On the Arrow property for symmetric classes of choice functions

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Introduction

In terms of choice functions, Arrow impossibility theorem states that there is no non-dictatorial *aggregation rule* which satisfies IIA and *unanimity* conditions and preserves the set of all rational choice functions on a finite set of at least three alternatives.

S. Shelah proved (2005) that Arrow theorem can be extended to the case when the choice functions are not rational in a very general setting. We obtained a refined version of this theorem containing a complete characterization of all symmetric sets of choice functions that have the Arrow property.

References.

S. Shelah. On the Arrow property. *Advances in Applied Mathematics*, 34:217–251, 2005.

N. Poliakov.On the Galois correspondence for classes of discrete functions and a generalization of Arrow theorem. *Journal of Mathematical Sciences*, to appear.

Basic definitions

Individual choices

Let A be a nonempty finite set of alternatives.
For any natural number r the symbol $[A]^r$ denote the set of all r-element subsets of A:

$$[A]^r = \{B \subseteq A \colon |B| = r\}$$

and the symbol $\mathfrak{C}_r(A)$ denote the set of all choice function defined on $[A]^r$

$$\mathfrak{C}_r(A) = \{\mathfrak{c} \in {}^{[A]^r}A \colon (\forall p \in [A]^r)\, \mathfrak{c}(p) \in p\}.$$

Functions $\mathfrak{c} \in \mathfrak{C}_r(A)$ represent individual choices of "voters".

Individual choices

A function $\mathfrak{c} \in \mathfrak{C}_r(A)$ is called *rational* if there is a linear order $\leq$ on A such that $\mathfrak{c}(q)$ is the maximal element of q, i.e.

$$(\forall q \in [A]^r)(\forall x \in q)\, x \leq \mathfrak{c}(q).$$

The set of all rational function $\mathfrak{c} \in \mathfrak{C}_r(A)$ is denoted by $\mathfrak{R}_r(A)$.

Individual choices

A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is called *symmetric* if for any function $\mathfrak{c} \in \mathfrak{D}$ and permutation $\sigma \in S_A$ the function $\mathfrak{c}_\sigma$ defined by

$$(\forall p \in [A]^r)\, \mathfrak{c}_\sigma(p) = \sigma^{-1}\mathfrak{c}(\sigma p),$$

belongs to $\mathfrak{D}$.

Informally, a symmetric set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ represents a set of individual choices coordinated by the same "common principle".

For example, the set $\mathfrak{R}_r(A)$ is symmetric.

Individual choices

Other natural examples:

- the set of all function $\mathfrak{c} \in \mathfrak{C}_r(A)$ such that $\mathfrak{c}(q)$ is the median element in q according to some ordering (r is odd);
- the set

$$\{\mathfrak{c} \in \mathfrak{C}_2(A)\colon (\exists x \in A)(\forall y \in A \setminus \{x\})\, \mathfrak{c}(\{x, y\}) = x\},$$

- let $\prec$ be a strict partial order on A and $\mathfrak{C}_r^{\prec}(A)$ a set of all functions $\mathfrak{c} \in \mathfrak{C}_r(A)$ such that $\mathfrak{c}(p)$ is some non-dominated elements of p, i.e.

$$(\forall x \in p)\, \mathfrak{c}(p) \not\prec x.$$

Let W be a set of strict partial order on A closed under isomorphisms. The set $\bigcup_{\prec \in W} C_r^{\prec}(A)$ is symmetric.

etc.

Aggregation rules

For any natural number $n \geq 1$ a function

$$f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$$

is called an (n-ary) *aggregation rule*.

The set of all aggregation rules is denoted by $\mathcal{O}(A, r)$.

Aggregation rules

Definition 1.
An aggregation rule $f \in \mathcal{O}(A, r)$ is *normal* if for all $p \in [A]^r$ there is a function $f_p \colon p^n \to p$ such that

1. $f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(p) = f_p(\mathfrak{c}_1(p), \mathfrak{c}_2(p), \ldots, \mathfrak{c}_n(p))$ for all $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{C}_r(A)$,
2. $\bigvee_{i<n} f_p(a_1, a_2, \ldots, a_n) = a_i$ for all $a_1, a_2, \ldots, a_n \in p$.

We denote the set of all normal aggregation rules $f \in \mathcal{O}(A, r)$ by $\mathcal{N}(A, r)$.

Aggregation rules

Remark. Item 1 of this Definition 1 means that the aggregation rule f has the *IIA property* (Independence of Irrelevant Alternatives).
Item 2 is slightly stronger than the *unanimity* condition, i.e. Item 2 implies that for all $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{C}_r(A)$, $p \in [A]^r$ and $a \in p$

$$\mathfrak{c}_1(p) = \mathfrak{c}_2(p) = \ldots = \mathfrak{c}_n(p) = a \to f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(p) = a.$$

Item 2 can be replaced by

2'. $f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(p) \in \{\mathfrak{c}_1(p), \mathfrak{c}_2(p), \ldots, \mathfrak{c}_n(p)\}$.

Aggregation rules

Definition 2.
An aggregation rule $f\colon (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is called

- *simple* if f is normal and f_p does not depend on p, i.e.

$$(\forall p, q \in [A]^r)(\forall \mathbf{a} \in p^n \cap q^n) f_p(\mathbf{a}) = f_q(\mathbf{a});$$

- *dictatorial* (or *monarchical*) if f is a projection, i.e.

$$(\exists j < n)(\forall \mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{C}_r(A)) f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) = \mathfrak{c}_j.$$

The set of all simple (dictatorial) aggregation rules $f \in \mathcal{O}(A, r)$ is denoted by $\mathcal{S}(A, r)$ (respectively $\mathcal{M}(A, r)$).

Aggregation rules

Remark.

1. Any dictatorial aggregation rule is simple.
2. If $f \in \mathcal{O}(A, 2)$, then the following conditions are equivalent
   - f is normal,
   - f is simple,
   - f satisfies IIA and unanimity.
3. If $2 < r \leq |A|$, then

$$\mathcal{M}(A, r) \subsetneq \mathcal{S}(A, r) \subsetneq \mathcal{N}(A, r) \subsetneq \mathcal{O}(A, r)$$

Preservation relation

Definition 3.
Let $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ and $f \in \mathcal{O}(A, r)$. We say that f *preserves* $\mathfrak{D}$ (or f is a *polymorphism* of $\mathfrak{D}$) and $\mathfrak{D}$ is *preserved* (or *closed*) under f if

$$f(\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) \in \mathfrak{D} \text{ for all } \mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n \in \mathfrak{D}.$$

The set of all $f \in \mathcal{O}(A, r)$ that preserves $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is denoted by $\operatorname{Pol} \mathfrak{D}$.

Arrow property

In terms of choice functions, Arrow impossibility theorem asserts that if $|A| \geq 3$, then any normal aggregation rule which preserves the set $\mathfrak{R}_2(A)$ (of all rational function $\mathfrak{c} \in \mathfrak{C}_2(A)$), is dictatorial, i.e.

$$\mathrm{Pol}\,\mathfrak{R}_2(A) \cap \mathcal{N}(A,2) = \mathcal{M}(A,2).$$

Definition 4.
A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ *has the Arrow property* if

$$\mathrm{Pol}\,\mathfrak{D} \cap \mathcal{N}(A,r) = \mathcal{M}(A,r).$$

Main theorem

Shelah's theorem

S. Shelah proved that Arrow theorem can be extended to the case when the individual choices are not rational in a very general setting.

Theorem (S. Shelah, 2005)

There are natural numbers r_1, r_2 *(e.g.* $r_1 = r_2 = 7$*) such that for any natural number* r, $r_1 \leq r \leq |A| - r_2$, *any non-empty proper symmetric subset* $\mathfrak{D}$ *of the set* $\mathfrak{C}_r(A)$ *has the Arrow property.*

We proved that, if $|A| \geq 5$, this theorem is true if $r_1 = 3$ and $r_2 = 0$. Conversely, if either $r = 2$, or $r = 3$ and $|A| = 4$, then there are non-empty proper symmetric subsets $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$ which do not have the Arrow property.

Exceptional cases

Let $|A| = 4$ and let K be the *Klein four-group* of permutations of A.
For any sets $p, q \in [A]^3$ there is only one permutation $\sigma_{p,q} \in K$ for which

$$q = \sigma_{p,q}(p).$$

We denote by the symbol $\mathfrak{C}_3^K(A)$ the set of all function $\mathfrak{c} \in \mathfrak{C}_3(A)$ such that

$$\mathfrak{c}(q) = \sigma_{p,q}\mathfrak{c}(p) \text{ for all } p, q \in [A]^3.$$

The set $\mathfrak{C}_3^K(A)$ is preserved under any simple binary function $f \in \mathcal{N}(A, 3)$ satisfying the condition

$$\sigma f_q(\mathbf{a}) = f_{\sigma q}(\sigma \mathbf{a}) \text{ for all } q \in [A]^3,\ \mathbf{a} \in q^2 \text{ and } \sigma \in K.$$

Exceptional cases

Table : $A = \{a, b, c, d\}$, $\mathfrak{C}_3^K(A) = \{\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2\}$

q	$\mathfrak{c}_0(q)$	$\mathfrak{c}_1(q)$	$\mathfrak{c}_2(q)$
$\{a, b, c\}$	a	b	c
$\{a, b, d\}$	b	a	d
$\{a, c, d\}$	c	d	a
$\{b, c, d\}$	d	c	b

Table : $f \in \mathrm{Pol}\, \mathfrak{C}_3^K(A) \cap \mathcal{N}(A, 3)$, $f \notin \mathcal{M}(A, 3)$

f_q	a	b	c	d
a	a	a	c	d
b	b	b	c	d
c	a	b	c	c
d	a	b	d	d

f	$\mathfrak{c}_0$	$\mathfrak{c}_1$	$\mathfrak{c}_2$
$\mathfrak{c}_0$	$\mathfrak{c}_0$	$\mathfrak{c}_0$	$\mathfrak{c}_2$
$\mathfrak{c}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_2$
$\mathfrak{c}_2$	$\mathfrak{c}_0$	$\mathfrak{c}_1$	$\mathfrak{c}_2$

Exceptional cases

Next we define the sets

$$\mathfrak{C}_2^0(A), \mathfrak{C}_2^1(A) \subseteq \mathfrak{C}_2(A)$$

for any set A, $|A| \geq 2$.
Let $a \in A$, $i \in \{0, 1\}$ and $\mathfrak{c} \in \mathfrak{C}_2(A)$.
Let

$$Z_a^{\mathfrak{c}} = \{b \in A \setminus \{a\} \colon \mathfrak{c}(\{a, b\}) = a\},$$

$$W_i^{\mathfrak{c}} = \{a \in A \colon |Z_a^{\mathfrak{c}}| = i \pmod 2)\},$$

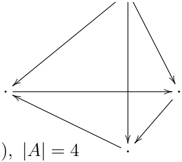
$$\mathfrak{C}_2^i(A) = \{\mathfrak{c} \in \mathfrak{C}_2(A) \colon W_{(1-i)}^{\mathfrak{c}} = \emptyset\}.$$

Exceptional cases

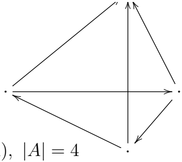
Remark.

Any function $\mathfrak{c} \in \mathfrak{C}_2(A)$ may be represented by the *tournament* $\Gamma_{\mathfrak{c}} = (A, E)$ where $E = \{(a, b) \in A^2 \colon a \neq b \wedge \mathfrak{c}(\{a, b\}) = b\}$. The sets $\mathfrak{C}_2^0(A)$ and $\mathfrak{C}_2^1(A)$ are the sets of all functions $\mathfrak{c} \in \mathfrak{C}_2(A)$ such that the *indegree* of any node of the tournament $\Gamma_{\mathfrak{c}}$ is even (respectively, odd).

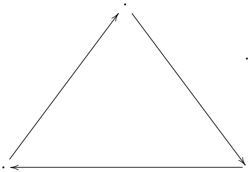
Exceptional cases



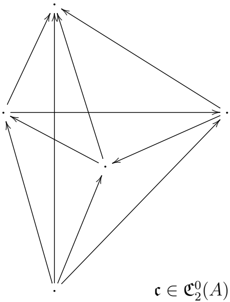
$\mathfrak{c} \in \mathfrak{C}_2^0(A), \ |A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^1(A), \ |A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^1(A), \ |A| = 3$



$\mathfrak{c} \in \mathfrak{C}_2^0(A), \ |A| = 5$

Exceptional cases

Proposition

1. *The sets* $\mathfrak{C}_2^0(A)$ *and* $\mathfrak{C}_2^1(A)$ *are symmetric,*
2. $\mathfrak{C}_2^0(A) \neq \emptyset$ *iff* n *equals* 0 *or* $1 \pmod 4$*,*
3. $\mathfrak{C}_2^1(A) \neq \emptyset$ *iff* n *equals* 0 *or* $3 \pmod 4$*,*
4. $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A) \neq \mathfrak{C}_2(A)$*.*

Each of the set $\mathfrak{C}_2^0(A)$, $\mathfrak{C}_2^1(A)$, $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$ is preserved, for example, under the (simple) ternary function $\ell \in \mathcal{N}(A, 2)$ defined by

$$\ell_q(x, x, y) = \ell_q(x, y, x) = \ell_q(y, x, x) = y$$

for all $q \in [A]^2$ and $x, y \in q$.

Main theorem

The main theorem states that there is no other "special cases".

Theorem

Let A be a finite set, r a natural number, and $\mathfrak{D}$ a non-empty proper symmetric subset of the set $\mathfrak{C}_r(A)$. Then the set $\mathfrak{D}$ does not has the Arrow property if and only if one of the following conditions holds:

1. $r = 2$, $|A|$ *equals* 0 *or* $1 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A)$,
2. $r = 2$, $|A|$ *equals* 0 *or* $3 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^1(A)$,
3. $r = 2$, $|A| = 0 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$,
4. $r = 3$, $|A| = 4$, *and* $\mathfrak{D} = \mathfrak{C}_3^K(A)$.

Outline of proof

Basic observations

We use the basic concepts of a *clone*. In universal algebra, a clone $\mathcal{F}$ on a set X is a set of functions $f\colon X^n \to X$, $n < \omega$, such that

1. $\mathcal{F}$ contains all the projections $\pi_i^m\colon X^m \to X$ $(1 \leq m < \omega$, $1 \leq i \leq m)$, defined by

$$\pi_i^m(x_1, x_2, \dots, x_m) = x_i \text{ for all } x_1, x_2, \dots, x_m \in X$$

2. $\mathcal{F}$ is closed under superposition: if $f, g_1, g_2 \dots, g_m \in \mathcal{F}$ and f is m-ary, and g_j is n-ary for every j, then the function $h\colon X^n \to X$, defined by
$h(x_1, x_2, \dots, x_n) =$

$$= f(g_1(x_1, x_2, \dots, x_n), g_2(x_1, x_2, \dots, x_n), \dots, g_m(x_1, x_2, \dots, x_n))$$

for all $x_1, x_2, \dots, x_n \in X$, is in $\mathcal{F}$.

Basic observations

Proposition

1. *The set* $\mathcal{N}(A,r)$ *is a clone on* $\mathfrak{C}_r(A)$.
2. *For any set* $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ *the set* $\operatorname{Pol}\mathfrak{D}$ *is a clone on* $\mathfrak{C}_r(A)$.
3. *For any clone* $\mathcal{F} \subseteq \mathcal{N}(A,r)$ *and any set* $p \in [A]^r$ *the set* $\{f_p \colon f \in \mathcal{F}\}$ *is a clone on* p.
4. *Let* $\mathfrak{D}$ *be a symmetrical subset of* $\mathfrak{C}(A,r)$ *and* $\mathcal{F} = \operatorname{Pol}\mathfrak{D} \cap \mathcal{N}(A,r)$. *Then the following condition holds:*

(∗) *for all n-ary function* $f \in \mathcal{F}$ *and all permutation* $\sigma \in S_A$ *the function* f^σ *defined by*

$$(\forall p \in [A]^r)\,(\forall \mathbf{a} \in p^n)\; f^\sigma_p(\mathbf{a}) = \sigma^{-1} f_{\sigma p}(\sigma \mathbf{a})$$

belongs to $\mathcal{F}$.

Basic observations

Thus we can consider Shelah theorem as a special result of theory of closed classes of discrete functions (also called functions of k-valued logic).

Some results related to this theory are relevant to our studies.

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Basic observations

The theory of closed classes of discrete functions uses the following concepts.
The set of all finitary function on A is denoted by $\mathcal{O}(A)$.
A n-ary function $f \in \mathcal{O}(A)$ *preserves* (or is a *polymorphism* of) a m-ary predicate $P \subseteq A^m$ if for any sequence of m-tuple

$$\begin{gathered}(a_{11}, a_{12}, \dots, a_{1m}),\\ (a_{21}, a_{22}, \dots, a_{2m}),\\ \dots\dots\dots,\\ (a_{21}, a_{22}, \dots, a_{2m})\end{gathered}$$

belonging to P, the m-tuple

$$f(a_{11}, a_{21}, \dots, a_{n1}), f(a_{12}, a_{22}, \dots, a_{n2}), \dots, f(a_{1m}, a_{2m}, \dots, a_{nm})$$

belong to P.

Basic observations

The set of all functions $f \in \mathcal{O}(A)$ which preserve a predicate P is denoted by $\mathrm{Pol}\, P$, and the set of all predicates $P \in \bigcup\limits_{n<\omega} \mathcal{P}(A^n)$ which is preserved under a function f is denoted by $\mathrm{Inv}\, f$.

For any set $\mathcal{F} \subseteq \mathcal{O}(A)$ and $\mathbb{P} \subseteq \bigcup\limits_{n<\omega} \mathcal{P}(A^n)$ we denote

$$\mathrm{Inv}\,\mathcal{F} = \bigcap_{f \in \mathcal{F}} \mathrm{Inv}\, f, \quad \mathrm{Pol}\,\mathbb{P} = \bigcap_{P \in \mathbb{P}} \mathrm{Pol}\, P.$$

Remark.

The pair $(\mathrm{Inv}, \mathrm{Pol})$ is a Galois connection between the Boolean lattices of $\mathcal{P}(\mathcal{O}(A))$ and $\mathcal{P}(\bigcup\limits_{n<\omega} \mathcal{P}(A^n))$.

The set $\mathcal{F} \subseteq \mathcal{O}(A)$ is Galois-closed iff $\mathcal{F}$ is a clone.

Steps of proof

The main idea of our proof is characterizing the set of all unary predicate $P \in \mathrm{Inv}\,\mathcal{F}$ for any clone $\mathcal{F} \subseteq \mathcal{N}(A, r)$ which satisfies condition $(*)$ (next we call this clones *Shelah clones*). We will illustrate our method by the outline of proof of the following weaker version of Main Theorem:

Theorem

Let A be a finite set, $|A| \geq 5$, r a natural number, $r \geq 3$, and $\mathfrak{D}$ a non-empty proper symmetric subset of $\mathfrak{C}_r(A)$. Then any simple aggregation rule in $\mathrm{Pol}\,\mathfrak{D}$ is dictatorial.

Steps of proof

Definition 5.
Let Q be a finite set. A function $f \in \mathcal{O}(A)$ preserves a set of function $D \subseteq {}^{Q}A$ if f preserves the predicate

$$P_D = \{(d(q_1), d(q_2), \dots, d(q_{|Q|}))\colon d \in D\}$$

for some enumeration of Q: $Q = \{q_1, q_2, \dots, q_{|Q|}\}$.

We will write $D \in \mathrm{Inv}\, f$ instead of $P_D \in \mathrm{Inv}\, f$.

Remark.
A function $f \in \mathcal{O}(A)$ preserves a set of function $D \subseteq {}^{Q}A$ iff for all $d_1, d_2, \dots, d_n \in D$ the function $f(d_1, d_2, \dots, d_n)$ is in D.

Steps of proof

For any simple n-ary function $f \in \mathcal{O}(A, r)$ and set $\mathcal{F} \subseteq \mathcal{S}(A, r)$ we denote

$$f^- = \{g \in \mathcal{O}(A) \colon (\forall p \in [A]^r)\, g \restriction p^n = f_p\}, \quad \mathcal{F}^- = \bigcup_{f \in \mathcal{F}} f^-.$$

Proposition

For any $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ and $\mathcal{F} \subseteq S(A, r)$

$$\mathfrak{D} \in \mathrm{Inv}\, \mathcal{F} \leftrightarrow \mathfrak{D} \in \mathrm{Inv}\, \mathcal{F}^-$$

Remark.
In left part of this formula $\mathfrak{D}$ is considered as an unary predicate on $\mathfrak{C}_r(A)$ and in right part as a set of functions in ${}^{[A]^r}A$ (i.e. $\binom{n}{r}$)-ary predicate on A).

Steps of proof

Next we prove three theorems: Theorem on Shelah clones, Preservation Theorem, and Theorem on symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$.

Notation.

For any natural number n and set $\mathcal{G} \subseteq \mathcal{O}(A)$ we denote

$$A^n_r = \{\mathbf{a} \in A^n \colon |\operatorname{ran} \mathbf{a}| = r\},$$

$$A^n_{<r} = \{\mathbf{a} \in A^n \colon |\operatorname{ran} \mathbf{a}| < r\},$$

$$\mathcal{G}_{[n]} = \mathcal{G} \cap {}^{A^n}A.$$

Steps of proof

We say that a clone $\mathcal{F} \subseteq \mathcal{O}(A)$ satisfies condition

Δ^e_k, if there is a natural number $i < k$ such that for any $\mathbf{a} \in A^k_k$ and $a \in \operatorname{ran} \mathbf{a}$ there is a function $w \in \mathcal{F}_{[k]}$ such that

$$w(\mathbf{a}) = a \text{ and } w(\mathbf{b}) = b_i$$

for any sequence $\mathbf{b} = (b_0, b_1, \ldots, b_{r-1}) \in A^k_{<k}$;

Δ^∂, if for any $\mathbf{a} \in A^3_3$ and $a \in \operatorname{ran} \mathbf{a}$ there is a function $w \in \mathcal{F}_{[3]}$ such that

$$w(\mathbf{a}) = a \text{ and } w(x, x, y) = w(x, y, x) = w(y, x, x) = x$$

for all $x, y \in A$;

Δ^2, if for any $\mathbf{a}, \mathbf{b} \in A^2_2$, $\operatorname{ran} \mathbf{a} \neq \operatorname{ran} \mathbf{b}$, and $a \in \operatorname{ran} \mathbf{a}$, $b \in \operatorname{ran} \mathbf{b}$ there is a function $w \in \mathcal{F}_{[2]}$ such that

$$w(\mathbf{a}) = a, w(\mathbf{b}) = b \text{ and } w(x, x) = x \text{ for all } x \in A.$$

Theorem on Shelah clones

Theorem

Let A be a finite set, $|A| \geq 5$, and let r be a natural number, $3 \leq r \leq |A|$.
Then for any Shelah clone $\mathcal{F} \subseteq \mathcal{S}(A, r)$, $\mathcal{F} \neq \mathcal{M}(A, r)$, the clone $\mathcal{F}^{-}$ satisfies on of the conditions Δ^2, Δ^∂, Δ_k^e for some k, $3 \leq k \leq r$.

Preservation theorem

Notation.
For any elements $p, q \in Q$, $a, b \in A$ and permutation $\sigma \in S_A$ we denote

$$H_0(p,q,\sigma) = \{h \in {}^Q\!A \colon h(q) = \sigma h(p)\};$$

$$H_1(p,q,a,b) = \{h \in {}^Q\!A \colon h(p) = a \vee h(q) = b\};$$

$$\mathbb{H}_{\leftrightarrow} = \{H_0(p,q,\sigma) \colon p,q \in Q, p \neq q, \sigma \in S_A\};$$

$\mathbb{H}_{=} = \{H_0(p,q,\mathrm{Id}) \colon p,q \in Q, p \neq q\}$, where Id is the identity permutation;

$$\mathbb{H}_{\vee} = \{H_1(p,q,a,b) \colon p,q \in Q, p \neq q, a,b \in A\}.$$

Preservation theorem

For any set $H \subseteq {}^{Q}A$, set $Q' \subseteq Q$, set $B \subseteq A$, element $q \in Q$ and natural number r we denote

$$H^{+}_{Q'} = \{h \in {}^{Q}A \colon h \restriction Q' \in H \restriction Q'\};$$
$$H(q) = \{h(q) \colon h \in H\};$$
$$H^{+} = \{h \in {}^{Q}A \colon (\forall q \in Q)\, h(q) \in H(q)\};$$
$$H^{-1}(B) = \{q \in Q \colon H(q) \in B\};$$
$$H^{-1}(< r) = \{q \in Q \colon |H(q)| < r\}.$$

Preservation theorem

Theorem

Let A, Q be a finite sets, $\mathcal{F}$ be a clone on A and H be a subset of ${}^{Q}A$. Let $H \in \operatorname{Inv} \mathcal{F}$. Then,

1. *if $\mathcal{F}$ satisfies condition Δ_k^e for some natural number $k \geq 3$, then there is a set $\mathbb{H} \subseteq \mathbb{H}_{\leftrightarrow}$ such that $H = H^+ \cap H^+_{H^{-1}(<k)} \cap \bigcap \mathbb{H}$,*
2. *if $\mathcal{F}$ satisfies condition Δ^{∂}, then there is a set $\mathbb{H} \subseteq \mathbb{H}_{\leftrightarrow} \cup \mathbb{H}_{\vee}$ such that $H = H^+ \cap \bigcap \mathbb{H}$,*
3. *if $\mathcal{F}$ satisfies condition Δ^2, then there is a set $\mathbb{H} \subseteq \mathbb{H}_{=}$ such that $H = H^+ \cap \bigcap \{H^+_{H^{-1}(B)} : B \in [A]^2\} \cap \bigcap \mathbb{H}$.*

Theorem on symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$

Theorem

Let A be a finite set, $|A| \geq 5$, and let r be a natural number, $2 \leq r < |A|$. Let $\mathfrak{D}$ be a symmetric subset of $\mathfrak{C}_r(A)$. Then $\mathfrak{D} \cap H = \emptyset$ for any $H \in \mathbb{H}_{\leftrightarrow} \cup \mathbb{H}_{\vee}$.

In other terms, in conditions of the Theorem, for any pair of different sets $p, q \in [A]^r$ there are two different elements $a, b \in p$ and four function $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4$ such that

$$\mathfrak{c}_1(p) = \mathfrak{c}_2(p) = a, \mathfrak{c}_1(q) \neq \mathfrak{c}_2(q),$$

$$\mathfrak{c}_3(p) = \mathfrak{c}_4(p) = b, \mathfrak{c}_3(q) \neq \mathfrak{c}_4(q).$$

Proof of the weaker version of Main Theorem

The weaker version of Main Theorem immediately follows from the above theorems.
Really, in conditions of the theorem we have:

- $\mathfrak{D} \cap H = \emptyset$ for any $H \in \mathbb{H}_{\leftrightarrow} \cup \mathbb{H}_{\vee}$ (from Theorem on symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$),
- $\mathfrak{D}^{-1}(< r) = \emptyset$, $\mathfrak{D}^{-1}(B) = \emptyset$ for any $B \in [A]^2$ (obviusly),
- $\mathfrak{D} = \mathfrak{D}^+ = \mathfrak{C}_r(A)$ (from Theorem on Shelah clones and Preservation theorem), a contradiction.

Corrolary: Impossibility theorem for symmetric class of choice function

Impossibility theorem

Our Main Theorem is not formulated as an "impossibility theorem", because it demonstrates that some of the sets of choice functions do not satisfy the Arrow property. However, by considering aggregation rules that satisfy some additional condition, we can formulate a corollary that is an impossibility theorem for all non-empty proper symmetric subsets $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$.

Impossibility theorem

Proposition

Let A be a finite set. Let $\mathfrak{D}$ be a non-empty proper symmetric subset of the set $\mathfrak{C}_2(A)$. Let $\mathfrak{D}$ do not has the Arrow property. Then if $|A| \geq 5$, the clone $\mathrm{Pol}\,\mathfrak{D} \cap \mathcal{N}(A, r)$ is generated by the normal (simple) function $\ell\colon (\mathfrak{C}_2(A))^3 \to \mathfrak{C}_2(A)$ defined by

$$\ell_p(x, x, y) = \ell_p(x, y, x) = \ell_p(y, x, x) = y$$

for all $p \in [A]^2$ and $x, y \in p$.

(A clone $\mathcal{F}$ is *generated* by a function $f \in \mathcal{O}(X)$ if $\mathcal{F}$ is the minimal clone on X which contains f).

Impossibility theorem

We will call a normal aggregation rule $f\colon (\mathfrak{C}_2(A))^n \to \mathfrak{C}_2(A)$ *conjectural*, if there exist a set $I \subseteq \{0, 1, \dots, n-1\}$, $|I|$ is odd, such that

$$f_q(a_0, a_1, \dots, a_{n-1}) = a_j \leftrightarrow |\{i \in I\colon a_i = a_j\}| \text{ is odd}$$

for all $q \in [A]^2$, $a_0, a_1, \dots, a_{n-1} \in q$ and $j \in \{0, 1, \dots, n-1\}$.

Proposition

The clone $\mathcal{F}$ on $\mathfrak{C}_2(A)$ generated by the function ℓ is the set of all conjectural aggregation rules.

Impossibility theorem

Theorem

Let A be a finite set, $|A| \geq 5$, and let $\mathfrak{D}$ be a non-empty proper symmetric subset of the set $\mathfrak{C}_r(A)$ for some natural number r. Then there exists no normal non-dictatorial and non-conjectural aggregation rule f which preserves the set $\mathfrak{D}$.

Thank you!