x}^1,\mathbf{x}^2)\right\}$$
$$= \sum\left\{\omega(n)\ ;\ n\in[1\ldots N] \text{ and } \mathbf{y}^1_n=\mathbf{x}^1\right\} \underset{(\dagger)}{=} \mu_1(\mathbf{x}^1).$$

Here $(*)$ is by equation (10), and both $(\dagger)$ are by equation (1). Thus, $\mu^{(1)}=\mu_1$. Likewise, $\mu^{(2)}=\mu_2$.

(b) Note that $\widetilde{\mu}=(\widetilde{\mu}^{(1)},\widetilde{\mu}^{(2)})\in\mathbb{R}^{\mathcal{K}}$. Thus, for both $n\in\{1,2\}$ and all $k\in\mathcal{K}_n$, we have $x_k\cdot\widetilde{\mu}_k=x^n_k\cdot\widetilde{\mu}^{(n)}_k$. Thus, for any $q\in(0,1]$, we have

$$\begin{aligned}
\{k\in\mathcal{K}\ ;\ x_k\cdot\widetilde{\mu}_k\geq q\} &= \left\{k\in\mathcal{K}_1\ ;\ x^1_k\cdot\widetilde{\mu}^{(1)}_k\geq q\right\}\sqcup\left\{k\in\mathcal{K}_2\ ;\ x^2_k\cdot\widetilde{\mu}^{(2)}_k\geq q\right\}.\\
\text{Thus,}\quad |\mathcal{K}|\cdot g_q(\mathbf{x},\mu) &= |\{k\in\mathcal{K}\ ;\ x_k\cdot\widetilde{\mu}_k\geq q\}|\\
&= \left|\left\{k\in\mathcal{K}_1\ ;\ x^1_k\cdot\widetilde{\mu}^{(1)}_k\geq q\right\}\right|+\left|\left\{k\in\mathcal{K}_2\ ;\ x^2_k\cdot\widetilde{\mu}^{(2)}_k\geq q\right\}\right|\\
&= |\mathcal{K}_1|\cdot g_q(\mathbf{x}^1,\mu^{(1)})+|\mathcal{K}_2|\cdot g_q(\mathbf{x}^2,\mu^{(2)}).
\end{aligned}$$

now divide both sides by $|\mathcal{K}|$ and substitute defining equation (C7) to prove part (b).

(c) Let $\mathbf{x} := (\mathbf{x}^1, \mathbf{x}^2)$ and $\mathbf{y} := (\mathbf{y}^1, \mathbf{y}^2)$ be elements of $\mathcal{X}$. Thus, $\mathbf{x}^1, \mathbf{y}^1 \in \mathcal{X}_1$ and $\mathbf{x}^2, \mathbf{y}^2 \in \mathcal{X}_2$. If $\mathbf{d} = \mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu)$, and $\mathbf{d}^n = \mathbf{g}(\mathbf{y}^n, \mu^{(n)}) - \mathbf{g}(\mathbf{x}^n, \mu^{(n)})$ for $n \in \{1, 2\}$, then part (b) implies that

$$\begin{aligned} \mathbf{d} &= \mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu) \qquad \text{(C9)} \\ &= s_1\, \mathbf{g}(\mathbf{y}^1, \mu^{(1)}) - s_1\, \mathbf{g}(\mathbf{x}^1, \mu^{(1)}) + s_2\, \mathbf{g}(\mathbf{y}^2, \mu^{(2)}) - s_2\, \mathbf{g}(\mathbf{x}^2, \mu^{(2)}) \;=\; s_1\, \mathbf{d}^1 + s_2\, \mathbf{d}^2. \end{aligned}$$

But for any $\mathbf{d} \in \mathbb{R}^{\mathcal{Q}}$, we have:

$$\begin{aligned} &\Big(\mathbf{d} \in \mathcal{D}(F, \mathcal{X}, \mu)\Big) \iff \Big(\exists\, \mathbf{x} \in F(\mathcal{X}, \mu) \text{ and } \mathbf{y} \in \mathcal{X} \text{ such that } \mathbf{d} = \mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu)\Big) \\ &\underset{(*)}{\iff} \left(\begin{array}{l} \text{There exist } \mathbf{x}^1 \in F(\mathcal{X}_1, \mu^{(1)}), \quad \mathbf{x}^2 \in F(\mathcal{X}_2, \mu^{(2)}), \quad \mathbf{y}^1 \in \mathcal{X}_1, \text{ and } \mathbf{y}^2 \in \mathcal{X}_2 \\ \text{such that } \mathbf{d} = \mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu), \text{ where } \mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) \text{ and } \mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2) \end{array}\right) \\ &\underset{(\dagger)}{\iff} \Big(\exists\, \mathbf{d}^1 \in \mathcal{D}(F, \mathcal{X}_1, \mu^{(1)}) \text{ and } \mathbf{d}^2 \in \mathcal{D}(F, \mathcal{X}_2, \mu^{(2)}) \text{ such that } \mathbf{d} = s_1 \mathbf{d}^1 + s_2 \mathbf{d}^2\Big). \end{aligned}$$

Here $(*)$ is because $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ and $F(\mathcal{X}, \mu) = F(\mathcal{X}_1, \mu^{(1)}) \times F(\mathcal{X}_2, \mu^{(2)})$ (because F is decomposable), while $(\dagger)$ is by equation (C9). ◇ `Claim 1`

For any $N \in \mathbb{N}$, let $\Delta_N := \left\{\mathbf{s} \in \mathbb{R}_+^N \,;\, \sum_{n=1}^N s_n = 1\right\}$ be the N-simplex.

Claim 2: *Let $N \geq 2$, and let $\mathbf{d}^1, \ldots, \mathbf{d}^N \in \mathcal{D}_\omega(F, \mathfrak{X})$. There exists a dense subset $\mathcal{S} \subseteq \Delta_N$ such that, for any $\mathbf{s} = (s_1, \ldots, s_N) \in \mathcal{S}$, we have* $\displaystyle\sum_{n=1}^N s_n \mathbf{d}^n \;\in\; \mathcal{D}_\omega(F, \mathfrak{X})$.

Proof: For all $n \in [1...N]$, there exists $(\mathcal{X}_n, \mu_n) \in \Delta_\omega(\mathfrak{X})$ such that $\mathbf{d}^n \in \mathcal{D}(F, \mathcal{X}_n, \mu_n)$. Let $K_n = |\mathcal{K}_n|$ for $n \in [1...N]$. Define

$$\mathcal{S} \quad := \quad \left\{\frac{(m_1 K_1,\ m_2 K_2,\ \ldots, m_N K_N)}{m_1 K_1 + \cdots + m_N K_N} \,;\, m_1, \ldots, m_N \in \mathbb{N}\right\}. \qquad \text{(C10)}$$

Then $\mathcal{S}$ is a dense subset of Δ_N.

Let $\mathbf{s} \in \mathcal{S}$ and let $m_1, \ldots, m_N$ be as in eqn.(C10). Define

$$\mathcal{X} \;:=\; \underbrace{\mathcal{X}_1 \times \cdots \times \mathcal{X}_1}_{m_1} \times \underbrace{\mathcal{X}_2 \times \cdots \times \mathcal{X}_2}_{m_2} \times \cdots\cdots \times \underbrace{\mathcal{X}_N \times \cdots \times \mathcal{X}_N}_{m_N}$$

Then $\mathcal{X} \in \mathfrak{X}$ (by definition of $\mathfrak{X}$). For all $n \in [2 \ldots N]$, define $M_n := m_1 + \cdots + m_{n-1}$. Inductive application of Claim 2(a) yields some $\mu \in \Delta_\omega(\mathcal{X})$ such that $\mu^{(j)} = \mu_1$ for all $j \in [1 \ldots m_1]$, and $\mu^{(j)} = \mu_2$ for all $j \in [m_1{+}1 \ldots m_1{+}m_2]$, and more generally, $\mu^{(j)} = \mu_n$ for all $n \in [2 \ldots N]$ and all $j \in [M_n{+}1 \ldots M_n{+}m_n]$. Next, inductive application of Claim 2(c) implies that $s_1 \mathbf{d}^1 + \cdots + s_N \mathbf{d}^N \in \mathcal{D}(F, \mathcal{X}, \mu)$. Thus, equation (C8) implies that $s_1 \mathbf{d}^1 + \cdots + s_N \mathbf{d}^N \in \mathcal{D}_\omega(F, \mathfrak{X})$. ◇ `Claim 2`

Now, $\mathcal{P}_\omega(\mathfrak{X}, F)$ is the closure of $\mathcal{D}_\omega(\mathfrak{X}, F)$. Thus, for any $\mathbf{d}^1, \ldots, \mathbf{d}^N \in \mathcal{D}_\omega(\mathfrak{X}, F)$ and any $(s_1, \ldots, s_N) \in \Delta_N$, Claim 2 implies that $s_1 \mathbf{d}^1 + \cdots + s_N \mathbf{d}^N \in \mathcal{P}_\omega(\mathfrak{X}, F)$. Thus $\mathcal{P}_\omega(\mathfrak{X}, F)$ contains the convex hull $\text{conv}[\mathcal{D}_\omega(\mathfrak{X}, F)]$. Thus, $\mathcal{D}_\omega(\mathfrak{X}, F) \subseteq \text{conv}[\mathcal{D}_\omega(\mathfrak{X}, F)] \subseteq \mathcal{P}_\omega(\mathfrak{X}, F) = \overline{\mathcal{D}_\omega(\mathfrak{X}, F)}$; hence $\mathcal{P}_\omega(\mathfrak{X}, F) = \overline{\text{conv}[\mathcal{D}_\omega(\mathfrak{X}, F)]}$, so $\mathcal{P}_\omega(\mathfrak{X}, F)$ is the closure of a convex set, and thus, itself convex.

(b) Let $\mathcal{X}_1, \ldots, \mathcal{X}_N$ be judgement spaces, and suppose $\mathfrak{X} := \langle \mathcal{X}_1, \ldots, \mathcal{X}_N \rangle$. Define

$$\mathcal{E} \quad := \quad \bigcup_{n=1}^{N} \bigcup_{\mu \in \Delta_\omega(\mathcal{X}_n)} \mathcal{D}(F, \mathcal{X}_n, \mu).$$

Then $\mathcal{E}$ is a finite set, because for each $n \in [1 \ldots N]$, the set $\Delta_\omega(\mathcal{X}_n)$ is finite (because ω is finitary), and for each $\mu \in \Delta_\omega(\mathcal{X}_n)$, the set $\mathcal{D}(F, \mathcal{X}_n, \mu)$ is finite (because $\mathcal{X}_n$ is finite). Thus, $\mathrm{conv}(\mathcal{E})$ is the convex hull of a finite set, and thus, a closed, convex polytope in $\mathbb{R}^{\mathcal{Q}}$.

For any $\mathcal{X} \in \mathfrak{X}$, if $\mathcal{X} = \mathcal{X}_1^{m_1} \times \mathcal{X}_2^{m_2} \times \cdots \times \mathcal{X}_N^{m_N}$, then Claim 1(c) implies that every element of $\mathcal{D}_\omega(F, \mathcal{X})$ is a rational convex combination of elements from $\mathcal{E}$. Thus, $\mathcal{D}_\omega(F, \mathcal{X}) \subseteq \mathrm{conv}(\mathcal{E})$. This holds for all $\mathcal{X} \in \mathfrak{X}$; thus, $\mathcal{D}_\omega(F, \mathfrak{X}) \subseteq \mathrm{conv}(\mathcal{E})$. Since $\mathrm{conv}(\mathcal{E})$ is closed, we deduce that $\mathrm{cl}\,(\mathcal{D}_\omega(F, \mathfrak{X})) \subseteq \mathrm{conv}(\mathcal{E})$.

Conversely, Claim 2 shows that every convex combination of elements from $\mathcal{E}$ lies in $\mathrm{cl}\,(\mathcal{D}_\omega(F, \mathfrak{X}))$. Thus, $\mathrm{conv}(\mathcal{E}) \subseteq \mathrm{cl}\,(\mathcal{D}_\omega(F, \mathfrak{X}))$.

Thus, $\mathrm{conv}(\mathcal{E}) = \mathrm{cl}\,(\mathcal{D}_\omega(F, \mathfrak{X})) = \mathcal{P}_\omega(F, \mathfrak{X})$. Thus, $\mathcal{P}_\omega(F, \mathfrak{X})$ is a convex polytope. □

Let Γ be the vector space of all real-valued functions on $(0, 1]$ of the form $\sum_{n=1}^{N} a_n \mathbf{1}_{(q_n, r_n]}$, where $N \in \mathbb{N}$, and $a_n \in \mathbb{R}$ and $0 \leq q_n < r_n \leq 1$ for all $n \in [1...N]$. Let Γ_+ be the set of all elements of Γ which are nonnegative everywhere. A *positive linear functional* on Γ is a linear function $P : \Gamma \longrightarrow {}^*\mathbb{R}$ such that $P(\gamma) \geq 0$ for all $\gamma \in \Gamma_+$. Let Γ_+^* be the set of all positive linear functionals on Γ. The next result is used in the proof of Lemma C.8 (below), which is necessary to prove Proposition C.1 and Theorem 4.2(a). It is also used to prove Theorems 6.3 and 6.4(a), as well as Propositions 5.2(a) and 6.2.

Lemma C.7 *Let Φ_{OND} be the set of odd, nondecreasing functions from $[-1, 1]$ into ${}^*\mathbb{R}$.*

(a) *There is a bijective correspondence between Φ_{OND} and Γ_+^* defined as follows. Given any $\phi \in \Phi_{\text{OND}}$, define $\phi^* : \Gamma \longrightarrow {}^*\mathbb{R}$ by first defining $\phi^*\left(\mathbf{1}_{(q,r]}\right) := \phi(r) - \phi(q)$ for all $q < r \in [0, 1]$, and then extending ϕ^* to all of Γ by linearity. Then $\phi^* \in \Gamma_+^*$.*

To invert this map, suppose $P \in \Gamma_+^*$. Define $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$ by first setting $\phi(q) := P(\mathbf{1}_{(0,q]})$ for all $q \in [0, 1]$, and then defining $\phi(q) := -\phi(-q)$ for all $q \in [-1, 0]$. Then $\phi \in \Phi_{\text{OND}}$, and $\phi^* = P$.

Let $\mathcal{X}$ be a judgement space, and let $\mu \in \Delta(\mathcal{X})$.

(b) *For all $\mathbf{x} \in \mathcal{X}$ and any $\phi \in \Phi_{\text{OND}}$, we have $\gamma_{\mu,\mathbf{x}} \in \Gamma_+$, and $\phi^*(\gamma_{\mathbf{x},\mu}) = \sum_{k \in \mathcal{M}(\mathbf{x},\mu)} \phi\,|\widetilde{\mu}_k|$.*

(c) *Thus, for any $\phi \in \Phi_{\text{OND}}$, we have $F_\phi(\mathcal{X}, \mu) \;=\; \underset{\mathbf{x} \in \mathcal{X}}{\mathrm{argmax}}\; \phi^*(\gamma_{\mu,\mathbf{x}})$.*

Proof: (a) Given $\phi \in \Phi_{\text{OND}}$, it is easy to check that $\phi^* \in \Gamma_+^*$.

Conversely, let $P \in \Gamma_+^*$ and define ϕ as above. By construction, ϕ is odd and $\phi(0) = 0$. To see that ϕ is nondecreasing, let $q < r \in (0, 1]$. Then

$$\phi(r) \;=\; P(\mathbf{1}_{(0,r]}) \;=\; P\left(\mathbf{1}_{(0,q]} + \mathbf{1}_{(q,r]}\right) \;=\; P(\mathbf{1}_{(0,q]}) + P(\mathbf{1}_{(q,r]}) \underset{(*)}{\geq} P(\mathbf{1}_{(0,q]}) \;=\; \phi(q). \tag{C11}$$

$(*)$ because $P(\mathbf{1}_{(q,r]}) \geq 0$ because P is positive, and $\mathbf{1}_{(q,r]} \in \Gamma_+$. Equation (C11) also shows that $P\left(\mathbf{1}_{(q,r]}\right) = \phi(r) - \phi(q)$ for $q < r \in (0, 1]$; thus $P = \phi^*$.

(b) For any $\mathbf{x} \in \mathcal{X}$, defining equation (5) can be rewritten: $\gamma_{\mathbf{x},\mu} = \sum_{k \in \mathcal{M}(\mathbf{x},\mu)} \mathbf{1}_{(0,|\widetilde{\mu}_k|]}$. Thus,

$$\phi^*(\gamma_{\mathbf{x},\mu}) = \sum_{k \in \mathcal{M}(\mathbf{x},\mu)} \phi^*(\mathbf{1}_{(0,|\widetilde{\mu}_k|]}) = \sum_{k \in \mathcal{M}(\mathbf{x},\mu)} \phi\,|\widetilde{\mu}_k|.$$

(c) follows from part (b) and Lemma 3.3. □

The next lemma is actually one component of Proposition 5.2(a) and Theorem 4.2(a). We prove it here because it is necessary for the proof of Proposition C.1.

Lemma C.8 *For any judgement problem $(\mathcal{X}, \mu)$ and any gain function $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$, we have $F_\phi(\mathcal{X}, \mu) \subseteq \text{SSME}\,(\mathcal{X}, \mu) \subseteq \text{SME}\,(\mathcal{X}, \mu)$.*

Proof: (by contrapositive) Let $\mathbf{y} \in \mathcal{X}$, and suppose $\mathbf{y} \notin \text{SSME}\,(\mathcal{X}, \mu)$; we will show that $\mathbf{y} \notin F_\phi(\mathcal{X}, \mu)$.

Define $\phi^* : \Gamma \longrightarrow {}^*\mathbb{R}$ as in Lemma C.7(a). For any $\rho \in \Delta_{\mathbb{Q}}(\mathcal{X})$, we have $\gamma_{\mu,\rho} \in \Gamma_+$, and

$$\phi^*\left(\gamma_{\mu,\rho}\right) \;\underset{(\diamond)}{=\!=}\; \phi^*\left(\sum_{\mathbf{x} \in \mathcal{X}} \rho(\mathbf{x}) \cdot \gamma_{\mu,\mathbf{x}}\right) \;=\; \sum_{\mathbf{x} \in \mathcal{X}} \rho(\mathbf{x}) \cdot \phi^*\left(\gamma_{\mu,\mathbf{x}}\right). \tag{C12}$$

where $(\diamond)$ is by eqn.(D1) defining $\gamma_{\mu,\rho}$.

If $\mathbf{y} \notin \text{SSME}\,(\mathcal{X}, \mu)$, then there exists $\rho \in \Delta_{\mathbb{Q}}(\mathcal{X})$ such that $\rho \underset{\mu}{\triangleright} \mathbf{y}$. Thus, $\gamma_{\mu,\rho} \geq \gamma_{\mu,\mathbf{y}}$, and there exist $q_1 < q_2 \in [0, 1]$ such that $\gamma_{\mu,\rho}(r) > \gamma_{\mu,\mathbf{y}}(r)$ for all $r \in [q_1, q_2]$. For any $\epsilon > 0$, let $\mathcal{R}_\epsilon := \{r \in [q_1, q_2]\ ;\ \gamma_{\mu,\rho}(r) - \gamma_{\mu,\mathbf{y}}(r) > \epsilon\}$. If ϵ is small enough, then there exist $r_1 < r_2 \in [q_1, q_2]$ such that $(r_1, r_2] \in \mathcal{R}_\epsilon$; thus, $\gamma_{\mu,\rho} - \gamma_{\mu,\mathbf{y}} \geq \epsilon \cdot \mathbf{1}_{(r_1,r_2]}$. Thus,

$$\begin{aligned}
\phi^*\left(\gamma_{\mu,\rho}\right) - \phi^*\left(\gamma_{\mu,\mathbf{y}}\right) \;&=\; \phi^*\left(\gamma_{\mu,\rho} - \gamma_{\mu,\mathbf{y}}\right) \;\underset{(*)}{\geq}\; \phi^*\left(\epsilon \mathbf{1}_{(r_1,r_2]}\right) \\
&=\; \epsilon \cdot \phi^*\left(\mathbf{1}_{(r_1,r_2]}\right) \;=\; \epsilon\left(\phi(r_2) - \phi(r_1)\right) \;\underset{(\dagger)}{>}\; 0.
\end{aligned} \tag{C13}$$

Here, $(*)$ is because ϕ^* is positive and $\gamma_{\mu,\rho} - \gamma_{\mu,\mathbf{y}} \geq \epsilon \cdot \mathbf{1}_{(r_1,r_2]}$, while $(\dagger)$ is because ϕ is strictly increasing. Combine (C12) and (C13) to get

$$\sum_{\mathbf{x} \in \mathcal{X}} \rho(\mathbf{x}) \cdot \phi^*\left(\gamma_{\mu,\mathbf{x}}\right) \quad > \quad \phi^*\left(\gamma_{\mu,\mathbf{y}}\right).$$

Thus, there is some $\mathbf{x} \in \mathcal{X}$ such that $\phi^*(\gamma_{\mu,\mathbf{x}}) > \phi^*(\gamma_{\mu,\mathbf{y}})$. Thus, $\mathbf{y} \notin F_\phi(\mathcal{X}, \mu)$, by Lemma C.7(c).

It is clear from their definitions that $\text{SSME}(\mathcal{X}, \mu) \subseteq \text{SME}(\mathcal{X}, \mu)$ (or see the proof of Proposition 5.1 below.) □

Proof of Proposition C.1. "$\Longleftarrow$" If ϕ is strictly increasing, then for all $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta_\omega(\mathcal{X})$, Lemma C.8 says $F_\phi(\mu) \subseteq \text{SME}(\mathcal{X}, \mu)$, and thus, $F(\mu) \subseteq \text{SME}(\mathcal{X}, \mu)$. Thus, F is supermajority efficient on $\Delta_\omega(\mathfrak{X})$.

"$\Longrightarrow$" Lemma C.6(b) says $\mathcal{P}_\omega(F, \mathfrak{X})$ is a closed, convex polytope in $\mathbb{R}^{\mathcal{Q}}$. If F is SME, then Lemma C.5 implies that $\mathcal{P}_\omega(F, \mathfrak{X}) \cap \mathbb{R}^{\mathcal{Q}}_+ = \{\mathbf{0}\}$. Thus, Lemma C.4 yields an all-positive vector $\mathbf{v} \in \mathbb{R}^{\mathcal{Q}}_+$ such that $\mathbf{v} \bullet \mathbf{p} \leq 0$ for all $\mathbf{p} \in \mathcal{P}_\omega(F, \mathfrak{X})$.

Now we define $\phi : \mathcal{Q}_\omega \longrightarrow \mathbb{R}$ as follows: for all $r \in \mathcal{Q}_\omega$, define

$$\phi(r) \;:=\; \sum_{\substack{q \in \mathcal{Q} \\ q \leq r}} v_q \text{ if } r \geq 0, \text{ and } \phi(r) \;:=\; -\phi(-r) \text{ if } r \leq 0. \tag{C14}$$

Then ϕ is odd. Also, ϕ is increasing on $\mathcal{Q}_\omega$, because $v_q > 0$ for all $q \in \mathcal{Q}_\omega$.

Let $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta_\omega(\mathcal{X})$. For any $\mathbf{x} \in \mathcal{X}$, recall $\mathcal{M}(\mu, \mathbf{x}) := \{k \in \mathcal{K};\; x_k \widetilde{\mu}_k \geq 0\}$.

Claim 1: *For any $\mathbf{x} \in \mathcal{X}$, we have $|\mathcal{K}| \cdot \mathbf{v} \bullet \mathbf{g}(\mathbf{x}, \mu) = \sum_{k \in \mathcal{M}(\mathbf{x}, \mu)} \phi\, |\widetilde{\mu}_k|$.*

Proof: For any $r, q \in [-1, 1]$, define $\delta^r_q := 1$ if $r \geq q$, whereas $\delta^r_q := 0$ if $r < q$. Then

$$\begin{aligned} |\mathcal{K}| \cdot \mathbf{v} \bullet \mathbf{g}(\mathbf{x}, \mu) &= |\mathcal{K}| \cdot \sum_{q \in \mathcal{Q}} v_q \cdot g_q(\mathbf{x}, \mu) \underset{(\diamond)}{=} \sum_{q \in \mathcal{Q}} v_q \cdot \#\{k \in \mathcal{K} \;;\; x_k\, \widetilde{\mu}_k \geq q\} \\ &= \sum_{q \in \mathcal{Q}} v_q \sum_{k \in \mathcal{K}} \delta_q^{x_k\, \widetilde{\mu}_k} = \sum_{k \in \mathcal{K}} \sum_{q \in \mathcal{Q}} v_q\, \delta_q^{x_k\, \widetilde{\mu}_k} \\ &= \sum_{k \in \mathcal{K}} \sum_{\substack{q \in \mathcal{Q} \\ q \leq x_k\, \widetilde{\mu}_k}} v_q \underset{(*)}{=} \sum_{\substack{k \in \mathcal{K} \\ x_k\, \widetilde{\mu}_k \geq 0}} \phi(x_k\, \widetilde{\mu}_k) = \sum_{k \in \mathcal{M}(\mathbf{x}, \mu)} \phi|\widetilde{\mu}_k|. \end{aligned}$$

Here, $(\diamond)$ is by definition (C7), while $(*)$ is by definition (C14). ◇ Claim 1

We must show that $F(\mathcal{X}, \mu) \subseteq F_\phi(\mathcal{X}, \mu)$. To see this, let $\mathbf{x} \in F(\mathcal{X}, \mu)$. For any other $\mathbf{y} \in \mathcal{X}$, we have $\mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu) \in \mathcal{D}_\omega(F, \mathfrak{X}) \subseteq \mathcal{P}_\omega(F, \mathfrak{X})$, so $\mathbf{v} \bullet (\mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu)) \leq 0$, and hence $\mathbf{v} \bullet \mathbf{g}(\mathbf{y}, \mu) \leq \mathbf{v} \bullet \mathbf{g}(\mathbf{x}, \mu)$. Thus, $F(\mathcal{X}, \mu) \subseteq \underset{\mathbf{x} \in \mathcal{X}}{\text{argmax}}\, (\mathbf{v} \bullet \mathbf{g}(\mu, \mathbf{x}))$. Thus, Claim 1 and Corollary 3.3(a) imply that $F(\mathcal{X}, \mu) \subseteq F_\phi(\mathcal{X}, \mu)$, as desired.

Unique minimal cover. Let $\mathcal{G}$ be the set of additive majority rules which cover F on $\Delta_\omega(\mathfrak{X})$, and which are defined by real-valued gain functions. We have just shown that $\mathcal{G} \neq \emptyset$.

Claim 2: *$\mathcal{G}$ is finite.*

Proof: Recall that $\mathfrak{X}$ is finitely generated —say it is generated by the judgement spaces $\mathcal{X}_1, \ldots, \mathcal{X}_N$. For all $n \in [1 \ldots N]$, the spaces $\mathcal{X}_n$ is finite (it is a subset of a finite-dimensional Hamming cube). Thus, since ω has finite support, the set $\Delta_\omega(\mathcal{X}_n)$ is finite. Thus, if $\mathcal{F}_n$ is the set of all possible judgement aggregation rules from $\Delta_\omega(\mathcal{X}_n)$ into $\mathcal{X}_n$, then $\mathcal{F}_n$ is also finite.

Any judgement aggregation rule F on $\Delta_\omega(\mathfrak{X})$ which satisfies Decomposition is obtained by choosing one rule $F_n \in \mathcal{F}_n$ for each $n \in [1 \ldots N]$. Thus, the set of all aggregation rules on $\Delta_\omega(\mathfrak{X})$ satisfying Decomposition is also finite. In particular, this means the set $\mathcal{G}$ is finite. ◇ `Claim 2`

Claim 2 implies that we can write $\mathcal{G} = \{F_{\phi_1}, F_{\phi_2}, \ldots, F_{\phi_L}\}$, for some gain functions $\phi_1, \phi_2, \ldots, \phi_L : [-1, 1] \longrightarrow \mathbb{R}$. Now define $\phi := \sum_{\ell=1}^{L} \phi_\ell$.

Claim 3: *F_ϕ covers F on $\Delta_\omega(\mathfrak{X})$.*

Proof: Let $\mathcal{X} \in \mathfrak{X}$ and let $\mu \in \Delta_\omega(\mathcal{X})$. Then for all $\ell \in [1 \ldots L]$, we have $F(\mathcal{X}, \mu) \subseteq F_{\phi_\ell}(\mathcal{X}, \mu)$, by definition of the set $\mathcal{G}$. This means: for all $\mathbf{x} \in F(\mathcal{X}, \mu)$ and all other $\mathbf{y} \in \mathcal{X}$, we have $\phi_\ell(\widetilde{\mu}) \bullet \mathbf{x} \geq \phi_\ell(\widetilde{\mu}) \bullet \mathbf{y}$. By summing these inequalities over all $\ell \in [1 \ldots L]$, it follows that $\phi(\widetilde{\mu}) \bullet \mathbf{x} \geq \phi(\widetilde{\mu}) \bullet \mathbf{y}$. This holds for all $\mathbf{x} \in F(\mathcal{X}, \mu)$ and all other $\mathbf{y} \in \mathcal{X}$; it follows that $F(\mathcal{X}, \mu) \subseteq F_\phi(\mathcal{X}, \mu)$, as desired. ◇ `Claim 3`

Claim 4: *For all $\ell \in [1 \ldots L]$, the rule F_{ϕ_ℓ} covers F_ϕ on $\Delta_\omega(\mathfrak{X})$.*

Proof: (by contrapositive) Let $\mathcal{X} \in \mathfrak{X}$, let $\mu \in \Delta_\omega(\mathcal{X})$, and suppose $\mathbf{x} \notin F_{\phi_\ell}(\mathcal{X}, \mu)$ for some $\ell \in [1 \ldots L]$. We will show that $\mathbf{x} \notin F_\phi(\mathcal{X}, \mu)$.

Let $\mathbf{y} \in F(\mathcal{X}, \mu)$. Then for all $\ell' \in [1 \ldots L]$, we have $\mathbf{y} \in F_{\phi_{\ell'}}(\mathcal{X}, \mu)$, and thus, $\phi_{\ell'}(\widetilde{\mu}) \bullet \mathbf{y} \geq \phi_{\ell'}(\widetilde{\mu}) \bullet \mathbf{x}$. In particular, $\mathbf{y} \in F_{\phi_\ell}(\mathcal{X}, \mu)$, whereas $\mathbf{x} \notin F_{\phi_\ell}(\mathcal{X}, \mu)$; thus, $\phi_\ell(\widetilde{\mu}) \bullet \mathbf{y} > \phi_\ell(\widetilde{\mu}) \bullet \mathbf{x}$. By summing these inequalities over all $\ell' \in [1 \ldots L]$, it follows that $\phi(\widetilde{\mu}) \bullet \mathbf{y} > \phi(\widetilde{\mu}) \bullet \mathbf{x}$. Thus, $\mathbf{x} \notin F_\phi(\mathcal{X}, \mu)$. ◇ `Claim 4`

Claims 3 and 4 imply that F_ϕ is a minimal covering of F by an additive majority rule. Minimality implies uniqueness.

Finally, if G is any additive majority rule on $\Delta_\omega(\mathfrak{X})$, then Lemma C.3 says that G can be represented with a real-valued gain function. □

Proof of Theorem 4.1. Let $\mathfrak{X} := \langle \mathcal{X} \rangle$. Fix $N \in \mathbb{N}$, and define $\omega(n) := 1$ for all $n \in [1 \ldots N]$ while $\omega(0) := 1$ for all $n > N$; then $\Delta_\omega(\mathfrak{X}) = \Delta_N\langle \mathcal{X} \rangle$. Now apply Proposition C.1. □

The proofs of Theorem 4.2(c) and Proposition 6.2 use the following result.

Lemma C.9 *Let $F, G : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be two judgement aggregation rules. Suppose $F(\mu) \subseteq G(\mu)$ for all $\mu \in \Delta(\mathcal{X})$, and G is monotone, and F is upper hemicontinuous. Then $F(\mu) = G(\mu)$ for all $\mu \in \Delta(\mathcal{X})$.*

Proof: Let $\mu \in \Delta(\mathcal{X})$. We have $F(\mu) \subseteq G(\mu)$ by hypothesis; we must show $F(\mu) \supseteq G(\mu)$. So, let $\mathbf{x} \in G(\mu)$; we will show that $\mathbf{x} \in F(\mu)$. Let $\delta_{\mathbf{x}} \in \Delta(\mathcal{X})$ be the unanimous profile at $\mathbf{x}$. For all $n \in \mathbb{N}$, define $\mu_n := (1 - \frac{1}{n})\mu + \frac{1}{n}\delta_{\mathbf{x}}$. Then μ_n is more supportive of $\mathbf{x}$ than μ, so $G(\mu_n) = \{\mathbf{x}\}$ because G is monotone. Thus, $F(\mu_n) = \{\mathbf{x}\}$ because $F \subseteq G$, and $F(\mu_n)$ must be nonempty. However, $\lim_{n\to\infty} \mu_n = \mu$, and F is upper hemicontinuous. Thus, $\mathbf{x} \in F(\mu)$, as desired. □

Theorem 4.2 is obtained by using an ultrapower construction to "stitch together" the gain functions defined in Proposition C.1 for every possible choice of weight function ω and finitely generated judgement monoid $\mathfrak{X}$.

Proof of Theorem 4.2. (a) Lemma C.2 says that any additive majority rule is decomposable, while Lemma C.8 says it is SME.

(b) "$\Longleftarrow$" exactly the same proof as Proposition C.1.

"$\Longrightarrow$" Let $\mathrm{FGSM}(\mathfrak{X})$ be the set of all finitely generated sub-monoids of $\mathfrak{X}$. Let $\mathcal{I} := \Omega \times \mathrm{FGSM}(\mathfrak{X})$. In effect, $\mathcal{I}$ is the set of possible "inputs" to Proposition C.1.

For any finite collection $\mathcal{T} := \{(\mathcal{X}_1, \mu_1),\ (\mathcal{X}_2, \mu_2),\ \ldots, (\mathcal{X}_N, \mu_N)\} \subset \Delta(\mathfrak{X})$, define $\mathcal{I}_{\mathcal{T}} := \{(\omega, \mathfrak{Y}) \in \mathcal{I};\ \ \mathcal{X}_n \in \mathfrak{Y}$ and $\mu_n \in \Delta_\omega(\mathcal{X}_n)$ for all $n \in [1 \ldots N]\}$. Then let $\mathfrak{E} := \{\mathcal{J} \subseteq \mathcal{I};\ \mathcal{I}_{\mathcal{T}} \subseteq \mathcal{J}$ for some nonempty finite $\mathcal{T} \subset \Delta(\mathfrak{X})\}$.

Claim 1: $\mathfrak{E}$ *is a free filter.*

Proof: We must check axioms (F0)-(F2) from Appendix A.

(F0) Let $\mathcal{T} := \{(\mathcal{X}_1, \mu_1),\ (\mathcal{X}_2, \mu_2),\ \ldots, (\mathcal{X}_N, \mu_N)\} \subset \Delta(\mathfrak{X})$. Fix $\omega \in \Omega$, and suppose $\mathrm{supp}(\omega) = [1 \ldots W]$. Let $M := W \cdot |\mathcal{X}_1| \cdot |\mathcal{X}_2| \cdots |\mathcal{X}_N|$, and let $\beta : [1 \ldots M] \longrightarrow [1 \ldots W] \times \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N$ be some bijection. Define $\omega' \in \Omega$ as follows: for all $m \in [1 \ldots M]$, if $\beta(m) = (w, \mathbf{x}_1, \ldots, \mathbf{x}_N)$, then let $\omega'(m) := \omega(w) \cdot \mu_1(\mathbf{x}_1) \cdots \mu_N(\mathbf{x}_N)$. Let $\mathfrak{Y} \subset \mathfrak{X}$ be any finitely generated sub-monoid of $\mathfrak{X}$ containing the finitely generated monoid $\langle \mathcal{X}_1, \ldots, \mathcal{X}_N \rangle$. Then it is easy to check that $(\mathcal{X}_n, \mu_n) \in \Delta_{\omega'}(\mathfrak{Y})$ for all $n \in [1 \ldots N]$. Thus, $(\omega', \mathfrak{Y}) \in \mathcal{I}_{\mathcal{T}}$.

We can repeat this construction for any $\omega \in \Omega$; thus, $\mathcal{I}_{\mathcal{T}}$ is infinite. This holds for any finite $\mathcal{T} \subset \Delta(\mathfrak{X})$. Every element of $\mathfrak{E}$ must contain $\mathcal{I}_{\mathcal{T}}$ for some finite $\mathcal{T} \subset \Delta(\mathfrak{X})$; thus, every element of $\mathfrak{E}$ is infinite.

(F1) Let $\mathcal{E}, \mathcal{F} \in \mathfrak{E}$. Then there exist finite sets $\mathcal{S}, \mathcal{T} \subset \Delta(\mathfrak{X})$ such that $\mathcal{I}_{\mathcal{S}} \subseteq \mathcal{E}$ and $\mathcal{I}_{\mathcal{T}} \subseteq \mathcal{F}$. But then $\mathcal{S} \cup \mathcal{T}$ is also finite, and $\mathcal{I}_{\mathcal{S}\cup\mathcal{T}} = \mathcal{I}_{\mathcal{S}} \cap \mathcal{I}_{\mathcal{T}} \subseteq \mathcal{E} \cap \mathcal{F}$; thus, $\mathcal{E} \cap \mathcal{F} \in \mathfrak{E}$.

(F2) Suppose $\mathcal{E} \in \mathfrak{E}$ and $\mathcal{E} \subseteq \mathcal{D}$. Then there is some finite $\mathcal{T} \subset \Delta(\mathfrak{X})$ such that $\mathcal{I}_{\mathcal{T}} \subseteq \mathcal{E}$. But then $\mathcal{I}_{\mathcal{T}} \subseteq \mathcal{D}$; thus $\mathcal{D} \in \mathfrak{E}$ also. ◇ Claim 1

Now Claim 1 and the Ultrafilter Lemma yields a free ultrafilter $\mathfrak{F}$ with $\mathfrak{E} \subseteq \mathfrak{F}$. Let ${}^*\mathbb{R}$ be the hyperreal field defined by $\mathfrak{F}$ (see Appendix A). We define $\phi : [-1,1] \longrightarrow {}^*\mathbb{R}$ as follows. For all $(\omega, \mathfrak{Y}) \in \mathcal{I}$, Proposition C.1 yields an odd, strictly increasing function $\phi_{\omega,\mathfrak{Y}} : \mathcal{Q}_\omega \longrightarrow \mathbb{R}$. Recall that $\mathcal{Q}_\omega$ is a finite subset of $[-1,1]$ (because ω has finite support). Thus, we can extend $\phi_{\omega,\mathfrak{Y}}$ to an odd, continuous increasing function $\phi_{\omega,\mathfrak{Y}} : [-1,1] \longrightarrow \mathbb{R}$, by linearly interpolating the values between the points in $\mathcal{Q}_\omega$. Now, for any $r \in [-1,1]$, define $\widehat{\phi}(r) \in \mathbb{R}^{\mathcal{I}}$ by:

$$\widehat{\phi}(r)_{\omega,\mathfrak{Y}} \; := \; \phi_{\omega,\mathfrak{Y}}(r), \quad \text{for all } (\omega, \mathfrak{Y}) \in \mathcal{I}. \tag{C15}$$

Then define $\phi(r) \in {}^*\mathbb{R}$ to be the $(\underset{\mathfrak{F}}{\sim})$-equivalence class of $\hat{\phi}(r)$.

Claim 2: *ϕ is odd and strictly increasing.*

Proof: Odd. Let $r \in [-1,1]$. For all $(\omega, \mathfrak{Y}) \in \mathcal{I}$, we have $\phi_{\omega,\mathfrak{Y}}(-r) = -\phi_{\omega,\mathfrak{Y}}(r)$, because $\phi_{\omega,\mathfrak{Y}}$ is odd by construction. But $\mathcal{I} \in \mathfrak{F}$, by definition of $\mathfrak{F}$. Thus, $\phi(-r) = -\phi(r)$.

Increasing. Let $q, r \in [-1,1]$, with $q < r$. For all $(\omega, \mathfrak{Y}) \in \mathcal{I}$, we have $\phi_{\omega,\mathfrak{Y}}(q) < \phi_{\omega,\mathfrak{Y}}(r)$, because $\phi_{\omega,\mathfrak{Y}}$ is increasing by construction. But $\mathcal{I} \in \mathfrak{F}$ by definition of $\mathfrak{F}$. Thus, we obtain $\phi(q) < \phi(r)$, by defining formula (A1). ◇ Claim 2

Claim 3: *For any $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta(\mathcal{X})$, we have $F(\mathcal{X}, \mu) \subseteq F_\phi(\mathcal{X}, \mu)$.*

Proof: Let $\mathbf{x} \in F(\mathcal{X}, \mu)$; we must show that $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$. For all $(\omega, \mathfrak{Y}) \in \mathcal{I}_{\{(\mathcal{X},\mu)\}}$, Proposition C.1 says that $F(\mathcal{X}, \mu) \subseteq F_{\phi_{\omega,\mathfrak{Y}}}(\mathcal{X}, \mu)$; thus, for all $\mathbf{y} \in \mathcal{X}$, we have $(\mathbf{x} - \mathbf{y}) \bullet \phi_{\omega,\mathfrak{Y}}(\mu) \geq 0$. Thus, if we define $\mathcal{I}_\mathbf{y} := \{(\omega, \mathfrak{Y}) \in \mathcal{I};\ (\mathbf{x} - \mathbf{y}) \bullet \phi_{\omega,\mathfrak{Y}}(\mu) \geq 0\}$, then $\mathcal{I}_\mathbf{y} \supseteq \mathcal{I}_{\{(\mathcal{X},\mu)\}}$. But $\mathcal{I}_{\{(\mathcal{X},\mu)\}} \in \mathfrak{F}$; thus, $\mathcal{I}_\mathbf{y} \in \mathfrak{F}$, by axiom (F2) from Appendix A. Thus, $(\mathbf{x} - \mathbf{y}) \bullet \phi(\mu) \geq 0$, by defining formula (A1). This holds for all $\mathbf{y} \in \mathcal{X}$; thus, $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$.

This holds for all $\mathbf{x} \in F(\mathcal{X}, \mu)$; thus, $F(\mathcal{X}, \mu) \subseteq F_\phi(\mathcal{X}, \mu)$. ◇ Claim 3

Since $F(\mathcal{X}, \mu)$ must be nonempty, we must have $F(\mathcal{X}, \mu) = F_\phi(\mathcal{X}, \mu)$ whenever $F_\phi(\mathcal{X}, \mu)$ is single-valued. But ϕ is strictly increasing, so F_ϕ is single-valued on a dense, open subset of $\Delta(\mathcal{X})$, by Proposition 3.4(b).

(c) For all $(\omega, \mathfrak{Y}) \in \mathcal{I}$, Proposition C.1 says that we can choose the gain functions $\phi_{\omega,\mathfrak{Y}}$ such that the additive majority rule $F_{\phi_{\omega,\mathfrak{Y}}}$ is the *minimal* covering of F on $\Delta_\omega(\mathfrak{Y})$. Suppose we perform the construction in part (b) using this set of gain functions; we claim the resulting rule F_ϕ is a minimal covering of F on $\Delta(\mathfrak{X})$.

To see this, let $\psi : [-1,1] \longrightarrow {}^*\mathbb{R}$ be another gain function, such that the additive majority rule F_ψ covers F on $\Delta(\mathfrak{X})$. Let $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta(\mathcal{X})$; we must show that $F_\phi(\mathcal{X}, \mu) \subseteq F_\psi(\mathcal{X}, \mu)$. So, let $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$; we will show that $\mathbf{x} \in F_\psi(\mathcal{X}, \mu)$.

For all $\mathbf{y} \in \mathcal{X}$, let $\mathcal{I}_{\mathbf{x},\mathbf{y}} := \{(\omega, \mathfrak{Y}) \in \mathcal{I};\ \mathbf{x} \bullet \phi_{\omega,\mathfrak{Y}}(\widetilde{\mu}) \geq \mathbf{y} \bullet \phi_{\omega,\mathfrak{Y}}(\widetilde{\mu})\}$. Then $\mathcal{I}_{\mathbf{x},\mathbf{y}} \in \mathfrak{F}$, by formula (A1) (because $\mathbf{x} \bullet \phi(\mu) \geq \mathbf{y} \bullet \phi(\mu)$, because $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$). Thus, if we define

$$\mathcal{I}_\mathbf{x} \; := \; \left\{(\omega, \mathfrak{Y}) \in \mathcal{I}_{\{(\mathcal{X},\mu)\}} \; ; \; \mathbf{x} \in F_{\phi_{\omega,\mathfrak{Y}}}(\mathcal{X}, \mu)\right\} \quad = \quad \mathcal{I}_{\{(\mathcal{X},\mu)\}} \cap \bigcap_{\mathbf{y} \in \mathcal{X}} \mathcal{I}_{\mathbf{x},\mathbf{y}},$$

then $\mathcal{I}_{\mathbf{x}} \in \mathfrak{F}$, by axiom (F1) (because it is a finite intersection of $\mathfrak{F}$-elements, because $\mathcal{X}$ is a finite set). Thus, $\mathcal{I}_{\mathbf{x}}$ is nonempty, by axiom (F0).

Now, for all $(\omega, \mathfrak{Y}) \in \mathcal{I}_{\{(\mathcal{X},\mu)\}}$, the rule F_ψ covers F on $\Delta_\omega(\mathfrak{Y})$. Thus, F_ψ also covers $F_{\phi_{\omega,\mathfrak{Y}}}$ (because $F_{\phi_{\omega,\mathfrak{Y}}}$ is the minimal covering of F on $\Delta_\omega(\mathfrak{Y})$). Thus, if we take any $(\omega, \mathfrak{Y}) \in \mathcal{I}_{\mathbf{x}}$, we obtain $\mathbf{x} \in F_{\phi_{\omega,\mathfrak{Y}}}(\mathcal{X}, \mu) \subseteq F_\psi(\mathcal{X}, \mu)$. Thus, $\mathbf{x} \in F_\psi(\mathcal{X}, \mu)$.

This argument works for all $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$; we conclude that $F_\phi(\mathcal{X}, \mu) \subseteq F_\psi(\mathcal{X}, \mu)$, as desired.

(d) "$\Longrightarrow$" This follows from part (a).

"$\Longleftarrow$" Let ϕ be the gain function constructed in part (b). Then F_ϕ covers F. The rule F_ϕ is monotone, by Proposition 3.4(a). Meanwhile F is uhc by hypothesis. Thus, Lemma C.9 implies that $F(\mu) = F_\phi(\mu)$ for all $\mu \in \Delta(\mathcal{X})$. □

Proof of Proposition 4.4. The proof strategy is almost identical to the proof of Theorem 4.2(b). Let $\mathfrak{X}$ be the set of *all* judgement aggregation problems. (Thus, $\mathfrak{X}$ is closed under Cartesian products.) Define $\mathcal{I}$ as in the proof of Theorem 4.2, and for each $(\omega, \mathfrak{Y}) \in \mathcal{I}$, let $\phi_{\omega,\mathfrak{Y}} : \mathcal{Q}_\omega \longrightarrow \mathbb{R}$ be the gain function from Lemma C.3. Then define $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$ as in Eq.(C15). The proof of Claim 2 is exactly as before. But we replace Claim 3 with the following:

Claim 1: *For any $\mathcal{X} \in \mathfrak{X}$ and $\mu \in \Delta(\mathcal{X})$, we have $F_\phi(\mathcal{X}, \mu) = F_\psi(\mathcal{X}, \mu)$.*

Proof: Suppose $\mathbf{x} \in F_\psi(\mathcal{X}, \mu)$ and $\mathbf{z} \notin F_\psi(\mathcal{X}, \mu)$. We must show that $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$ and $\mathbf{z} \notin F_\phi(\mathcal{X}, \mu)$. For all $(\omega, \mathfrak{Y}) \in \mathcal{I}_{\{(\mathcal{X},\mu)\}}$, Lemma C.3 says that $F_\psi(\mathcal{X}, \mu) = F_{\phi_{\omega,\mathfrak{Y}}}(\mathcal{X}, \mu)$; thus, for all $\mathbf{y} \in \mathcal{X}$, we have $(\mathbf{x}-\mathbf{y}) \bullet \phi_{\omega,\mathfrak{Y}}(\mu) \geq 0$, and furthermore, $(\mathbf{x}-\mathbf{z}) \bullet \phi_{\omega,\mathfrak{Y}}(\mu) > 0$. Thus, if we define $\mathcal{I}_{\mathbf{y},\geq} := \{(\omega, \mathfrak{Y}) \in \mathcal{I};\ (\mathbf{x} - \mathbf{y}) \bullet \phi_{\omega,\mathfrak{Y}}(\mu) \geq 0\}$, then $\mathcal{I}_{\mathbf{y},\geq} \supseteq \mathcal{I}_{\{(\mathcal{X},\mu)\}}$. Also, if we define $\mathcal{I}_{\mathbf{z},>} := \{(\omega, \mathfrak{Y}) \in \mathcal{I};\ (\mathbf{x} - \mathbf{z}) \bullet \phi_{\omega,\mathfrak{Y}}(\mu) > 0\}$, then $\mathcal{I}_{\mathbf{z},>} \supseteq \mathcal{I}_{\{(\mathcal{X},\mu)\}}$. But $\mathcal{I}_{\{(\mathcal{X},\mu)\}} \in \mathfrak{F}$; thus, we get $\mathcal{I}_{\mathbf{y},\geq} \in \mathfrak{F}$ and $\mathcal{I}_{\mathbf{z},>} \in \mathfrak{F}$, by axiom (F2) from Appendix A. Thus, $(\mathbf{x} - \mathbf{y}) \bullet \phi(\mu) \geq 0$, by the defining formula (A1). This holds for all $\mathbf{y} \in \mathcal{X}$; thus, $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$. Likewise, $(\mathbf{x} - \mathbf{z}) \bullet \phi(\mu) > 0$ by the defining formula (A1), so $\mathbf{z} \notin F_\phi(\mathcal{X}, \mu)$. ◇ `Claim 1`

□

D Proofs from Section 5

It will be convenient to introduce another definition of SSME. For any probability measure $\rho \in \Delta(\mathcal{X})$, and any $q \in (0, 1]$, define

$$\gamma_{\mu,\rho}(q) \quad := \quad \sum_{\mathbf{x} \in \mathcal{X}} \rho(\mathbf{x}) \cdot \gamma_{\mu,\mathbf{x}}(q). \tag{D1}$$

For example, fix $N \in \mathbb{N}$ and suppose $\rho(\mathbf{x}) = N_{\mathbf{x}}/N$ for all $\mathbf{x} \in \mathcal{X}$, where $\{N_{\mathbf{x}}\}_{\mathbf{x}\in\mathcal{X}}$ is a collection of natural numbers such that $\sum_{\mathbf{x}\in\mathcal{X}} N_{\mathbf{x}} = N$. Let $\nu \in \Delta(\mathcal{X}^N)$ by a profile such that $\nu^{(n)} = \mu$ for all $n \in [1 \ldots N]$, and let $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N) \in \mathcal{X}^N$ be such that, for all $\mathbf{x} \in \mathcal{X}$, we have $\#\{n \in [1 \ldots N];\ \mathbf{y}^n = \mathbf{x}\} = N_{\mathbf{x}}$. Then $\gamma_{\mu,\rho} = \gamma_{\nu,\mathbf{y}}/N$

For any $\rho_1, \rho_2 \in \Delta(\mathcal{X})$, write "$\rho_1 \underset{\mu}{\triangleright} \rho_2$" if $\gamma_{\mu,\rho_1}(q) \geq \gamma_{\mu,\rho_2}(q)$ for all $q \in (0,1]$, with at least one strict inequality. Then $\underset{\mu}{\triangleright}$ is a partial order on $\Delta(\mathcal{X})$. For any $\mathbf{x} \in \mathcal{X}$, define the *unanimous* profile $\delta_{\mathbf{x}} \in \Delta(\mathcal{X})$ by setting $\delta_{\mathbf{x}}(\mathbf{x}) := 1$, while $\delta_{\mathbf{x}}(\mathbf{y}) := 0$, for all $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$. Clearly $\gamma_{\mu,\delta_{\mathbf{x}}} = \gamma_{\mu,\mathbf{x}}$. We say $\mathbf{x} \in \mathcal{X}$ is *strongly supermajority efficient* (SSME) if $\delta_{\mathbf{x}}$ is undominated in $(\Delta(\mathcal{X}), \underset{\mu}{\triangleright})$. It is easy to check that this is equivalent to the definition given in Section 5.

Proof of Proposition 5.1. SSME $(\mathcal{X}, \mu)$ is always nonempty because it is obtained by maximizing over a finite set. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, observe that

$$\Big(\mathcal{M}(\mathbf{x},\mu) \supset \mathcal{M}(\mathbf{y},\mu)\Big) \Longrightarrow \Big(\mathbf{x} \underset{\mu}{\triangleright} \mathbf{y}\Big) \Longrightarrow \Big(\delta_{\mathbf{x}} \underset{\mu}{\triangleright} \delta_{\mathbf{y}}\Big).$$

Thus, if $\delta_{\mathbf{y}}$ is undominated in $(\Delta(\mathcal{X}), \underset{\mu}{\triangleright})$, then $\mathbf{y}$ must be undominated in $(\mathcal{X}, \underset{\mu}{\triangleright})$. Likewise if $\mathbf{y}$ is undominated in $(\mathcal{X}, \underset{\mu}{\triangleright})$, then $\mathbf{y}$ must be Condorcet admissible in $\mathcal{X}$. Thus, SSME $(\mathcal{X}, \mu) \subseteq$ SME $(\mathcal{X}, \mu) \subseteq$ Cond $(\mathcal{X}, \mu)$. □

Proof of Proposition 5.2. (a) "[iii] $\Longrightarrow$ [ii]" is immediate, while "[ii] $\Longrightarrow$ [i]" follows immediately from Lemma C.8. It remains to prove "[i] $\Longrightarrow$ [iii]".

Let $\mathcal{Q}(\mu) := \{|\widetilde{\mu}_k|\ ;\ k \in \mathcal{K}\}$, and write $\mathcal{Q} = \{q_1, q_2, \ldots, q_N\}$, where $0 \leq q_1 < q_2 < \cdots < q_N \leq 1$. Define $q_0 := 0$. Let Γ be the space of functions defined above Lemma C.7. Define a linear injective map $f_\mu : \mathbb{R}^N \longrightarrow \Gamma$ by setting $f_\mu(r_1, \ldots, r_N) := r_1 \mathbf{1}_{(0,q_1]} + r_2 \mathbf{1}_{(q_1,q_2]} + \cdots + r_N \mathbf{1}_{(q_{N-1},q_N]}$ for all $(r_1, \ldots, r_N) \in \mathbb{R}^N$. Let $\Gamma_\mu := f_\mu[\mathbb{R}^N]$ (a linear subspace of Γ); then $\gamma_{\mu,\rho} \in \Gamma_\mu$ for all $\rho \in \Delta(\mathcal{X})$. Let $g_\mu := f_\mu^{-1} : \Gamma_\mu \longrightarrow \mathbb{R}^N$, and then define $G_\mu : \Delta(\mathcal{X}) \longrightarrow \mathbb{R}^N$ by $G_\mu(\rho) := g_\mu(\gamma_{\mu,\rho})$. Let $\mathcal{P}_\mu := G_\mu[\Delta(\mathcal{X})]$; then $\mathcal{P}_\mu$ is a convex, compact polytope in $\mathbb{R}^N$. For all $\mathbf{x} \in \mathcal{X}$, let $G_\mu(\mathbf{x}) := g_\mu(\gamma_{\mu,\mathbf{x}}) \in \mathcal{P}_\mu$. Let "$<$" be the Pareto relation on $\mathbb{R}^N$. Fix $\mathbf{x} \in \mathcal{X}$. We have

$$\Big(\mathbf{x} \underset{\mu}{\triangleleft} \rho\Big) \iff \Big(G_\mu(\mathbf{x}) < G_\mu(\rho)\Big), \quad \text{for all } \rho \in \Delta(\mathcal{X}).$$

$$\text{Thus,} \quad \Big(\mathbf{x} \in \text{SSME}\,(\mathcal{X},\mu)\Big) \iff \Big(G_\mu(\mathbf{x}) \text{ is Pareto optimal in } \mathcal{P}_\mu\Big).$$

Lemma C.4 now yields a strictly positive vector $\mathbf{v} \in \mathbb{R}^N_+$ such that

$$\mathbf{v} \bullet G_\mu(\mathbf{x}) \quad \geq \quad \mathbf{v} \bullet \mathbf{p}, \quad \text{for all } \mathbf{p} \in \mathcal{P}_\mu. \tag{D2}$$

Construct $\phi \in \Phi_I$ such that $\phi(q_n) - \phi(q_{n-1}) = v_n$ for all $n \in [1...N]$. (For example, ϕ could be piecewise linear, with vertices at $q_1, \ldots, q_N$.) Let $\phi^* : \Gamma \longrightarrow \mathbb{R}$ be the positive linear functional defined by ϕ, as in Lemma C.7(a). Then for all $\mathbf{y} \in \mathcal{X}$, if $G_\mu(\mathbf{y}) = \mathbf{p} \in \mathcal{P}_\mu$, then $\gamma_{\mu,\mathbf{y}} = f_\mu(\mathbf{p})$, so that

$$\phi^*(\gamma_{\mu,\mathbf{y}}) \;=\; \phi^*\left(f_\mu(\mathbf{p})\right) \;=\; \phi^*\left(\sum_{n=1}^N p_n \mathbf{1}_{(q_{n-1},q_n]}\right) \;=\; \sum_{n=1}^N p_n \phi^*\left(\mathbf{1}_{(q_{n-1},q_n]}\right) \tag{D3}$$

$$= \sum_{n=1}^N p_n \left(\phi(q_n) - \phi(q_{n-1})\right) \;=\; \sum_{n=1}^N p_n v_n \;=\; \mathbf{v} \bullet \mathbf{p} \;=\; \mathbf{v} \bullet G_\mu(\mathbf{y}).$$

$$\text{Thus,} \qquad \Big(\mathbf{v} \bullet G_\mu(\mathbf{x}) \geq \mathbf{v} \bullet G_\mu(\mathbf{y})\Big) \quad \Longleftrightarrow \quad \Big(\phi^*(\gamma_{\mu,\mathbf{x}}) \geq \phi^*(\gamma_{\mu,\mathbf{y}})\Big).$$

Thus, equation (D2) implies that $\phi^*(\gamma_{\mu,\mathbf{x}}) \geq \phi^*(\gamma_{\mu,\mathbf{y}})$ for all $\mathbf{y} \in \mathcal{X}$. Thus, Lemma C.7(c) says that $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$, as desired.

(b) Let $\mathbf{x}, \mathbf{y} \in \mathrm{SSME}(\mathcal{X}, \mu)$. If $\gamma_{\mathbf{x},\mu} \neq \gamma_{\mathbf{y},\mu}$, then $G_\mu(\mathbf{x}) \neq G_\mu(\mathbf{y})$. Thus, in the proof of (a), we can choose some $\mathbf{v} \in \mathbb{R}^N_+$ satisfying equation (D2), and such that $\mathbf{v} \bullet G_\mu(\mathbf{x}) > \mathbf{v} \bullet G_\mu(\mathbf{y})$. Thus, if $\phi \in \Phi_I$ is the function defined by $\mathbf{v}$, then equation (D3) implies that $\phi^*(\gamma_{\mu,\mathbf{x}}) > \phi^*(\gamma_{\mu,\mathbf{y}})$. Thus, Lemma C.7(c) says that $\mathbf{y} \notin F_\phi(\mathcal{X}, \mu)$. □

Judgement	Partition	$\mu(\bullet)$	$a\sim$ c,d,e,f	$b\sim$ c,d,e,f	$a \sim g$	$b \sim g$	$a \sim h$	$b \sim h$	c,d,e,f $\not\sim g,h$	$a \sim b$	$c\sim d$ $\sim e\sim f$	$g \sim h$
$\mathbf{t}$	$\{a,b,c,d,e,f,g,h\}$	0.2	1	1	1	1	1	1	−1	1	1	1
$\mathbf{u}$	$\{a,b,g,h\},\{c,d,e,f\}$	0.45	−1	−1	1	1	1	1	1	1	1	1
$\mathbf{v}$	$\{a,b,c,d,e,f\},\{g,h\}$	0.35	1	1	−1	−1	−1	−1	1	1	1	1
$\mathbf{w}$	$\{a,c,d,e,f\},\{b,g,h\}$	0	1	−1	−1	1	−1	1	1	−1	1	1
$\mathbf{w}'$	$\{b,c,d,e,f\},\{a,g,h\}$	0	−1	1	1	−1	1	−1	1	−1	1	1
$\mathbf{y}$	$\{a,b\},\{c,d,e,f\},\{g,h\}$	0	−1	−1	−1	−1	−1	−1	1	1	1	1
$\mathbf{z}_1$	$\{a\},\{c,d,e,f\},\{b,g,h\}$	0	−1	−1	−1	1	−1	1	1	−1	1	1
$\mathbf{z}_2$	$\{b\},\{c,d,e,f\},\{a,g,h\}$	0	−1	−1	1	−1	1	−1	1	−1	1	1
$\mathbf{z}_3$	$\{a,b,g\},\{c,d,e,f\},\{h\}$	0	−1	−1	1	1	−1	−1	1	1	1	−1
$\mathbf{z}_4$	$\{a,b,h\},\{c,d,e,f\},\{g\}$	0	−1	−1	−1	−1	1	1	1	1	1	−1
		$\widetilde{\mu}_k$	0.1	0.1	0.3	0.3	0.3	0.3	0.6	1	1	1
		$\mathrm{Maj}_k(\mu)$	1	1	1	1	1	1	1	1	1	1
		# coords	4	4	1	1	1	1	8	1	6	1

Table 1: The proof of Proposition 5.3: an example where $\mathrm{SSME}(\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}, \mu) \neq \mathrm{SME}(\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}, \mu)$. For brevity, we have grouped several coordinates of $\mathcal{K}$ in some columns. For example, the column labelled "$a \sim g, h$" corresponds to the two assertions $a \sim g$ and $a \sim h$, while "$\underset{c,d,e,f}{a\sim}$" corresponds to the four assertions: $a \sim c$, $a \sim d$, $a \sim e$, and $a \sim f$. The last row ("# coords") indicates the number of coordinates represented by each column.

Proof of Proposition 5.3. Let $\mathcal{A} := \{a,b,c,d,e,f,g,h\}$, and let $\mathcal{K}$ be the set of all cardinality-2 subsets of $\mathcal{A}$ (so $|\mathcal{K}| = 28$). For any equivalence relation "$\sim$" on $\mathcal{A}$, we define the element $\mathbf{x}^\sim \in \{\pm 1\}^{\mathcal{K}}$ by setting $x^\sim_{\{i,j\}} := 1$ if $i \sim j$, while $x^\sim_{\{i,j\}} := -1$ if $i \not\sim j$, for all $i, j \in \mathcal{A}$. Let $\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}} := \{\mathbf{x}^\sim;\ \text{"}\sim\text{" is any equivalence relation on } \mathcal{A}\}$.

Let $\mathbf{t} \in \mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}$ correspond to the "total" equivalence relation where all elements are equivalent (i.e. with a single equivalence class $\{a,b,c,d,e,f,g,h\}$). Let $\mathbf{u} \in \mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}$ correspond to the equivalence class partition $\{a,b,g,h\},\{c,d,e,f\}$. Let $\mathbf{v} \in \mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}$ correspond to the equivalence class partition $\{a,b,c,d,e,f\},\{g,h\}$. Let $\mu \in \Delta(\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}})$ be the profile

	$q \in (0, 0.1]$	$q \in (0.1, 0.3]$	$q \in (0.3, 0.6]$	$q \in (0.6, 1]$
$\gamma_{\mu,\mathbf{t}}(q)$	20	12	8	8
$\gamma_{\mu,\mathbf{u}}(q)$	20	20	16	8
$\gamma_{\mu,\mathbf{v}}(q)$	24	16	16	8
$\gamma_{\mu,\mathbf{w}}(q) = \gamma_{\mu,\mathbf{w}'}(q)$	21	17	15	7
$\gamma_{\mu,\mathbf{y}}(q)$	16	16	16	8
$\gamma_{\mu,\mathbf{z}_1}(q) = \cdots = \gamma_{\mu,\mathbf{z}_4}(q)$	17	17	15	7
$\gamma_{\mu,\rho}(q)$	22	18	16	8

Table 2: The proof of Proposition 5.3: an example where SSME $(\mathcal{X}_{\mathcal{A}}^{\mathrm{eq}}, \mu) \neq$ SME $(\mathcal{X}_{\mathcal{A}}^{\mathrm{eq}}, \mu)$.

such that $\mu(\mathbf{t}) = 0.2$, $\mu(\mathbf{u}) = 0.45$, $\mu(\mathbf{v}) = 0.35$, and $\mu(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{X}_{\mathcal{A}}^{\mathrm{eq}} \setminus \{\mathbf{t}, \mathbf{u}, \mathbf{v}\}$. The structure of μ and $\widetilde{\mu}$ is described in Table 1. For future reference, Table 1 also describes a few other elements of $\mathcal{X}_{\mathcal{A}}^{\mathrm{eq}}$, namely $\mathbf{w}$, $\mathbf{w}'$, $\mathbf{y}$, and $\mathbf{z}_1, \ldots, \mathbf{z}_4$. We will be particularly interested in $\mathbf{w}$ (which corresponds to the partition $\{a, c, d, e, f\}, \{b, g, h\}$) and $\mathbf{w}'$ (which corresponds to the partition $\{b, c, d, e, f\}, \{a, g, h\}$). The functions $\gamma_{\mathbf{t},\mu}$, $\gamma_{\mathbf{u},\mu}$, $\gamma_{\mathbf{v},\mu}$, $\gamma_{\mathbf{w},\mu} = \gamma_{\mathbf{w}',\mu}$ $\gamma_{\mathbf{y},\mu}$, and $\gamma_{\mathbf{z}_1,\mu} = \cdots = \gamma_{\mathbf{z}_4,\mu}$ are described in Table 2.

Claim 1: $\mathbf{w}, \mathbf{w}' \in$ SME $(\mathcal{X}_{\mathcal{A}}^{\mathrm{eq}}, \mu)$.

Proof: (by contradiction) Suppose not. Then there is some $\mathbf{x} \in \mathcal{X}_{\mathcal{A}}^{\mathrm{eq}}$ such that $\gamma_{\mu,\mathbf{x}} \geq \gamma_{\mu,\mathbf{w}} = \gamma_{\mu,\mathbf{w}'}$, with at least one strict inequality. This means that

$$\gamma_{\mathbf{x},\mu}(q) \;\geq\; \begin{cases} 7 & \text{if} \quad 1 \geq q > 0.6; \\ 15 & \text{if} \quad 0.6 \geq q > 0.3; \\ 17 & \text{if} \quad 0.3 \geq q > 0.1; \\ 21 & \text{if} \quad 0.1 \geq q > 0; \end{cases} \qquad \text{and at least one of these inequalities is strict.} \tag{D4}$$

We will show that this is impossible. There are three cases.

Case 1. If $\gamma_{\mu,\mathbf{x}}(1) = 8$, then $\mathbf{x}$ must support all the assertions "$a \sim b$", "$c \sim d \sim e \sim f$", and "$g \sim h$". Thus, $\mathbf{x}$ must be one of $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ or $\mathbf{y}$. But if $\gamma_{\mu,\mathbf{x}}(0.6) \geq 15$, then $\mathbf{x} \neq \mathbf{t}$. Likewise, if $\gamma_{\mu,\mathbf{x}}(0.3) \geq 17$, then $\mathbf{x} \neq \mathbf{v}$ or $\mathbf{y}$. And if $\gamma_{\mu,\mathbf{x}}(0.1) \geq 21$, then $\mathbf{x} \neq \mathbf{u}$. Contradiction.

Case 2. If $\gamma_{\mu,\mathbf{x}}(0.6) = 16$, then $\mathbf{x}$ must support all the assertions "$a \sim b$", "$c \sim d \sim e \sim f$", and "$g \sim h$", and *also* assert "$c, d, e, f \not\sim g, h$". Thus, $\mathbf{x}$ must be one of $\mathbf{u}$, $\mathbf{v}$ or $\mathbf{y}$. Now the analysis proceeds as in *Case 1.*

Case 3. Suppose $\gamma_{\mu,\mathbf{x}}(q) = 7$ for all $q \in (0.6, 1]$, and $\gamma_{\mu,\mathbf{x}}(q) = 15$ for all $q \in (0.3, 0.6]$. Then $\mathbf{x}$ must support all 8 of the coordinates encoding the assertions "$c, d, e, f \not\sim g, h$", and must support exactly 7 out of the 8 coordinates encoding the assertions "$a \sim b$", "$c \sim d \sim e \sim f$", and "$g \sim h$". Transitivity makes it impossible to assert only 5 out the 6 coordinates asserting "$c \sim d \sim e \sim f$". Thus:

$$\text{Either } \mathbf{x} \text{ asserts "}a \sim b\text{", or } \mathbf{x} \text{ asserts "}g \sim h\text{", but not both.} \tag{D5}$$

Now, also suppose that $\gamma_{\mu,\mathbf{x}}(0.3) \geq 17$. Then $\mathbf{x}$ must support at least two out of the four coordinates encoding the assertions "$a, b \sim g, h$". Transitivity and statement

(D5) imply that $\mathbf{x}$ encodes *exactly* two of these assertions. Thus, $\mathbf{x}$ must be one of $\mathbf{z}_1, \ldots, \mathbf{z}_4$, $\mathbf{w}$, or $\mathbf{w}'$. But from Table 2, it is clear that $\gamma_{\mu,\mathbf{z}_i}(q) = 17$ for all $q \in [0, 0.3)$ and all $i \in \{1, 2, 3, 4\}$. Thus, $\mathbf{z}_1, \ldots, \mathbf{z}_4$ violate the last inequality in (D4). Thus, $\mathbf{x}$ must be one of $\mathbf{w}$ or $\mathbf{w}'$. But then no inequality in (D4) is strict. Contradiction. $\diamond$ `Claim 1`

Now, let $\rho := \frac{1}{2}\delta_{\mathbf{u}} + \frac{1}{2}\delta_{\mathbf{v}}$; then $\rho \in \Delta(\mathcal{X})$, and $\gamma_{\rho,\mu} = (\gamma_{\mathbf{u},\mu} + \gamma_{\mathbf{v},\mu})/2$. From the last line of Table 2, we see that $\gamma_{\rho,\mu}(q) > \gamma_{\mathbf{w},\mu}(q) = \gamma_{\mathbf{w}',\mu}(q)$ for all $q \in [0, 1]$. Thus, $\rho \underset{\mu}{\triangleright} \delta_{\mathbf{w}}, \delta_{\mathbf{w}'}$; thus, $\mathbf{w}, \mathbf{w}' \notin \mathrm{SSME}\,(\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}, \mu)$. Thus, $\mathrm{SSME}\,(\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}, \mu) \neq \mathrm{SME}\,(\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}, \mu)$, so $\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}$ is not neat. $\square$

Remark. In the proof of Proposition 5.3, note that $\mathbf{w}$ and $\mathbf{w}'$ violate unanimity in the coordinate "$a \sim b$". This shows that *supermajority efficiency does not always respect unanimity on* $\mathcal{X}^{\mathrm{eq}}_{\mathcal{A}}$. (In contrast, it can be shown that supermajority efficiency *does* always respect unanimity on $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$.)

E Proofs from Section 6

Proof of Proposition 6.1. Let μ and ν be generic profiles in $\Delta(\mathcal{X})$ such that μ is majority determinate and ν is supermajority indeterminate. Let $\beta := \sup\{r \in [0, 1];\ r\nu + (1 - r)\,\mu$ is majority determinate$\} = \sup\{r \in [0, 1];\ \mathrm{Maj}\,(r\nu + (1 - r)\,\mu) \cap \mathcal{X} \neq \varnothing\}$. Clearly $\beta < 1$, and, for some $k \in \mathcal{K}$, we have

$$\beta\widetilde{\nu}_k + (1 - \beta)\,\widetilde{\mu}_k = 0.$$

Generically, this k is unique. In this case, it is easily verified that for sufficiently small $\epsilon > 0$ and $\alpha = \beta + \epsilon$, the profile $\alpha\nu + (1 - \alpha)\,\mu$ is barely Condorcet inconsistent. $\square$

Proof of Proposition 6.2. "(a) $\iff$ (c)" Let $\mu \in \Delta(\mathcal{X})$, and let $\mathbf{x} \in \mathrm{SME}\,(\mathcal{X}, \mu)$. Then for all $\mathbf{y} \in \mathcal{X}$, we have $\mathbf{y} \in \mathrm{SME}\,(\mathcal{X}, \mu)$ iff $\mathbf{y} \underset{\mu}{\equiv} \mathbf{x}$, because $\mathcal{X}$ is supermajority determinate. Thus, F satisfies SME and SMEQ if and only if $F(\mu) = \mathrm{SME}\,(\mathcal{X}, \mu)$ for all $\mu \in \Delta(\mathcal{X})$.

"(d) $\Longrightarrow$ (a)" F_ϕ is SME by Lemma C.8, and F_ϕ is SMEQ by Lemma C.7(c) and supermajority determinacy.

"(a) $\Longrightarrow$ (d)" F_ϕ is SME by Lemma C.8, while F is SME by hypothesis. Thus, for any $\mu \in \Delta(\mathcal{X})$, any $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$, and any $\mathbf{y} \in F(\mu)$, we have $\mathbf{y} \underset{\mu}{\equiv} \mathbf{x}$, because $\mathcal{X}$ is supermajority determinate. Thus, $F_\phi(\mathcal{X}, \mu) \subseteq F_\phi(\mu)$ because F is SMEQ by hypothesis; conversely, $F(\mu) \subseteq F_\phi(\mathcal{X}, \mu)$ because F_ϕ is SMEQ by Lemma C.7(c).

"(b) $\Longrightarrow$ (d)" Let $\mu \in \Delta(\mathcal{X})$. Lemma C.8 says $F_\phi(\mathcal{X}, \mu) \subseteq \mathrm{SME}\,(\mathcal{X}, \mu)$. But $\mathcal{X}$ is supermajoritary determinate, so $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y} \in \mathrm{SME}\,(\mathcal{X}, \mu)$. Thus, Lemma

C.7(c) implies that $F_\phi(\mathcal{X},\mu) = \mathrm{SME}(\mathcal{X},\mu)$. Thus, if $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is any other supermajority efficient rule, then $F(\mu) \subseteq F_\phi(\mathcal{X},\mu)$.

This argument holds for all $\mu \in \Delta(\mathcal{X})$. However, F_ϕ is monotone, by Proposition 3.4(a). Thus, if F is also upper hemicontinuous, then Lemma C.9 says that $F = F_\phi$.

"(d) $\Longrightarrow$(b)" If $F = F_\phi$ for *any* gain function ϕ, then in particular this holds for any continuous, real-valued gain functin ϕ. Then Lemma C.8 says F is SME, while Proposition 3.5 say F is uhc. □

The proof of Theorem 6.3 makes use of a key technical result (Proposition F.5). The proof of Proposition F.5 is rather lengthy and involves several auxiliary results, so it is relegated to Appendix F (below). The proof of Theorem 6.3 also uses the next result, which is of independent interest.

Proposition E.1 *Let $(\mathcal{X},\mu)$ be a judgement problem. The following are equivalent.*

(a) *$(\mathcal{X},\mu)$ is supermajority determinate.*

(b) *For any $\mathbf{x},\mathbf{y} \in \mathrm{SSME}(\mathcal{X},\mu)$, we have $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$.*

(c) *$F_\phi(\mathcal{X},\mu) = F_\psi(\mathcal{X},\mu)$ for any gain functions $\phi,\psi : [-1,1]\longrightarrow {}^*\mathbb{R}$.*

(d) *$F_\phi(\mathcal{X},\mu) = F_\psi(\mathcal{X},\mu)$ for any continuous, real-valued gain functions $\phi,\psi : [-1,1]\longrightarrow\mathbb{R}$.*

Proof: "(a)$\Longrightarrow$(b)" follows from Proposition 5.1 and the definition of supermajority determinacy. "(b) $\Longrightarrow$ (c)" follows from Proposition 5.2(a) and Lemma C.7(c). "(c) $\Longrightarrow$ (d)" is obvious. "(d) $\Longrightarrow$ (b)" follows from Proposition 5.2(b) (by contrapositive). It remains to prove "(b)$\Longrightarrow$(a)". So, suppose $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$ for all $\mathbf{x},\mathbf{y} \in \mathrm{SSME}(\mathcal{X},\mu)$.

Claim 1: $\mathrm{SME}(\mathcal{X},\mu) \subseteq \mathrm{SSME}(\mathcal{X},\mu)$.

Proof: (by contrapositive) If $\mathbf{z} \in \mathcal{X}\setminus\mathrm{SSME}(\mathcal{X},\mu)$, then there exists $\rho \in \Delta(\mathcal{X})$ such that $\mathbf{z} \underset{\mu}{\lhd} \rho$ —i.e. such that $\gamma_{\mu,\mathbf{z}} \le \gamma_{\mu,\rho}$ with at least one strict inequality. Without loss of generality, suppose that ρ is $(\underset{\mu}{\rhd})$-maximal in $\Delta(\mathcal{X})$. Then $\mathrm{supp}(\rho) \subseteq \mathrm{SSME}(\mathcal{X},\mu)$. Fix $\mathbf{y} \in \mathrm{SSME}(\mathcal{X},\mu)$; then our hypothesis says that $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$ for all other $\mathbf{x} \in \mathrm{SSME}(\mathcal{X},\mu)$. Thus, $\gamma_{\mu,\rho} = \gamma_{\mu,\mathbf{y}}$ (because $\gamma_{\mu,\rho}$ is a convex combination of $\{\gamma_{\mu,\mathbf{x}};\ \mathbf{x} \in \mathrm{SSME}(\mathcal{X},\mu)\}$). Thus, $\gamma_{\mu,\mathbf{z}} \le \gamma_{\mu,\mathbf{y}}$, with at least one strict inequality. Thus, $\mathbf{z} \underset{\mu}{\lhd} \mathbf{y}$. Thus, $\mathbf{z} \notin \mathrm{SME}(\mathcal{X},\mu)$. ◇ Claim 1

Claim 1 implies that $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$ for all $\mathbf{x},\mathbf{y} \in \mathrm{SME}(\mathcal{X},\mu)$. Thus, $(\mathcal{X},\mu)$ is supermajority determinate. □

Proof of Theorem 6.3. "$\Longrightarrow$" If $\mu \in \Delta(\mathcal{X})$ and $\mathrm{Median}(\mathcal{X},\mu) = \{\mathbf{x},\mathbf{y}\}$, then $\{\mathbf{x},\mathbf{y}\} \subseteq \mathrm{SME}(\mathcal{X},\mu)$. Thus, $\gamma_{\mathbf{x},\mu} = \gamma_{\mathbf{y},\mu}$, because $\mathcal{X}$ is supermajority determinate.

"$\Longleftarrow$" Suppose $\mathcal{X}$ is friendly. Let $\phi : [-1,1]\longrightarrow\mathbb{R}$ be any continuous, real-valued gain function. For any $\mathbf{x},\mathbf{y} \in \mathcal{X}$, define ${}^{o}\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathcal{C};\ \mathrm{Median}(\mathcal{X},\mathbf{c}) = \{\mathbf{x},\mathbf{y}\}\}$, and define $\mathcal{C}^\phi_{\mathbf{x}} := \{\mathbf{c} \in \mathcal{C}\ ;\ \mathbf{x} \in F_\phi(\mathbf{c})\}$ and $\mathcal{B}^\phi_{\mathbf{x},\mathbf{y}} := \mathcal{C}^\phi_{\mathbf{x}} \cap \mathcal{C}^\phi_{\mathbf{y}}$.

Claim 1: ${}^{o}\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^\phi_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^\phi_{\mathbf{x}})$.

Proof: Let $\mathbf{b} \in {}^o\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}}$; then $\mathbf{b} = \widetilde{\mu}$ for some $\mu \in \Delta(\mathcal{X})$ such that $\text{Median}(\mathcal{X}, \mu) = \{\mathbf{x}, \mathbf{y}\}$. Thus, the simplicity of $\mathcal{X}$ implies that $\gamma_{\mathbf{x},\mu} = \gamma_{\mathbf{y},\mu}$. There are now two cases:

- If $\mathbf{x} \in F_\phi(\mathcal{X}, \mu)$, then Lemma C.7(c) says that $\mathbf{y} \in F_\phi(\mathcal{X}, \mu)$ also; thus, $\mathbf{b} \in \mathcal{B}^\phi_{\mathbf{x},\mathbf{y}}$.
- If $\mathbf{x} \notin F_\phi(\mathcal{X}, \mu)$, then $\mathbf{b} \in \mathcal{C} \setminus \mathcal{C}^\phi_{\mathbf{x}}$.

Thus, $\mathbf{b} \in \mathcal{B}^\phi_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^\phi_{\mathbf{x}})$. This holds for all $\mathbf{b} \in {}^o\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}}$. ◇ Claim 1

Proposition F.5 and Claim 1 together imply that $F_\phi(\mathcal{X}, \mu) = \text{Median}(\mathcal{X}, \mu)$ for all $\mu \in \Delta(\mathcal{X})$. But this argument works if ϕ is any continuous, real-valued gain function. Thus, $F_\phi(\mathcal{X}, \mu) = F_\psi(\mathcal{X}, \mu)$ for all continuous, real-valued gain functions $\phi, \psi : [-1, 1] \longrightarrow \mathbb{R}$. Thus, Proposition E.1 says that $\mathcal{X}$ is supermajority determinate. □

Proof of Theorem 6.4. (c) follows immediately by combining (a) and (b).

(a) Suppose $\mathcal{X}$ is proximal. We claim that $\mathcal{X}$ is friendly. To see this, let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, and let $\mu \in \Delta(\mathcal{X})$ with $\text{Median}(\mathcal{X}, \mu) = \{\mathbf{x}, \mathbf{y}\}$. Then the vector $\widetilde{\mu}$ supports the edge $\{\mathbf{x}, \mathbf{y}\}$. Thus, $\widetilde{\mu} \bullet (\mathbf{x} - \mathbf{y}) = 0$, and $\mathbf{x}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{y}$. Thus, $d(\mathbf{x}, \mathbf{y}) \leq 2$ (by proximality). There are now two cases.

Case 1. Suppose $d(\mathbf{x}, \mathbf{y}) = 1$. Then $\mathcal{K}_\pm(\mathbf{x}, \mathbf{y}) = \{i\}$ for some $i \in \mathcal{K}$. Thus, $0 = \widetilde{\mu} \bullet (\mathbf{x} - \mathbf{y}) = \widetilde{\mu}_i(x_i - y_i)$, while $x_i \neq y_i$; thus, $\widetilde{\mu}_i = 0$. Thus, $\widetilde{\mu}_i x_i = \widetilde{\mu}_i y_i$. Meanwhile, clearly, $\widetilde{\mu}_k x_k = \widetilde{\mu}_k y_k$ for all $k \in \mathcal{K} \setminus \mathcal{K}_\pm(\mathbf{x}, \mathbf{y})$. Thus, $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$.

Case 2. Suppose $d(\mathbf{x}, \mathbf{y}) = 2$. Then $\mathcal{K}_\pm(\mathbf{x}, \mathbf{y}) = \{i, j\}$ for some distinct $i, j \in \mathcal{K}$. Thus, $0 = \widetilde{\mu} \bullet (\mathbf{x} - \mathbf{y}) = \widetilde{\mu}_i(x_i - y_i) + \widetilde{\mu}_j(x_j - y_j)$. Thus, $\widetilde{\mu}_i(x_i - y_i) = -\widetilde{\mu}_j(x_j - y_j)$. There are now two subcases.

- If $x_i = x_j = -y_i = -y_j$, then $\widetilde{\mu}_i = -\widetilde{\mu}_j$. Thus, $\widetilde{\mu}_i x_i = \widetilde{\mu}_j y_j$, while $\widetilde{\mu}_i y_i = \widetilde{\mu}_j x_j$.
- If $x_i = -x_j = -y_i = y_j$, then $\widetilde{\mu}_i = \widetilde{\mu}_j$. Thus, $\widetilde{\mu}_i x_i = \widetilde{\mu}_j y_j$, while $\widetilde{\mu}_i y_i = \widetilde{\mu}_j x_j$.

In either case, we also have $\widetilde{\mu}_k x_k = \widetilde{\mu}_k y_k$ for all $k \in \mathcal{K} \setminus \mathcal{K}_\pm(\mathbf{x}, \mathbf{y})$. Thus, $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$.

In both cases, $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$. This holds whenever $\mathbf{x}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{y}$. We conclude that $\mathcal{X}$ is friendly. Thus, Theorem 6.3 says that $\mathcal{X}$ is supermajority determinate.

(b) (by contrapositive) Suppose that $\mathcal{X}$ is also thick but *not* proximal. We will show that $\mathcal{X}$ is not friendly, and thus, not supermajority efficient.

Let $\mathbf{x} \in \mathcal{X}$. For any $\mathbf{c} \in \mathcal{C}$, if $\mathbf{c} = \widetilde{\mu}$ for some $\mu \in \Delta(\mathcal{X})$, then define $\gamma_{\mathbf{c},\mathbf{x}} := \gamma_{\mu,\mathbf{x}}$. Given some other $\mathbf{y} \in \mathcal{X}$, we define $\mathcal{D}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathcal{C};\ \ \gamma_{\mathbf{c},\mathbf{x}} = \gamma_{\mathbf{c},\mathbf{y}}\}$. Note that $\mathcal{X}$ is friendly if and only if ${}^o\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{D}_{\mathbf{x},\mathbf{y}}$ for all $\mathbf{x}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{y}$ in $\mathcal{X}$.

Claim 1: *For any distinct* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, *we have* $\dim(\mathcal{D}_{\mathbf{x},\mathbf{y}}) \leq K - \lceil d(\mathbf{x}, \mathbf{y})/2 \rceil$.

Proof: Let $\mathcal{J} := \mathcal{K}_\pm(\mathbf{x}, \mathbf{y})$, and for any $\mathbf{c} \in \mathcal{C}$, let $\mathbf{c}_{\mathcal{J}} := (c_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$.

Let Π be the set of all permutations π of $\mathcal{J}$ such that $\mathcal{J} = \mathcal{J}_0 \sqcup \mathcal{J}_1 \sqcup \mathcal{J}_2 \sqcup \cdots \sqcup \mathcal{J}_N$, where $\pi(j) = j$ for all $j \in \mathcal{J}_0$, and where each of $\mathcal{J}_1, \ldots, \mathcal{J}_N$ is a π-orbit of even cardinality. For any $\pi \in \Pi$ and $\mathbf{c}_{\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$, define $\pi(\mathbf{c}_{\mathcal{J}}) := (c_{\pi(j)})_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$. Then define $\mathcal{D}^\pi_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathcal{C};\ \pi(\mathbf{c}_{\mathcal{J}}) = -\mathbf{c}_{\mathcal{J}}\}$.

Claim 1.1: $\dim(\mathcal{D}^\pi_{\mathbf{x},\mathbf{y}}) \leq K - \lceil d(\mathbf{x}, \mathbf{y})/2 \rceil$.

Proof: Suppose π has orbit decomposition $\mathcal{J} = \mathcal{J}_0 \sqcup \mathcal{J}_1 \sqcup \mathcal{J}_2 \sqcup \cdots \sqcup \mathcal{J}_N$, as described above. Then $\mathcal{D}^\pi_{\mathbf{x},\mathbf{y}}$ is the set of all $\mathbf{c} \in \mathcal{C}$ satisfying the following linear constraints:

(a) $c_j = 0$ for all $j \in \mathcal{J}_0$.

(b) For all $n \in [1 \ldots N]$, if $\mathcal{J}_n = \{j, \pi(j), \pi^2(j), \ldots, \pi^L(j)\}$ (for some $j \in \mathcal{J}$ and some odd $L \in \mathbb{N}$), then $c_j = -c_{\pi(j)} = c_{\pi^2(j)} = -c_{\pi^3(j)} = \cdots = -c_{\pi^L(j)}$.

Let $J_n := |\mathcal{J}_n|$ for all $n \in [0 \ldots N]$. Then (a) imposes J_0 linear constraints on $\mathbf{c}$, whereas (b) imposes $(J_n - 1)$ distinct linear constraints on $\mathbf{c}$ for each $n \in [1 \ldots N]$. These constraints are all linearly independent, because they involve distinct coordinats of $\mathbf{c}$. Thus, the total number of linearly independent constraints on $\mathbf{c}$ is

$$J_0 + (J_1 - 1) + \cdots + (J_N - 1) \;=\; J_0 + J_1 + \cdots + J_N - N \;=\; J - N,$$

where $J := |\mathcal{J}| = d(\mathbf{x}, \mathbf{y})$. Thus, $\dim(\mathcal{D}^\pi_{\mathbf{x},\mathbf{y}}) \leq K - J + N$. But $N \leq \lfloor J/2 \rfloor$, because $|\mathcal{J}_n| \geq 2$ for all $n \in [1 \ldots N]$. Thus, $K - J + N \leq K - J + \lfloor J/2 \rfloor = K - \lceil J/2 \rceil$.
$\bigtriangledown$ Claim 1.1

It is easy to check that $\gamma_{\mathbf{c},\mathbf{x}} = \gamma_{\mathbf{c},\mathbf{y}}$ if and only if $\pi(\mathbf{c}) = -\mathbf{c}$ for some $\pi \in \Pi$. Thus, $\mathcal{D}_{\mathbf{x},\mathbf{y}} = \bigcup_{\pi \in \Pi} \mathcal{D}^\pi_{\mathbf{x},\mathbf{y}}$, a finite union of sets of dimension $K - \lceil d(\mathbf{x}, \mathbf{y})/2 \rceil$ or less. Thus, $\dim(\mathcal{D}_{\mathbf{x},\mathbf{y}}) \leq K - \lceil d(\mathbf{x}, \mathbf{y})/2 \rceil$. $\diamondsuit$ Claim 1

Now, if $\mathcal{X}$ is *not* proximal, then there exist $\mathbf{x} \; \mathrm{I}_{\mathcal{X}} \; \mathbf{y} \in \mathcal{X}$ with $d(\mathbf{x}, \mathbf{y}) \geq 3$. Let $\mathcal{B}^m_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathcal{C};\; \{\mathbf{x}, \mathbf{y}\} \subseteq \text{Median}\,(\mathcal{X}, \mathbf{c})\}$. If $\mathcal{H}_0 := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}};\; (\mathbf{x} - \mathbf{y}) \bullet \mathbf{r} = 0\}$, then $\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{C} \cap \mathcal{H}_0$.

Claim 2: $\mathcal{C} \cap \mathcal{H}_0 \not\subseteq \mathcal{D}_{\mathbf{x},\mathbf{y}}$.

Proof: Let $\mathcal{H}_+ := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}};\; (\mathbf{x} - \mathbf{y}) \bullet \mathbf{r} > 0\}$, and let $\mathcal{H}_- := \{\mathbf{r} \in \mathbb{R}^{\mathcal{K}};\; (\mathbf{x} - \mathbf{y}) \bullet \mathbf{r} < 0\}$. Then $\mathbf{x} \in \mathcal{C} \cap \mathcal{H}_+$, and $\mathbf{y} \in \mathcal{C} \cap \mathcal{H}_-$. Thus, $\mathcal{C} \cap \mathcal{H}_+$ contains a relatively open neighbourhood around $\mathbf{x}$, while $\mathcal{C} \cap \mathcal{H}_-$ contains a relatively open neighbourhood around $\mathbf{y}$ (because $\mathcal{H}^+$ and $\mathcal{H}^-$ are open subsets of $\mathbb{R}^{\mathcal{K}}$.). But $\mathcal{C}$ is thick; thus $\mathcal{C} \cap \mathcal{H}_+$ and $\mathcal{C} \cap \mathcal{H}_-$ have nonempty interiors in $\mathbb{R}^{\mathcal{K}}$. Any line segment from any point in $\mathcal{C} \cap \mathcal{H}_+$ to any point $\mathcal{C} \cap \mathcal{H}_-$ must pass through $\mathcal{C} \cap \mathcal{H}_0$ somewhere. Thus, $\mathcal{C} \cap \mathcal{H}_0$ is a relatively open subset of $\mathcal{H}_0$; hence $\dim(\mathcal{C} \cap \mathcal{H}_0) = \dim(\mathcal{H}_0) = K - 1 > K - \lceil 3/2 \rceil \geq K - \lceil d(\mathbf{x}, \mathbf{y})/2 \rceil$. Thus, Claim 1 implies that $\mathcal{C} \cap \mathcal{H}_0 \not\subseteq \mathcal{D}_{\mathbf{x},\mathbf{y}}$. $\diamondsuit$ Claim 2

Recall that ${}^o\mathcal{B}^m_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathcal{C};\; \text{Median}\,(\mathcal{X}, \mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.

Claim 3: ${}^o\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \not\subseteq \mathcal{D}_{\mathbf{x},\mathbf{y}}$.

Proof: Let $\mathbf{b} \in {}^o\mathcal{B}^m_{\mathbf{x},\mathbf{y}}$. If $\mathbf{b} \notin \mathcal{D}_{\mathbf{x},\mathbf{y}}$, then we're done. So suppose $\mathbf{b} \in \mathcal{D}_{\mathbf{x},\mathbf{y}}$. Claim 2 yields some $\mathbf{c} \in (\mathcal{C} \cap \mathcal{H}_0) \setminus \mathcal{D}_{\mathbf{x},\mathbf{y}}$. For any $r \in [0, 1]$, define $\mathbf{c}^r := r\mathbf{c} + (1 - r)\mathbf{b}$. Observe that $\mathbf{c}^r \in \mathcal{C} \cap \mathcal{H}_0$, because $\mathbf{b} \in \mathcal{C} \cap \mathcal{H}_0$ and $\mathbf{c} \in \mathcal{C} \cap \mathcal{H}_0$, while $\mathcal{H}_0$ and $\mathcal{C}$ are convex.

Now, Median$\,(\mathcal{X}, \mathbf{b}) = \{\mathbf{x}, \mathbf{y}\}$, and the median rule is upper hemicontinuous (by Proposition 3.5). Thus, if r is close enough to zero, then Median$\,(\mathcal{X}, \mathbf{c}^r) \subseteq \{\mathbf{x}, \mathbf{y}\}$. But since $\mathbf{c}^r \in \mathcal{H}_0$, this implies that Median$\,(\mathcal{X}, \mathbf{c}^r) = \{\mathbf{x}, \mathbf{y}\}$; hence $\mathbf{c}^r \in {}^o\mathcal{B}^m_{\mathbf{x},\mathbf{y}}$. Finally, since $\mathbf{c} \notin \mathcal{D}_{\mathbf{xy}}$ we can choose r such that $\mathbf{c}^r \notin \mathcal{D}_{\mathbf{xy}}$. $\diamondsuit$ Claim 3

Claim 3 means that $\mathcal{X}$ is *not* friendly. Thus, the contrapositive of Theorem 6.3 says that $\mathcal{X}$ is *not* supermajority determinate. □

Proof of Proposition 6.5. Let $\mathbf{x}, \mathbf{z} \in \mathcal{X}_{I,J}^{\text{com}}$ with $d(\mathbf{x}, \mathbf{z}) \geq 3$; we must show that not($\mathbf{x} \, \mathrm{I}_{\mathcal{X}} \, \mathbf{z}$). It suffices to show that the line from $\mathbf{x}$ to $\mathbf{y}$ is not an edge of the polytope conv($\mathcal{X}$). Let

$$\begin{array}{llllll} \mathcal{K}_{++} & := & \{k \in \mathcal{K} \, ; \, x_k = 1 = z_k\}, & & \mathcal{K}_{+-} := & \{k \in \mathcal{K} \, ; \, x_k = 1, \; z_k = -1\} \\ \mathcal{K}_{--} & := & \{k \in \mathcal{K} \, ; \, x_k = -1 = z_k\}, & \text{and} & \mathcal{K}_{-+} := & \{k \in \mathcal{K} \, ; \, x_k = -1, \; z_k = 1\}. \end{array}$$

Let $K_{+-} := |\mathcal{K}_{+-}|$ and $K_{-+} := |\mathcal{K}_{-+}|$. Then we have $|\mathbf{x}| = |\mathcal{K}_{++}| + K_{+-}$ and $|\mathbf{z}| = |\mathcal{K}_{++}| + K_{-+}$. But $I \leq |\mathbf{x}| \leq J$ and $I \leq |\mathbf{z}| \leq J$ (by definition of $\mathcal{X}_{I,J}^{\text{com}}$); thus,

$$I - |\mathcal{K}_{++}| \;\leq\; K_{+-} \;\leq\; J - |\mathcal{K}_{++}| \quad \text{and} \quad I - |\mathcal{K}_{++}| \;\leq\; K_{-+} \;\leq\; J - |\mathcal{K}_{++}|. \qquad \text{(E1)}$$

Let $\mathcal{K}_{+-}^1, \mathcal{K}_{+-}^2 \subseteq \mathcal{K}_{+-}$ be two disjoint subsets such that $\mathcal{K}_{+-}^1 \sqcup \mathcal{K}_{+-}^2 = \mathcal{K}_{+-}$, and such that $|\mathcal{K}_{+-}^1| = \lfloor K_{+-}/2 \rfloor$ and $|\mathcal{K}_{+-}^2| = \lceil K_{+-}/2 \rceil$. (Thus, if $K_{+-} \geq 2$, then both sets are nonempty. If $K_{+-} = 1$, then $\mathcal{K}_{+-}^1 = \emptyset$ and $\mathcal{K}_{+-}^2 = \mathcal{K}_{+-}$. If $K_{+-} = 0$, then $\mathcal{K}_{+-}^1 = \emptyset = \mathcal{K}_{+-}^2$.) Likewise, let $\mathcal{K}_{-+}^1, \mathcal{K}_{-+}^2 \subseteq \mathcal{K}_{-+}$ be two disjoint subsets with $\mathcal{K}_{-+}^1 \sqcup \mathcal{K}_{-+}^2 = \mathcal{K}_{-+}$, and such that $|\mathcal{K}_{-+}^1| = \lfloor K_{-+}/2 \rfloor$ and $|\mathcal{K}_{-+}^2| = \lceil K_{-+}/2 \rceil$. Suppose without loss of generality that $K_{+-} \leq K_{-+}$. Then $|\mathcal{K}_{+-}^1| \leq |\mathcal{K}_{-+}^1|$ and $|\mathcal{K}_{+-}^2| \leq |\mathcal{K}_{-+}^2|$. Thus

$$\left.\begin{array}{lllllllll} K_{+-} & = & |\mathcal{K}_{+-}^1| + |\mathcal{K}_{+-}^2| & \leq & |\mathcal{K}_{+-}^1| + |\mathcal{K}_{-+}^2| & \leq & |\mathcal{K}_{-+}^1| + |\mathcal{K}_{-+}^2| & = & K_{-+}. \\ K_{+-} & = & |\mathcal{K}_{+-}^1| + |\mathcal{K}_{+-}^2| & \leq & |\mathcal{K}_{-+}^1| + |\mathcal{K}_{+-}^2| & \leq & |\mathcal{K}_{-+}^1| + |\mathcal{K}_{-+}^2| & = & K_{-+}. \end{array}\right\} \quad \text{(E2)}$$

Combining the inequalities (E1) and (E2), we obtain

$$\left.\begin{array}{rlllll} & I - |\mathcal{K}_{++}| & \leq & |\mathcal{K}_{+-}^1| + |\mathcal{K}_{-+}^2| & \leq & J - |\mathcal{K}_{++}|, \\ \text{and} & I - |\mathcal{K}_{++}| & \leq & |\mathcal{K}_{+-}^2| + |\mathcal{K}_{-+}^1| & \leq & J - |\mathcal{K}_{++}|. \end{array}\right\} \quad \text{(E3)}$$

Now define $\mathbf{y}^1, \mathbf{y}^2 \in \{\pm 1\}^{\mathcal{K}}$ as follows:

$$\left.\begin{array}{rl} y_k^1 := & \begin{cases} 1 & \text{if} \quad k \in \mathcal{K}_{+-}^1 \sqcup \mathcal{K}_{++} \sqcup \mathcal{K}_{-+}^2; \\ -1 & \text{if} \quad k \in \mathcal{K}_{+-}^2 \sqcup \mathcal{K}_{--} \sqcup \mathcal{K}_{-+}^1; \end{cases} \\ \text{and} \quad y_k^2 := & \begin{cases} 1 & \text{if} \quad k \in \mathcal{K}_{+-}^2 \sqcup \mathcal{K}_{++} \sqcup \mathcal{K}_{-+}^1; \\ -1 & \text{if} \quad k \in \mathcal{K}_{+-}^1 \sqcup \mathcal{K}_{--} \sqcup \mathcal{K}_{-+}^2. \end{cases} \end{array}\right\} \quad \text{(E4)}$$

Then the inequalities (E3) imply that $I \leq |\mathbf{y}^1| \leq J$ and $I \leq |\mathbf{y}^2| \leq J$, so $\mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}_{I,J}^{\text{com}}$.

Furthermore, note that, $K_{+-} + K_{-+} = d(\mathbf{x}, \mathbf{z}) \geq 3$, so either $K_{+-} \geq 2$ or $K_{-+} \geq 2$. Thus, either $\mathcal{K}_{+-}^1 \neq \emptyset \neq \mathcal{K}_{+-}^2$, or $\mathcal{K}_{-+}^1 \neq \emptyset \neq \mathcal{K}_{-+}^2$. Either way, it is clear from the defining equations (E3) that $\mathbf{y}^1, \mathbf{y}^2 \notin \{\mathbf{x}, \mathbf{z}\}$. However, for every $k \in \mathcal{K}$, we have

$$y_k^1 + y_k^2 \quad = \quad \left\{\begin{array}{rll} 2 & \text{if} & k \in \mathcal{K}_{++} \\ 0 & \text{if} & k \in \mathcal{K}_{+-} \sqcup \mathcal{K}_{-+} \\ -2 & \text{if} & k \in \mathcal{K}_{--} \end{array}\right\} \quad = \quad x_k + z_k.$$

Thus, $(\mathbf{y}^1 + \mathbf{y}^2)/2 = (\mathbf{x} + \mathbf{z})/2$, so $\operatorname{conv}\{\mathbf{x}, \mathbf{z}\} \cap \operatorname{conv}\{\mathbf{y}^1, \mathbf{y}^2\} \neq \emptyset$, so the line segment $\operatorname{conv}\{\mathbf{x}, \mathbf{z}\}$ is *not* an edge of the polytope conv($\mathcal{X}$); hence it can't be an internal edge. Thus, it is false that $\mathbf{x} \, \mathrm{I}_{\mathcal{X}} \, \mathbf{z}$. This argument works whenever $d(\mathbf{x}, \mathbf{y}) \geq 3$. Thus, $\mathcal{X}_{I,J}^{\text{com}}$ is proximal. □

Proof of Proposition 6.6. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \triangle_M^D$, so that $\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}, \widetilde{\mathbf{z}} \in \mathcal{X}_{D,M}^{\triangle}$ (as defined by equation (12)). We say $\mathbf{y}$ is ***between*** $\mathbf{x}$ and $\mathbf{z}$ if, for every $d \in [1 \ldots D]$, we have either $x_d \leq y_d \leq z_d$ or $x_d \geq y_d \geq z_d$.

Now suppose $\widetilde{\mathbf{x}} \, \mathrm{I}_{\mathcal{X}} \, \widetilde{\mathbf{z}}$. Thus, there exists some $\mu \in \Delta(\mathcal{X}_{D,M}^{\triangle})$ such that $\mathrm{Median}\left(\mathcal{X}_{D,M}^{\triangle}, \mu\right) = \{\mathbf{x}, \mathbf{z}\}$. Proposition 1 of Lindner et al. (2010) says that $\mathrm{Median}\left(\mathcal{X}_{D,M}^{\triangle}, \mu\right)$ is a *convex* subset of $\mathcal{X}_{D,M}^{\triangle}$, which means that, for any $\mathbf{y} \in \triangle_M^D$, if $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$, then $\widetilde{\mathbf{y}} \in \mathrm{Median}\left(\mathcal{X}_{D,M}^{\triangle}, \mu\right)$ also. Thus, if $\mathrm{Median}\left(\mathcal{X}_{D,M}^{\triangle}, \mu\right) = \{\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}\}$, then there are no other elements of $\triangle_M^D$, between $\mathbf{x}$ and $\mathbf{z}$. This means that there exist some $c, e \in [1 \ldots D]$ such that $x_c = z_c - 1$, $x_e = z_e + 1$, and $x_d = z_d$ for all $d \in [1 \ldots D] \setminus \{c, e\}$. Suppose $z_c = \ell$ and $x_e = n$. Inspecting equation (12), we see that $\widetilde{x}_{(c,\ell)} = -1 \neq \widetilde{z}_{(c,\ell)} = 1$ and $\widetilde{x}_{(e,n)} = 1 \neq \widetilde{z}_{(e,n)} = -1$, whereas $\widetilde{x}_{(d,m)} = \widetilde{z}_{(d,m)}$ for all other $(d, m) \in [1 \ldots D] \times [1 \ldots M] \setminus \{(c, \ell), (e, m)\}$. Thus, $d(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}) = 2$. This argument works whenever $\widetilde{\mathbf{x}} \, \mathrm{I}_{\mathcal{X}} \, \widetilde{\mathbf{z}}$. Thus, $\mathcal{X}_{D,M}^{\triangle}$ is proximal. □

Proof of Proposition 6.7(a). We will prove the case $|\mathcal{A}| = 4$; the proof for larger $|\mathcal{A}|$ is similar. So, let $\mathcal{A} = \{1, 2, 3, 4\}$. For any distinct $a, b, c, d \in \mathcal{A}$, let $\mathbf{x}^{abcd} \in \mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ be the element representing the ordering $a \succ b \succ c \succ d$. Observe that $d(\mathbf{x}^{1234}, \mathbf{x}^{4123}) = 3$ (because these two elements differ on exactly the propositions $1 \succ 4$, $2 \succ 4$, and $3 \succ 4$). However, we will show that $\mathbf{x}^{1234} \, \mathrm{I}_{\mathcal{X}} \, \mathbf{x}^{4123}$.

Let $\mathbf{c} := (\mathbf{x}^{1234} + \mathbf{x}^{4123})/2$. Then $\mathrm{Median}\,(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}, \mathbf{c}) = \{$all elements of $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ between $\mathbf{x}^{1234}$ and $\mathbf{x}^{4123}\}$ $=$ $\{$all elements of $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ such that $1 \succ 2 \succ 3\}$ $=$ $\{\mathbf{x}^{1234}, \mathbf{x}^{4123}, \mathbf{x}^{1243}, \mathbf{x}^{1423}\}$. Now, for any $\epsilon > 0$, define $\mathbf{c}^{\epsilon} := (1 - \epsilon)\,\mathbf{c} + \epsilon\,(\mathbf{x}^{3421} + \mathbf{x}^{3241})/2$. If ϵ is small enough, then $\mathrm{Median}\,(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}, \mathbf{c}) \subseteq \{\mathbf{x}^{1234}, \mathbf{x}^{4123}, \mathbf{x}^{1243}, \mathbf{x}^{1423}\}$, by upper hemicontinuity (Proposition 3.5). But it is easy to check that

$$\mathbf{c}^{\epsilon} \bullet \mathbf{c}^{1243} \;<\; \mathbf{c}^{\epsilon} \bullet \mathbf{x}^{1234} \;=\; \mathbf{c}^{\epsilon} \bullet \mathbf{x}^{4123} \;>\; \mathbf{c}^{\epsilon} \bullet \mathbf{x}^{1423}$$

Thus, $\mathrm{Median}\,(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}, \mathbf{c}) = \{\mathbf{x}^{1234}, \mathbf{x}^{4123}\}$. Thus, $\mathbf{x}^{1234} \, \mathrm{I}_{\mathcal{X}} \, \mathbf{x}^{4123}$.

Thus, $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is not proximal. But $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is thick (Nehring and Pivato, 2011, Example 3.3), so the contrapositive of Theorem 6.4(c) says $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is *not* supermajority determinate. □

Proposition E.2 *Let $\mathcal{S}$ be a finite set, with $S := |\mathcal{S}| \geq 4$. Let $\mathcal{X} := \{\mathbf{x}^s;\ s \in \mathcal{S}\}$. Then $\mathcal{X}$ is supermajority determinate.*

Proof: Let $\mu \in \Delta(\mathcal{X})$. We claim that $\mathrm{SME}\,(\mathcal{X}, \mu) = \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x})$. We will use the following easily verified fact:

$$\text{For any } k \in \mathcal{K}, \qquad \widetilde{\mu}_k \quad := \quad 2 \sum_{s \in k} \mu(\mathbf{x}^s) \; - \; 1. \tag{E5}$$

Claim 1: *Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.*

(a) *If $\mu(\mathbf{x}) = \mu(\mathbf{y})$, then $\gamma_{\mathbf{x},\mu} = \gamma_{\mathbf{y},\mu}$.*

(b) *If $\mu(\mathbf{x}) > \mu(\mathbf{y})$, then $\gamma_{\mathbf{x},\mu} \geq \gamma_{\mathbf{y},\mu}$, with at least one strict inequality.*

Proof: By the definition of $\mathcal{X}$, we have $\mathbf{x} = \mathbf{x}^s$ and $\mathbf{y} = \mathbf{x}^t$ for some $s, t \in \mathcal{S}$. Let $k \in \mathcal{K}$. If $s, t \in k$, then $x_k = y_k = 1$. If $s, t \notin k$, then $x_k = y_k = -1$. Either way, $\widetilde{\mu}_k x_k = \widetilde{\mu}_k y_k$. Thus, the coordinates of interest are those in the sets

$$\mathcal{K}_{st} := \{k \in \mathcal{K}\ ;\ s \in k \text{ and } t \notin k\} \quad \text{and} \quad \mathcal{K}_{ts} := \{k \in \mathcal{K}\ ;\ t \in k \text{ and } s \notin k\}$$

For any $k \in \mathcal{K}_{st}$, define $k' := \{t\} \sqcup k \setminus \{s\}$; then $k' \in \mathcal{K}_{ts}$. Likewise, for any $k \in \mathcal{K}_{ts}$, define $k' := \{s\} \sqcup k \setminus \{t\}$; then $k' \in \mathcal{K}_{st}$. The map $k \mapsto k'$ is an involution of $\mathcal{K}_{st} \sqcup \mathcal{K}_{ts}$—that is, a bijection from $\mathcal{K}_{st} \sqcup \mathcal{K}_{ts}$ to itself, such that $k'' = k$ for all $k \in \mathcal{K}_{st} \sqcup \mathcal{K}_{ts}$.

Claim 1.1: *(a) If $\mu(\mathbf{x}) = \mu(\mathbf{y})$, then $x_k \widetilde{\mu}_k = y_{k'} \widetilde{\mu}_{k'}$ for all $k \in \mathcal{K}_{st} \sqcup \mathcal{K}_{ts}$.*

(b) If $\mu(\mathbf{x}) > \mu(\mathbf{y})$, then $x_k \widetilde{\mu}_k > y_{k'} \widetilde{\mu}_{k'}$ for all $k \in \mathcal{K}_{st} \sqcup \mathcal{K}_{ts}$.

Proof: Suppose $k \in K_{st}$. Then $\widetilde{\mu}_{k'} \underset{(\diamond)}{=} \widetilde{\mu}_k + 2\mu(\mathbf{y}) - 2\mu(\mathbf{x}) \underset{(*)}{\leq} \widetilde{\mu}_k$, where $(\diamond)$ is by Equation (E5), and $(*)$ is because $\mu(\mathbf{x}) \geq \mu(\mathbf{y})$. Now, $k \in K_{st}$ and $k' \in \mathcal{K}_{ts}$, so $x_k = 1 = y_{k'}$, so $y_{k'} \widetilde{\mu}_{k'} \underset{(*)}{\leq} x_k \widetilde{\mu}_k$. Meanwhile, $x_{k'} = -1 = y_k$, so $x_{k'} \widetilde{\mu}_{k'} \underset{(*)}{\geq} y_k \widetilde{\mu}_k$.

On the other hand, if $k \in \mathcal{K}_{ts}$, then $k' \in \mathcal{K}_{st}$, and an identical argument implies that $\widetilde{\mu}_{k'} \underset{(*)}{\geq} \widetilde{\mu}_k$. Meanwhile, $x_{k'} = 1 = y_k$, so that $x_{k'} \widetilde{\mu}_{k'} \underset{(*)}{\geq} y_k \widetilde{\mu}_k$. Also, $y_{k'} = -1 = x_k$, so that $y_{k'} \widetilde{\mu}_{k'} \underset{(*)}{\leq} x_k \widetilde{\mu}_k$.

If $\mu(\mathbf{x}) = \mu(\mathbf{y})$, then all the inequalities $(*)$ are actually equalities; this proves (a). If $\mu(\mathbf{x}) > \mu(\mathbf{y})$, then all the inequalities $(*)$ are strict inequalities; this proves (b). ▽ `Claim 1.1`

If $\mu(\mathbf{x}) = \mu(\mathbf{y})$, then Claim 1.1(a) implies that $\gamma_{\mathbf{x},\mu} = \gamma_{\mathbf{y},\mu}$. This proves (a). If $\mu(\mathbf{x}) > \mu(\mathbf{y})$, then Claim 1.1(b) implies that $\gamma_{\mathbf{x},\mu}(q) > \gamma_{\mathbf{y},\mu}(q)$ for any q such that there is some $k \in \mathcal{K}_{st}$ with $\widetilde{\mu}_k > q > \widetilde{\mu}_{k'}$. This proves (b). ◇ `Claim 1`

Claim 1 implies that $\mathbf{x} \underset{\mu}{\trianglerighteq} \mathbf{y}$ if and only if $\mu(\mathbf{x}) \geq \mu(\mathbf{y})$. Thus, $\text{SME}(\mathcal{X}, \mu) = \underset{\mathbf{x} \in \mathcal{X}}{\text{argmax}}\, \mu(\mathbf{x})$. Thus, if $\mathbf{x}, \mathbf{y} \in \text{SME}(\mathcal{X}, \mu)$, then we must have $\mu(\mathbf{x}) = \mu(\mathbf{y})$, and then Claim 1(a) says $\gamma_{\mu,\mathbf{x}} = \gamma_{\mu,\mathbf{y}}$. □

F Identifying the median rule

The main result of this appendix (Proposition F.5) is a technical tool for determining when a judgement aggregation rule F actually is equal to the median rule. This result was used in the proof of Theorem 6.3 above. But before proving Proposition F.5, we must develop some preliminaries. (The results in this section are also important for the companion paper Nehring and Pivato (2012a).) Let $\mathcal{X}$ be a judgement space, and let $\mathcal{C} := \text{conv}(\mathcal{X})$. A judgement aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is a *tally rule* if there exists a function $\widetilde{F} : \text{conv}(\mathcal{X}) \rightrightarrows \mathcal{X}$ such that $F(\mu) = \widetilde{F}(\widetilde{\mu})$ for all $\mu \in \Delta(\mathcal{X})$. For example, any additive majority rule F_ϕ is a tally rule.

If F is a tally rule on $\mathcal{X}$ and $\mathbf{c} \in \operatorname{conv}(\mathcal{X})$, and $\mu \in \Delta(\mathcal{X})$ is any profile such that $\widetilde{\mu} = \mathbf{c}$, the we will often abuse notation and write "$F(\mathbf{c})$" to indicate $F(\mu)$. In particular, we will use this notation for the median rule. Recall the definition of monotonicity from the end of §3. The next result will be useful.

Lemma F.1 *Let F be a monotone tally rule on $\mathcal{X}$. Let $\mathbf{c} \in \mathcal{C}$, and let $\mathbf{x} \in F(\mathbf{c})$. Then for any $\epsilon \in (0, 1]$, $F(\epsilon\mathbf{x} + (1 - \epsilon)\mathbf{c}) = \{\mathbf{x}\}$.*

Proof: Let $\delta_{\mathbf{x}} \in \Delta(\mathcal{X})$ be the "unanimous" profile, such that $\delta_{\mathbf{x}}(\mathbf{x}) = 1$ while $\delta_{\mathbf{x}}(\mathbf{y}) = 0$ for all other $\mathbf{y} \in \mathcal{X}$. Let $\mu \in \Delta(\mathcal{X})$ be a profile such that $\widetilde{\mu} = \mathbf{c}$. Then for all $\epsilon > 0$, if $\mu^\epsilon := \epsilon\delta_{\mathbf{x}} + (1-\epsilon)\mu$, then $\widetilde{\mu}^\epsilon = \epsilon\mathbf{x} + (1-\epsilon)\mathbf{c}$. But monotonicity implies that $F(\widetilde{\mu}^\epsilon) = \{\mathbf{x}\}$. □

Let $\mathcal{C} := \operatorname{conv}(\mathcal{X})$, and let $F : \mathcal{C} \rightrightarrows \mathcal{X}$ be a tally rule. For any $\mathbf{x} \in \mathcal{X}$, we define

$$\mathcal{C}_{\mathbf{x}}^F \quad := \quad \{\mathbf{c} \in \mathcal{C}\ ;\ \mathbf{x} \in F(\mathbf{c})\} \quad \text{and} \quad {}^{o}\mathcal{C}_{\mathbf{x}}^F \quad := \quad \{\mathbf{c} \in \mathcal{C}\ ;\ F(\mathbf{c}) = \{\mathbf{x}\}\}.$$

Next, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we define $\mathcal{B}_{\mathbf{x},\mathbf{y}}^F \ := \ \mathcal{C}_{\mathbf{x}}^F \cap \mathcal{C}_{\mathbf{y}}^F \ = \ \{\mathbf{c} \in \mathcal{C}\ ;\ \mathbf{x}, \mathbf{y} \in F(\mathbf{c})\}$. Finally, let $\partial_* \mathcal{C}_{\mathbf{x}}^F$ be the relative boundary of $\mathcal{C}_{\mathbf{x}}^F$ as a subset of $\mathcal{C}$, and let $\operatorname{int}\left(\mathcal{C}_{\mathbf{x}}^F\right)$ be the relative interior of $\mathcal{C}_{\mathbf{x}}^F$. That is: $\partial_* \mathcal{C}_{\mathbf{x}}^F := \mathcal{C}_{\mathbf{x}}^F \cap \operatorname{cl}\left(\mathcal{C} \setminus \mathcal{C}_{\mathbf{x}}^F\right)$ and $\operatorname{int}\left(\mathcal{C}_{\mathbf{x}}^F\right) := \mathcal{C}_{\mathbf{x}}^F \setminus \partial_* \mathcal{C}_{\mathbf{x}}^F$.

Lemma F.2 *Let $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be an upper hemicontinuous, monotone tally rule, and let $\mathbf{x} \in \mathcal{X}$. Then:*

(a) $\operatorname{int}\left(\mathcal{C}_{\mathbf{x}}^F\right) = {}^{o}\mathcal{C}_{\mathbf{x}}^F$.

(b) ${}^{o}\mathcal{C}_{\mathbf{x}}^F$ *is connected.*

(c) $\mathcal{C}_{\mathbf{x}}^F = \operatorname{cl}\left({}^{o}\mathcal{C}_{\mathbf{x}}^F\right)$.

(d) $\mathcal{B}_{\mathbf{x},\mathbf{y}}^F \subseteq \partial_* \mathcal{C}_{\mathbf{x}}^F$ *for all* $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$.

Proof: (c) "$\supseteq$" Clearly, $\mathcal{C}_{\mathbf{x}}^F \supseteq {}^{o}\mathcal{C}_{\mathbf{x}}^F$. To show that $\mathcal{C}_{\mathbf{x}}^F \supseteq \operatorname{cl}\left({}^{o}\mathcal{C}_{\mathbf{x}}^F\right)$, it suffices to observe that $\mathcal{C}_{\mathbf{x}}^F$ is closed, because F is upper hemicontinuous by hypothesis.

"$\subseteq$" Let $\mathbf{c} \in \mathcal{C}_{\mathbf{x}}^F$. For any $r \in (0, 1)$, let $\mathbf{c}^r := r\mathbf{x} + (1 - r)\mathbf{c}$. Clearly, $\lim_{r \to 0} \mathbf{c}^r = \mathbf{c}$. Thus, $\mathbf{c} \in \operatorname{cl}\left({}^{o}\mathcal{C}_{\mathbf{x}}^F\right)$, because Lemma F.1 says that that $\mathbf{c}^r \in {}^{o}\mathcal{C}_{\mathbf{x}}^F$ for all $r > 0$.

(b) For any $\mathbf{c} \in {}^{o}\mathcal{C}_{\mathbf{x}}^F$, the proof of part (c) shows that the line segment from $\mathbf{x}$ to $\mathbf{c}$ is in ${}^{o}\mathcal{C}_{\mathbf{x}}^F$. Thus, ${}^{o}\mathcal{C}_{\mathbf{x}}^F$ is path-connected, hence connected.

(d) Let $\mathbf{b} \in \mathcal{B}_{\mathbf{x},\mathbf{y}}^F$. Then $\mathbf{b} \in \mathcal{C}_{\mathbf{x}}^F$ and $\mathbf{b} \in \mathcal{C}_{\mathbf{y}}^F$. Thus, part (c) implies that $\mathbf{b} \in \operatorname{cl}\left({}^{o}\mathcal{C}_{\mathbf{y}}^F\right)$. But ${}^{o}\mathcal{C}_{\mathbf{y}}^F \subset \mathcal{C} \setminus \mathcal{C}_{\mathbf{x}}^F$. Thus, $\mathbf{b}$ is in both $\mathcal{C}_{\mathbf{x}}^F$ and $\operatorname{cl}\left(\mathcal{C} \setminus \mathcal{C}_{\mathbf{x}}^F\right)$. Thus, $\mathbf{b} \in \partial_* \mathcal{C}_{\mathbf{x}}^F$.

(a) To see $\operatorname{int}\left(\mathcal{C}_{\mathbf{x}}^F\right) \supseteq {}^{o}\mathcal{C}_{\mathbf{x}}^F$, note that

$${}^{o}\mathcal{C}_{\mathbf{x}}^F \quad := \quad \{\mathbf{c} \in \mathcal{C}\ ;\ F(\mathbf{c}) = \mathbf{x}\} \quad = \quad \mathcal{C} \setminus \bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathcal{C}_{\mathbf{y}}^F. \tag{F1}$$

Now $\bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathcal{C}_{\mathbf{y}}^F$ is closed because $\mathcal{X}$ is finite and $\mathcal{C}_{\mathbf{y}}^F$ is closed for any $\mathbf{y} \in \mathcal{X}$ (because F is upper hemicontinuous). Thus, eqn.(F1) makes ${}^{o}\mathcal{C}_{\mathbf{x}}^F$ a relatively open subset of $\mathcal{C}$; thus, ${}^{o}\mathcal{C}_{\mathbf{x}}^F \subseteq \operatorname{int}\left(\mathcal{C}_{\mathbf{x}}^F\right)$.

To see $\mathrm{int}\left(\mathcal{C}^F_{\mathbf{x}}\right) \subseteq \mathscr{C}^F_{\mathbf{x}}$, note that

$$\begin{aligned}
\mathrm{cl}\left(\mathcal{C} \setminus \mathscr{C}^F_{\mathbf{x}}\right) \underset{(*)}{=} \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathcal{C}^F_{\mathbf{y}} \underset{(\diamond)}{=} \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathrm{cl}\left(\mathscr{C}^F_{\mathbf{y}}\right) \\
\underset{(\dagger)}{=} \mathrm{cl}\left(\bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathscr{C}^F_{\mathbf{y}}\right) \subseteq \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^F_{\mathbf{x}}\right).
\end{aligned} \tag{F2}$$

Here, $(*)$ is by eqn.(F1), $(\diamond)$ is by applying part (c) to each $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$, and $(\dagger)$ is because $\mathcal{X}$ is finite.

Taking the complement of both sides of (F2), we get $\mathcal{C} \setminus \mathrm{cl}\left(\mathcal{C} \setminus \mathscr{C}^F_{\mathbf{x}}\right) \supseteq \mathcal{C} \setminus \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^F_{\mathbf{x}}\right)$, which is equivalent to $\mathrm{int}\left(\mathscr{C}^F_{\mathbf{x}}\right) \supseteq \mathrm{int}\left(\mathcal{C}^F_{\mathbf{x}}\right)$, which means $\mathscr{C}^F_{\mathbf{x}} \supseteq \mathrm{int}\left(\mathcal{C}^F_{\mathbf{x}}\right)$ (because $\mathscr{C}^F_{\mathbf{x}}$ is relatively open). □

Lemma F.3 *Let $F, G : \Delta(\mathcal{X}) \rightrightarrows \mathbb{R}$ be upper hemicontinuous, monotone tally rules. The following are equivalent:*

- **(a)** $\mathcal{C}^F_{\mathbf{x}} \subseteq \mathcal{C}^G_{\mathbf{x}}$ *for all* $\mathbf{x} \in \mathcal{X}$.
- **(b)** $\mathscr{C}^F_{\mathbf{x}} \subseteq \mathscr{C}^G_{\mathbf{x}}$ *for all* $\mathbf{x} \in \mathcal{X}$.
- **(c)** $F = G$.

Proof: The statement "(c) $\Longrightarrow$ (b)" is immediate. The statement "(b) $\Longrightarrow$ (a)" follows Lemma F.2(c). It remains to show "(a) $\Longrightarrow$ (c)". So suppose $\mathcal{C}^F_{\mathbf{x}} \subseteq \mathcal{C}^G_{\mathbf{x}}$ for all $\mathbf{x} \in \mathcal{X}$.

Claim 1: *For all $\mathbf{x} \in \mathcal{X}$, we have:* (i) $\mathscr{C}^G_{\mathbf{x}} \subseteq \mathscr{C}^F_{\mathbf{x}}$, *and* (ii) $\mathrm{cl}\left(\mathscr{C}^G_{\mathbf{x}}\right) \subseteq \mathrm{cl}\left(\mathscr{C}^F_{\mathbf{x}}\right)$.

Proof: (i) Suppose $\mathbf{c} \in \mathscr{C}^G_{\mathbf{x}}$. Then for all $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$, $\mathbf{c} \notin \mathcal{C}^G_{\mathbf{y}}$; hence the contrapositive of hypothesis (a) says $\mathbf{c} \notin \mathcal{C}^F_{\mathbf{y}}$, so $\mathbf{y} \notin F(\mathbf{c})$. Thus, if $\mathbf{x} \notin F(\mathbf{c})$, then $F(\mathbf{c}) = \emptyset$, which is impossible. Thus, $\mathbf{x} \in F(\mathbf{c})$, which means that $F(\mathbf{c}) = \{\mathbf{x}\}$; hence $\mathbf{c} \in \mathscr{C}^F_{\mathbf{x}}$.

(ii) Follows immediately by taking the closure in part (i). ◇ `Claim 1`

Claim 2: *For all $\mathbf{x} \in \mathcal{X}$, we have $\mathcal{C}^G_{\mathbf{x}} = \mathcal{C}^F_{\mathbf{x}}$.*

Proof: Combining Claim 1(ii) and Lemma F.2(c), we get $\mathcal{C}^G_{\mathbf{x}} \subseteq \mathcal{C}^F_{\mathbf{x}}$. But $\mathcal{C}^G_{\mathbf{x}} \supseteq \mathcal{C}^F_{\mathbf{x}}$ by hypothesis. Thus, $\mathcal{C}^G_{\mathbf{x}} = \mathcal{C}^F_{\mathbf{x}}$. ◇ `Claim 2`

Thus, for any $\mathbf{c} \in \mathcal{C}$ and any $\mathbf{x} \in \mathcal{X}$, we have

$$\Big(\mathbf{x} \in F(\mathbf{c})\Big) \iff \Big(\mathbf{c} \in \mathcal{C}^F_{\mathbf{x}}\Big) \underset{(*)}{\iff} \Big(\mathbf{c} \in \mathcal{C}^G_{\mathbf{x}}\Big) \iff \Big(\mathbf{x} \in G(\mathbf{c})\Big),$$

where $(*)$ is by Claim 2. Thus, $F = G$. □

Now, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define $\mathcal{C}^m_{\mathbf{x}} := \{\mathbf{c} \in \mathcal{C}\ ;\ \mathbf{x} \in \text{Median}\,(\mathcal{X}, \mathbf{c})\}$ and $\mathcal{B}^m_{\mathbf{x},\mathbf{y}} := \mathcal{C}^m_{\mathbf{x}} \cap \mathcal{C}^m_{\mathbf{y}} = \{\mathbf{c} \in \mathcal{C};\ \{\mathbf{x}, \mathbf{y}\} \in \text{Median}\,(\mathcal{X}, \mathbf{c})\}$. Then $\mathcal{C}^m_{\mathbf{x}}$ and $\mathcal{B}^m_{\mathbf{x},\mathbf{y}}$ are closed, because the median rule is upper hemicontinuous by Proposition 3.5. Next, define ${}^o\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathcal{C};\ \text{Median}\,(\mathcal{X}, \mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$, a subset of $\mathcal{B}^m_{\mathbf{x},\mathbf{y}}$. Observe that $\mathbf{x}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{y}$ if and only if ${}^o\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \neq \emptyset$. Let $\mathcal{Y}(\mathbf{x}) := \{\mathbf{y} \in \mathcal{X};\ \mathbf{y}\ \mathrm{I}_{\mathcal{X}}\ \mathbf{x}\}$. Finally, the "internal boundary" of $\mathcal{C}^m_{\mathbf{x}}$ inside $\mathcal{C}$ is the set $\partial_* \mathcal{C}^m_{\mathbf{x}} := \mathcal{C}^m_{\mathbf{x}} \cap \text{cl}\,(\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}})$.

Lemma F.4 *For all* $\mathbf{x} \in \mathcal{X}$, *we have* $\partial_* \mathcal{C}^m_{\mathbf{x}} = \bigcup_{\mathbf{y} \in \mathcal{Y}(\mathbf{x})} \text{cl}\left({}^o\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}}\right)$.

Proof: "$\supseteq$" For all $\mathbf{y} \in \mathcal{Y}(\mathbf{x})$, we have ${}^o\!\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^m_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{C}^m_{\mathbf{x}}$, while $\mathcal{C}^m_{\mathbf{x}}$ is closed.

"$\subseteq$" Let $D := \dim(\mathcal{C})$. We begin with four claims.

Claim 1: $\text{cl}\,(\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}}) = \bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathcal{C}^m_{\mathbf{y}}$.

Proof: "$\subseteq$" $\mathcal{X} = \bigcup_{\mathbf{y} \in \mathcal{X}} \mathcal{C}^m_{\mathbf{y}}$, so $\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}} \subseteq \bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathcal{C}^m_{\mathbf{y}}$. Thus, $\text{cl}\,(\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}}) \subseteq \bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathcal{C}^m_{\mathbf{y}}$, because the right hand expression is closed (being a finite union of closed sets).

"$\supseteq$" Let $\mathbf{c} \in \bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathcal{C}^m_{\mathbf{y}}$. Then $\mathbf{c} \in \mathcal{C}^m_{\mathbf{y}}$ for some $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$. Then for all $\epsilon > 0$, Lemma F.1 says that $\text{Median}\,(\mathcal{X},\ \epsilon\mathbf{y} + (1-\epsilon)\mathbf{c}) = \{\mathbf{y}\}$, because the median rule is monotone, by Proposition 3.4(a). Thus, $\epsilon\mathbf{y} + (1-\epsilon)\mathbf{c} \in \mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}}$, for all $\epsilon > 0$. But $\epsilon\mathbf{y} + (1-\epsilon)\mathbf{c} \xrightarrow[\epsilon \to 0]{} \mathbf{c}$. Thus, $\mathbf{c} \in \text{cl}\,(\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}})$. ◇ Claim 1

Claim 2: *For any distinct* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$, *if* $\mathcal{B}^m_{\mathbf{x},\mathbf{y},\mathbf{z}} := \{\mathbf{c} \in \mathcal{C};\ \text{Median}\,(\mathcal{X}, \mathbf{c}) \supseteq \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}\}$, *then* $\dim(\mathcal{B}^m_{\mathbf{x},\mathbf{y},\mathbf{z}}) \leq D - 2$.

Proof: Let $\mathcal{A}$ be the affine subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\mathcal{X}$ (so $\mathcal{C} \subset \mathcal{A}$, and $D = \dim(\mathcal{A})$). Let $\mathcal{A}^0 := \text{span}(\mathcal{X} - \mathcal{X})$; then $\mathcal{A}^0$ is a linear subspace of $\mathbb{R}^{\mathcal{K}}$ parallel to $\mathcal{A}$ (and $\dim(\mathcal{A}^0) = D$ also). Define $\mathcal{A}^0_{\mathbf{x},\mathbf{y},\mathbf{z}} := \{\mathbf{a} \in \mathcal{A}^0;\ (\mathbf{x} - \mathbf{y}) \bullet \mathbf{a} = (\mathbf{x} - \mathbf{z}) \bullet \mathbf{a} = 0\}$. The vectors $(\mathbf{x}-\mathbf{y})$ and $(\mathbf{x}-\mathbf{z})$ are linearly independent elements of $\mathcal{A}^0$; thus, $\dim(\mathcal{A}^0_{\mathbf{x},\mathbf{y},\mathbf{z}}) = D-2$. Now let $\mathcal{A}_{\mathbf{x},\mathbf{y},\mathbf{z}} := \{\mathbf{a} \in \mathcal{A};\ (\mathbf{x}-\mathbf{y}) \bullet \mathbf{a} = (\mathbf{x}-\mathbf{z}) \bullet \mathbf{a} = 0\}$. Then $\mathcal{A}_{\mathbf{x},\mathbf{y},\mathbf{z}}$ is an affine subspace of $\mathbb{R}^{\mathcal{K}}$ parallel to $\mathcal{A}^0_{\mathbf{x},\mathbf{y},\mathbf{z}}$, so $\dim(\mathcal{A}_{\mathbf{x},\mathbf{y},\mathbf{z}}) = D - 2$ also. But $\mathcal{B}^m_{\mathbf{x},\mathbf{y},\mathbf{z}} \subseteq \mathcal{A}_{\mathbf{x},\mathbf{y},\mathbf{z}}$; thus, $\dim(\mathcal{B}^m_{\mathbf{x},\mathbf{y},\mathbf{z}}) \leq D - 2$. ◇ Claim 2

Claim 3: *If* $\mathbf{x} \in \mathcal{X}$ *and* $\mathbf{y} \in \mathcal{X} \setminus \mathcal{Y}(\mathbf{x})$, *then* $\dim(\mathcal{B}_{\mathbf{x},\mathbf{y}}) \leq D - 2$.

Proof: If $\mathbf{y} \in \mathcal{X} \setminus \mathcal{Y}(\mathbf{x})$, then there is no $\mu \in \Delta(\mathcal{X})$ such that $\text{Median}\,(\mathcal{X}, \mu) = \{\mathbf{x}, \mathbf{y}\}$. Thus, for all $\mathbf{b} \in \mathcal{B}^m_{\mathbf{x},\mathbf{y}}$, there is some $\mathbf{z} \in \mathcal{X} \setminus \{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{b} \in \mathcal{B}^m_{\mathbf{x},\mathbf{y},\mathbf{z}}$. In other words, $\mathcal{B}^m_{\mathbf{x},\mathbf{y}} = \bigcup_{\mathbf{z} \in \mathcal{X} \setminus \{\mathbf{x},\mathbf{y}\}} \mathcal{B}^m_{\mathbf{x},\mathbf{y},\mathbf{z}}$, a finite union of sets of dimension $D - 2$ or less (by Claim 2). Thus, $\dim(\mathcal{B}_{\mathbf{x},\mathbf{y}}) \leq D - 2$. ◇ Claim 3

Claim 4: $\dim(\partial_* \mathcal{C}^m_{\mathbf{x}}) = D - 1$.

Proof: Lemma F.2(a,c) implies that $\partial_* \mathcal{C}^m_{\mathbf{x}}$ is the boundary of $\mathscr{C}^m_{\mathbf{x}}$. But $\mathscr{C}^m_{\mathbf{x}}$ is a convex, relatively open subset of $\mathcal{C}$, by the formula (6) defining the median rule. Since $\dim(\mathcal{C}) = D$, we have $\dim(\partial_* \mathcal{C}^m_{\mathbf{x}}) = D - 1$. ◇ `Claim 4`

Now we have

$$\begin{aligned}
\partial_* \mathcal{C}^m_{\mathbf{x}} &= \mathcal{C}^m_{\mathbf{x}} \cap \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}}\right) \underset{(*)}{=} \mathcal{C}^m_{\mathbf{x}} \cap \left(\bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathcal{C}^m_{\mathbf{y}}\right) = \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \left(\mathcal{C}^m_{\mathbf{x}} \cap \mathcal{C}^m_{\mathbf{y}}\right) \\
&= \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathcal{B}^m_{\mathbf{x},\mathbf{y}} = \bigcup_{\mathbf{y}\in\mathcal{Y}(\mathbf{x})} \mathcal{B}^m_{\mathbf{x},\mathbf{y}} \cup \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\mathcal{Y}(\mathbf{x})} \mathcal{B}^m_{\mathbf{x},\mathbf{y}} \\
&= \overbrace{\bigcup_{\mathbf{y}\in\mathcal{Y}(\mathbf{x})} {}^{o}\mathcal{B}^m_{\mathbf{x},\mathbf{y}}}^{\text{(A)}} \cup \overbrace{\bigcup_{\mathbf{y}\in\mathcal{Y}(\mathbf{x})} \ \bigcup_{\mathbf{z}\in\mathcal{X}\setminus\{\mathbf{x},\mathbf{y}\}} \mathcal{B}^m_{\mathbf{x},\mathbf{y},\mathbf{z}}}^{\text{(B)}} \cup \overbrace{\bigcup_{\mathbf{y}\in\mathcal{X}\setminus\mathcal{Y}(\mathbf{x})} \mathcal{B}^m_{\mathbf{x},\mathbf{y}}}^{\text{(C)}},
\end{aligned}$$

where $(*)$ is by Claim 1. Now, $\mathcal{X}$ and $\mathcal{Y}(\mathbf{x})$ are finite sets, and Claim 2 says that every element in the union (B) is a set of dimension $D-2$ or less. Also, $\mathcal{X} \setminus \mathcal{Y}(\mathbf{x})$ is a finite set, and Claim 3 says that every element in the union (C) is a set of dimension $D-2$ or less. Thus Claim 4 implies that the unions (B) and (C) are nowhere dense in $\partial_* \mathcal{C}^m_{\mathbf{x}}$. Thus, the union (A) must be dense in $\partial_* \mathcal{C}^m_{\mathbf{x}}$. Thus,

$$\partial_* \mathcal{C}^m_{\mathbf{x}} = \mathrm{cl}\left(\bigcup_{\mathbf{y}\in\mathcal{Y}(\mathbf{x})} {}^{o}\mathcal{B}^m_{\mathbf{x},\mathbf{y}}\right) = \bigcup_{\mathbf{y}\in\mathcal{Y}(\mathbf{x})} \mathrm{cl}\left({}^{o}\mathcal{B}^m_{\mathbf{x},\mathbf{y}}\right),$$

because it is a finite union. □

We now come to the main result of this appendix.

Proposition F.5 *Let $\mathcal{X}$ be a judgement space, and let $G : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be an upper hemicontinuous, monotone tally rule. Suppose ${}^{o}\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^G_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}})$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Then $G(\mathcal{X}, \mu) = \mathrm{Median}\,(\mathcal{X}, \mu)$ for all $\mu \in \Delta(\mathcal{X})$*

Example F.6. Let $\phi : [-1, 1] \longrightarrow \mathbb{R}$ be a continuous, real-valued gain function, and let $\mathcal{X}$ be a judgement space. Then F_ϕ is upper hemicontinuous (by Proposition 3.5) and monotone (by Proposition 3.4(a)). Thus, if ${}^{o}\mathcal{B}^m_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^\phi_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^\phi_{\mathbf{x}})$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, then Proposition F.5 says that $F_\phi(\mathcal{X}, \mu) = \mathrm{Median}\,(\mathcal{X}, \mu)$ for all $\mu \in \Delta(\mathcal{X})$. ◇

Proof of Proposition F.5. According to Lemma F.3, it suffices to establish the following statement:

$$\mathcal{C}^G_{\mathbf{x}} \subseteq \mathcal{C}^m_{\mathbf{x}}, \qquad \text{for every } \mathbf{x} \in \mathcal{X}. \tag{F3}$$

To verify statement (F3), first note that

$$\begin{aligned}
\partial_* \mathcal{C}^m_{\mathbf{x}} &\underset{(*)}{=} \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathrm{cl}\left({}^{o}\mathcal{B}^m_{\mathbf{x},\mathbf{y}}\right) \underset{(\diamond)}{\subseteq} \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathrm{cl}\left(\mathcal{B}^G_{\mathbf{x},\mathbf{y}} \cup \mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}}\right) = \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathrm{cl}\left(\mathcal{B}^G_{\mathbf{x},\mathbf{y}}\right) \cup \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}}\right) \\
&\underset{(@)}{=} \bigcup_{\mathbf{y}\in\mathcal{X}\setminus\{\mathbf{x}\}} \mathcal{B}^G_{\mathbf{x},\mathbf{y}} \cup \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}}\right) \underset{(\dagger)}{\subseteq} \partial_* \mathcal{C}^G_{\mathbf{x}} \cup \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}}\right) \underset{(\ddagger)}{=} \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}}\right).
\end{aligned} \tag{F4}$$

Here, $(*)$ is by Lemma F.4, $(\diamond)$ is by the theorem hypothesis, and $(@)$ is because G is upper hemicontinuous. Next, $(\dagger)$ is by applying Lemma F.2(d) to $\partial_* \mathcal{C}^G_{\mathbf{x}}$. Finally $(\ddagger)$ is because $\partial_* \mathcal{C}^G_{\mathbf{x}} \subseteq \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}}\right)$ by definition.

It follows that

$$\mathrm{int}\left(\mathcal{C}^m_{\mathbf{x}}\right) \sqcup \mathrm{int}\left(\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}}\right) \quad = \quad \mathcal{C} \setminus \left(\partial_* \mathcal{C}^m_{\mathbf{x}}\right) \underset{(*)}{\supseteq} \mathcal{C} \setminus \mathrm{cl}\left(\mathcal{C} \setminus \mathcal{C}^G_{\mathbf{x}}\right) \quad = \quad \mathrm{int}\left(\mathcal{C}^G_{\mathbf{x}}\right),$$

where $(*)$ is by equation (F4). But $\mathrm{int}\left(\mathcal{C}^G_{\mathbf{x}}\right)$ is connected, by Lemma F.2(a,b). Thus, we must have either $\mathrm{int}\left(\mathcal{C}^G_{\mathbf{x}}\right) \subseteq \mathrm{int}\left(\mathcal{C}^m_{\mathbf{x}}\right)$, or $\mathrm{int}\left(\mathcal{C}^G_{\mathbf{x}}\right) \subseteq \mathrm{int}\left(\mathcal{C} \setminus \mathcal{C}^m_{\mathbf{x}}\right)$. However, $\mathbf{x} \in \mathrm{int}\left(\mathcal{C}^G_{\mathbf{x}}\right)$ and $\mathbf{x} \in \mathrm{int}\left(\mathcal{C}^m_{\mathbf{x}}\right)$ (by monotonicity). Thus, we must have $\mathrm{int}\left(\mathcal{C}^G_{\mathbf{x}}\right) \subseteq \mathrm{int}\left(\mathcal{C}^m_{\mathbf{x}}\right)$. Take the closures and apply Lemma F.2(c) to get $\mathcal{C}^G_{\mathbf{x}} \subseteq \mathcal{C}^m_{\mathbf{x}}$, as desired. □

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