Majority Rule in the Absence of a Majority

Klaus Nehring and Marcus Pivato

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Majoritarianism

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  - Principle that **the "most widely shared" view should prevail**

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1. **The Analytical Question:**
   What is "the most widely shared" view?
   - on complex issues, there may be none (total indeterminacy), or only a set of views can be identified as more or less predominant (partial indeterminacy)

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  1. **The Analytical Question:**
     What is "the most widely shared" view?
     - on complex issues, there may be none (total indeterminacy), or only a set of views can be identified as more or less predominant (partial indeterminacy)
  2. **The Normative Question:**
     Why should the most widely shared view prevail?
     - may invoke principles of democracy, self-governance, political stability etc.

- Here we shall focus on analytical question:
  What is Majority Rule without a Majority?

- stay agnostic about normative question
- in practice, many institutions seem to adopt majoritarian procedures
  - prima facie case for majoritarian committments,
    - but not clear how deep it is.

Framework I

- standard JA framework:
  individuals (voters) and the group hold judgments on a set of interdependent issues ("views")
  - K set of *issues*
  - $X \subseteq \{\pm 1\}^K$ set of *feasible* **views**
  - $x \in X$ particular *views ("sets of judgments") on* $x \in X$.

- shall describe anonymous **profiles** of views by measures $\mu \in \Delta(X)$
  - allow profiles to be real-valued
- (X, μ) "**JA problem**"

Framework II

Example: (Preference Aggregation over 3 Alternatives)

- $A = \{a, b, c\}$
- $K = \{ab, bc, ca\}$
  - The ranking *abc* corresponds to $(1, 1, -1)$, etc.
- Thus $X =: X_A^{pr}$ given by

$$\{\pm 1\}^K \setminus \{(1,1,1), (-1,-1,-1)\}.$$

- preference aggregation problem as *judgment aggregation* problem:
  - about competing views re how group should rank/choose
- not: as *welfare aggregation* problem:
  - about 'adding up' info about what is good for each individual into what is "good overall".
  - MAJ makes much less sense for WA than JA.

Framework III

- Systematic criteria to select among views in JA problems described by **aggregation rules**
  - Aggregation rule $F : (X, \mu) \mapsto F(X, \mu) \subseteq X$.
  - will consider different domains
    - X frequently fixed
  - leave domain unspecified for now to emphasize **single-profile issue**: *what views are majoritarian in the JA problem* (X, μ)?

The Program: Criteria for Majoritarianism

1. Plain Majoritarianism
2. Condorcet Consistency
   - transfer from voting literature
3. Condorcet Admissibility
   - defines MAJ *per se*
     - NehPivPup 2011
4. Supermajority Efficiency
   - MAJ *plus* Issue Parity
5. Additive Majority Rules
   - MAJ *plus* Issue Parity *plus cardinal tradeoffs.*

Majority Rule in the Presence of a Majority

Axiom

(Plain Majoritarianism)

If $\mu(x) > \frac{1}{2}$, *then* $F(X,\mu) = \{x\}$.

- view as definitional:
  If reject Plain M, simply reject Majoritarianism.
- Evident Problem: premise rarely satisfied if $K > 1$.

Condorcet Consistency I

- Useful piece of notation

$$\widetilde{\mu}_k := \sum_{x \in X} x_k \mu(x) = \mu(x : x_k = 1) - \mu(x : x_k = -1)$$

  - E.g.: If 57% affirm proposition k at μ, $\widetilde{\mu}_k = 0.14$

- $\mathcal{M}(x, \mu) := \{k \in K : x_k \widetilde{\mu}_k \geq 0\}$
  - those issues in which x aligned with majority

Condorcet Consistency II

- Condorcet Consistency: if majority judgment on each issue is consistent, this is the majority view.
  - $Maj(\mu) := \{x \in \{\pm 1\}^K : \mathcal{M}(x, \mu) = K\}$

Axiom (Condorcet Consistency)

If $Maj(\mu) \cap X \neq \varnothing$, *then* $F(X, \mu) \subseteq Maj(\mu)$.

- Obvious Limitation: "Condorcet Paradox" in JA
  - $Maj(\mu) \cap X = \varnothing$, unless X *median space*
    - median space: all 'minimally inconsistent subsets' have cardinality 2.

Condorcet Admissibility I

- **Condorcet Set** (NPP 2011):

$$x \in Cond(X, \mu) \text{ iff, for no } y \in X,\ \mathcal{M}(v, \mu) \supsetneq \mathcal{M}(x, \mu).$$

Axiom

Condorcet Admissibility $F(X, \mu) \subseteq Cond(X, \mu).$

- Claim in NPP 2011: this captures normative implications of Majoritarianism *per se.*

- Problem: outside median-spaces, $Cond(X, \mu)$ can easily be large.
  - But: additional considerations may favor some Condorcet admissible views over another
    - here: refine *Cond* based on considerations of "parity" among issues.

Supermajority Efficiency I

- Premise: Majoritarianism **plus Issue Parity**

- **Issue Parity**: "each issue counts equally"
  - sometimes, Parity may be justified by symmetries of judgment space X
    - e.g. preference aggregation, equivalence relations
  - but Parity has broader applicability
  - Parity not always plausible, e.g. truth-functional aggregation

Supermajority Efficiency II

Example: (Preference Aggregation over 3 Alternatives)

- $A = \{a, b, c\}$
- $X = X_A^{pr}$; (3-Permutahedron)
  - $K = \{ab, bc, ca\}$
- $\mu(a \succ b) = 0.75$;
  $\mu(b \succ c) = 0.7$;
  $\mu(c \succ a) = 0.55$
- $Cond(X, \mu) = \{abc, bca, cab\}$.

- Each Condorcet admissible ordering overrides one majority preference
- *Arguably, the ordering abc is the most widely supported (hence "most majoritarian") since it overrides the weakest majority*

Supermajority Efficiency III

- Argument via "Supermajority Dominance"
  - compare *abc* to *bca*
    - *abc* has advantage over *bca* on *ab* (at 0.75 vs. 0.25);
      bca has advantage over *abc* on *ca* (at 0.55 vs. 0.45);
    - since 0.75>0.55, *abc* **supermajority dominates** *bca*
  - dto. *abc* supermajority dominates *cab*
  - hence *abc* uniquely **supermajority efficient**

Supermajority Efficiency IV

- General idea: x supermajority dominates y at μ if it sacrifices smaller majorities for larger majorities.
  - assumes that each proposition $k \in K$ counts equally.

- For any threshhold $q \in [0, 1]$,

$$\gamma_{\mu,x}(q) := \#\{k \in K : x_k \widetilde{\mu}_k \geq q\}.$$

- x **supermajority-dominates** y at μ ($\boxed{\text{“}x \rhd_\mu y\text{”}}$)

  if, for all $q \in [0, 1]$, $\gamma_{\mu,x}(q) \geq \gamma_{\mu,y}(q)$, and,
  for some $q \in [0, 1]$, $\gamma_{\mu,x}(q) > \gamma_{\mu,y}(q)$.

  - for economists: note analogy to first-order stochastic dominance.

Supermajority Efficiency V

- x is **supermajority efficient** at μ ($\boxed{\text{“}x \in SME\,(X,\mu)\text{”}}$) if, for no $y \in X$, $y \rhd_\mu x$.

- In example: $SME\,(X,\mu) = \{abc\}$.

Supermajority Determinacy I

- In 3-permutahedron, for all $\mu \in \Delta(X)$, $SME(X, \mu)$ unique 'up to (non-generic) ties'
- such spaces *supermajority determinate*

- In paper, provide full characterization of supermajority-determinate spaces
  - interesting examples beyond median spaces

- *Most spaces not supermajority determinate*
  - E.g. permutahedron with $\#A > 3$

Additive Majority Rules I

- In general case, need to make tradeoffs between number and strength of majorities overruled
  - systematic tradeoff criterion described by "additive majority rules"
  - main result provides axiomatic foundation based on SME

Aggregation Rules

- Let $\mathfrak{X}$ be a family of spaces
    - e.g. $\mathfrak{X} = \{X\}$;
    - or $\mathfrak{X}$ =all finite JA spaces.

Definition

An **aggregation rule** is a correspondence $F : \bigsqcup_{X \in \mathfrak{X}} (X, \Delta(X)) \rightrightarrows \bigsqcup_{X \in \mathfrak{X}} X$ such that, for all X, $\mu \in \Delta(X)$ $F(X, \mu) \subseteq X$.

- Often simplify $F(X, \mu)$ to $F(\mu)$

Additive Majority Rules III

Definition

An aggregation rule F is an **additive majority rule** if there exists a function $\phi : [-1, +1] \longrightarrow {}^{*}\mathbb{R}$ such that, for all $X \in \mathfrak{X}$ and $\mu \in \Delta(X)$,

$$F_{\phi}(X, \mu) = \arg\max_{x \in X} \sum_{k \in K} \phi\left(x_k \widetilde{\mu}_k\right).$$

- ${}^{*}\mathbb{R}$ are the *hyperreal* numbers
  - extension of $\mathbb{R}$ containing infinites and infinitesimals
  - for now, focus on real-valued case

$$F_{\phi}(\mu) := \arg\max_{x \in X} \sum_{k \in K} \phi\left(x_k \widetilde{\mu}_k\right).$$

- key ingredient: **gain function** $\phi : [-1, +1] \to \mathbb{R}$

1. $\boxed{x_k \widetilde{\mu}_k}$ "majority advantage" for x on issue k
2. $\boxed{\phi\left(x_k \widetilde{\mu}_k\right)}$ is the alignment of x with μ on issue k;
   - by increasingness of ϕ, largest when $x_k = sgn(\widetilde{\mu}_k)$;
     - hence F_{ϕ} tries to align group view with issue-wise majorities; in particular, F_{ϕ} *Condorcet consistent*.
3. $\boxed{\sum_{k \in K} \phi\left(x_k \widetilde{\mu}_k\right)}$ measures overall alignment of x with profile μ
   - hence $F_{\phi}(\mu)$ choses group view(s) x that is *most representative* for distribution of individual views μ.

- this conceptual interpretation important complement to axiomatic foundation.
  - underlines *conceptual coherence and unity* of intuitive, pre-formal notion of "majoritarianism"

- $F_{\phi}(\mu)$ SME by increasingness of ϕ

- W.l.o.g. ϕ odd, i.e. $\phi(r) = -\phi(-r)$ for all $r \in [-1, +1]$.

Example

(Median Rule: $\phi = id$**)**;

$$F_{med}(\mu) := F_{id}(\mu) = \arg\max_{x \in X} \sum_{k \in K} x_k \widetilde{\mu}_k$$

- maximizes total number of votes for x over all issues.
  - in preference aggregation: Kemeny rule
    - axiomatized by HP Young
      – one of the (hidden) classics of social choice theory
  - widely studied as general-purpose aggregation rule (Barthelemy, Monjardet, Janowitz, ...)

- Axiomatized in master/companion paper NPiv 2011/13

- Here: leave ϕ open
  - ϕ describes how issue-wise majorities are traded off depending on their size.

- well-illustrated with *homogeneous rules* $H^d := F_{\phi^d}$, with

$$\phi^d(r) = sgn(r)\,|r|^d.$$

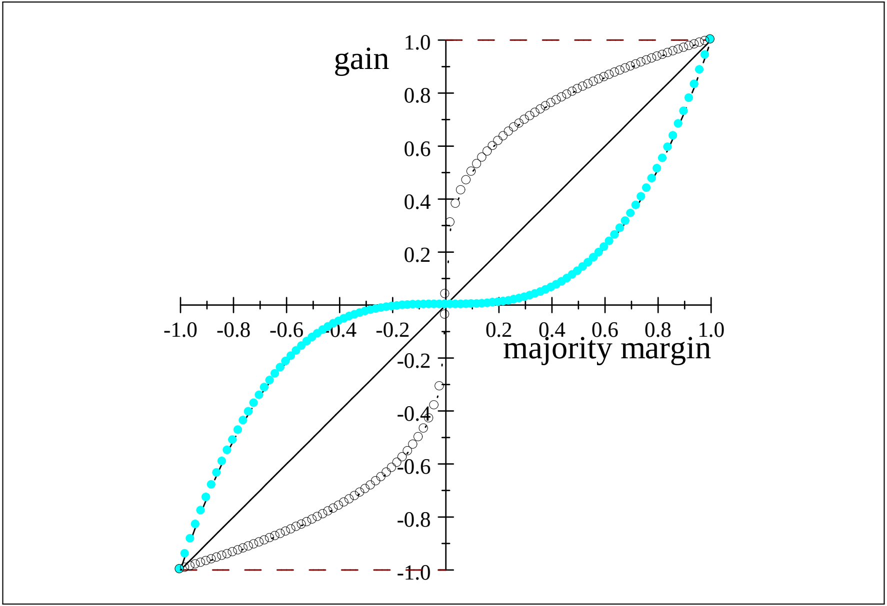
A One-Parameter Family

- $$\phi^d(r) = sgn(r)\,|r|^d.$$

-
  - $d = 1$ median rule
  - $d > 1$ inverse-S-shape; *consensus-oriented:*
    - priority to respect large majorities.
  - $d < 1$ S-shape: *breadth-oriented*
    - priority to respect as many majorities as possible.

- One majority of size $2r$ balances 2^d majorities of size r.
  - E.g. with $r = 2$, a 70% supermajority balances 4 60% majorities.

- Limiting cases:
  - $d \to \infty$ refinement of Ranked Pairs rule
  - $d \to 0$ refinement of Slater rule



Homogeneous Gain Functions for d=0, 0.3, 1, 3.

Hyperreal-Valued Gain Functions I

- other simple rules satisfy SME

Example

(Leximax) $xL_\mu y$ if there exist $\overline{q}$ such that $\gamma_{\mu,x}(q) = \gamma_{\mu,y}(q)$ for all $q > \overline{q}$, and $\gamma_{\mu,x}(q) > \gamma_{\mu,y}(q)$.

$$F_{lex\,\max}(X,\mu) := \{x \in X : \text{for no } y \in X, xL_\mu y\}$$

- Looks non-additive, but can be described by allowing ϕ to be hyperreal-valued.
  - Indeed, intuitively $F_{lex\,\max} = \lim_{d\to\infty} H^d$; hyperreals allow to state

$$F_{lex\,\max} = \lim H^{\lim_{d\to\infty} d}$$

Hyperreal-Valued Gain Functions II

- **hyperreals** $^*\mathbb{R}$:
  1. linearly ordered: can maximize
  2. group: can add
     - all that's needed for additive separable representation
  3. contains $\mathbb{R}$
  4. bonus: usual rules for arithmetic
     1. *field:* can multiply and divide
     2. *hyperreal field:* can exponentiate
  5. potential difficulty: no sups and infs in general

Hyperreal-Valued Gain Functions III

Example

$F_{\text{lexmin}} = F_{\phi^d}$, with d any infinite hyperreal $\omega > 0$.

- For verification, note that $r > s > 0$ implies $r^\omega > ns^\omega$ for all $n \in \mathbb{N}$.

Axiomatic Foundation I

- Need additional normative axiom: Decomposition
  - Natural setting: domains $\mathfrak{X}$ closed under Cartesian products.

Axiom

(Deomposition) *For any If* $X_1, X_2 \in \mathfrak{X}$:
$F(X_1 \times X_2, \mu) = F(X_1, marg_1 \mu) \times F(X_2, marg_2 \mu)$

- Interpretation: in the absence of any logical interconnection, the optimal group view can be determined by combining optimal group views in each component problem.
  - "optimal" could mean different things in different context; here "optimal" = "most majoritarian", "most widely supported"

Axiomatic Foundation II

We will present two representation theorems

1. Narrow domain: fixed finite population and a fixed judgment space
   - real-valued representation sufficient
2. Wide domains: variable population and variable judgment spaces.
   - the general, hyper-realvalued representation becomes indispensable.

- (1) is key building block for (2).

Decomposable Extensions

- Let $\langle X \rangle := \bigsqcup_{n\in\mathbb{N}} X^n$,

  $$\text{with } X^n := \underbrace{X \times X \times ... \times X}_{(n \text{ times})}$$

  - Interpretation: $\langle X \rangle$ consists of the combination of multiple instances of the same (isomorphic) judgment problem X with different views of the individuals in each instance
  - e.g. preference aggregation over ℓ alternatives.

- Given F on X, there exists unique separable aggregation rule $G = F^*$ on $\langle X \rangle$ such that $G(X, \cdot) = F$
  - F^* is the **decomposable extension** of F

Axiomatic Foundation IV

Fixed Population, Fixed Space

- anonyomous profiles generated from W voters:

$$\Delta_W(X) := \{\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} : x_i \in X \text{ for all } i\}$$

  - dto. $\Delta_W(\mathfrak{X})$

Theorem

Let X be any judgment space, $N \in \mathbb{N}$ a fixed number of voters, and F be any aggregation rule on $\Delta_N(X)$. Then the decomposable extension of F is SME if and only if there exists a real-valued gain-function ϕ such that $F \subseteq F_\phi$.

Axiomatic Foundation V

Variable Population, Variable Spaces

Theorem

Let $\mathfrak{X}$ be any domain of judgment spaces closed under Cartesian products, and F any decomposable aggregation rule on $\Delta(\mathfrak{X})$.

1. *F is SME if and only if there exists a hyperrealvalued gain function ϕ such that $F \subseteq F_\phi$.*
   In this case, for every $X \in \mathfrak{X}$, there exists a dense open set $\mathcal{O}_X \subseteq \Delta(X)$ such that, for all $\mu \in \mathcal{O}_X$,
   $$\#F_\phi(X,\mu) = 1, \text{ and thus } F(X,\mu) = F_\phi(X,\mu).$$
2. *If F is continuous (uhc), then $F = F_\phi$.*