Binary Aggregation with Integrity Constraints

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Binary aggregation studies problems in which individuals express yes/no choices over a number of possibly correlated issues, and these individual choices need to be aggregated into a collective choice. We show how classical frameworks of Social Choice Theory, particularly preference and judgment aggregation, can be viewed as binary aggregation problems by designing an appropriate set of integrity constraints.

We explore the generality of this framework, showing that it makes available useful techniques both to prove theoretical results and to analyse practical problems. We obtain new impossibility and characterisation theorems, we formulate a general definition of paradox that is independent of the domain under consideration, and we study the class of aggregation procedures of generalised dictatorships.

The Framework: Binary Aggregation

- A finite set $\mathcal{N}$ of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean *combinatorial domain*: $\mathcal{D}=D_1\times\ldots\times D_m$ with $|D_i|=2$

Definition 1. *An aggregation procedure is a function* $F : \mathcal{D}^\mathcal{N} \to \mathcal{D}$ *mapping each profile of ballots* $\underline{B} = (B_1, \ldots, B_n)$ *to an element of the domain* $\mathcal{D}$.

Integrity Constraints

We define a propositional language $\mathcal{L}$ to express integrity constraints on $D = \{0,1\}^m$ to express what is a rational ballot:

- One propositional symbol for every issue: $PS = \{p_1, \ldots, p_m\}$
- $\mathcal{L}_{PS}$ closing under connectives $\wedge, \vee, \neg, \to$ the set of atoms PS

Given IC $\in \mathcal{L}_{PS}$, a rational ballot is $B \in \mathrm{Mod}(\mathrm{IC})$

Town Council (with congested roads)

A council has to decide on whether to build a Shopping mall, a Train station, and a new Road. If there is both a train station and a shopping mall then a new road is necessary.
Propositional constraint: IC $= (p_S \wedge p_T) \to p_R$

Individual 1 submits $B_1 = (1,0,0)$: B_1 satisfies IC ✓
Individual 2 submits $B_2 = (1,1,1)$: $B_2 \models$ IC ✓
Individual 3 submits $B_3 = (0,1,0)$: $B_3 \models$ IC ✓

Majority aggregation outputs $(1,1,0)$: IC not satisfied (as are drivers...)

Definition of Paradox

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition 2. *A paradox is a triple* $(F, \underline{B}, \mathrm{IC})$*, where:*

- F *is an aggregation procedure*
- $\underline{B} = (B_1, \ldots, B_n)$ *a profile*
- IC $\in \mathcal{L}_{PS}$ *an integrity constraint*

such that $B_i \models$ *IC for all* $i \in \mathcal{N}$ *but* $F(\underline{B}) \not\models$ *IC.*

Characterisation Results

Given an integrity constraint, which conditions (e.g., classical social choice axioms) can we assume to avoid paradoxes?

Definition 3. F *is collectively rational (CR) for IC* $\in \mathcal{L}_{PS}$ *if for all profiles* $\underline{B}$ *such that* $B_i \models$ *IC for all* $i \in N$ *then* $F(\underline{B}) \models$ *IC.*

Proposition 1. *A quota rule is CR for a k-pclause IC if and only if* $\sum_j q_j < n+k$*, with* j *ranging over all issues that occur in IC and* n *being the number of individuals, or* $q_j = 0$ *for at least one issue* j *that occurs in IC.*

More on characterisation results: *Grandi and Endriss, Lifting Rationality Assumptions in Binary Aggregation. AAAI-2010*

Preference Aggregation

Embedding PA

If we represent preferences with linear orders, a social welfare function aggregates every profile of linear orders $(<_1, \ldots, <_n)$ into a collective order.

Linear order $<$ over alternatives $\mathcal{X}$ $\Leftrightarrow$ Ballot $B_<$ over issues $\mathcal{I} = \{ab \mid a \neq b \in \mathcal{X}\}$

Property of linear orders enforced with $\mathrm{IC}_<$:

Completeness and antisymmetry: $p_{ab} \leftrightarrow \neg p_{ba}$ for $a \neq b \in \mathcal{X}$
Transitivity: $p_{ab} \wedge p_{bc} \to p_{ac}$ for $a, b, c \in \mathcal{X}$ pairwise distinct

Social welfare function $\Leftrightarrow$ Binary aggregation proc. CR with respect to $\mathrm{IC}_<$

Axioms are preserved: unanimity, IIA, neutrality...

Condorcet Paradox

	ab	bc	ac
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- IC that is violated is $p_{ab} \wedge p_{bc} \to p_{ac}$

References: *Arrow (1951), Condorcet (18th century)*

Impossibility Result

A SWF is imposed for x and y if x is collectively preferred to y in every profile

Proposition 2. *Any anonymous, independent and monotonic SWF for more than 3 alternatives and 2 individuals is imposed.*

New proof method:
search for clashes between IC and axiomatic properties

- Use correspondence between A, I, M social welfare functions and A, I and M binary aggregation procedures that are CR wrt $\mathrm{IC}_<$
- A, I and M procedures are quota rules (Dietrich and List, 2007)
- Use characterisation result: quota rule lift IC iff satisfies property on quotas or the quota is zero for at least one issue
- $\mathrm{IC}_<$ does not satisfy this property therefore procedure is imposed!

Judgment Aggregation

Embedding JA

JA studies aggregation of judgments over sets of correlated propositions:

Judgment sets J over agenda Φ $\Leftrightarrow$ Ballot B_J over issues $\mathcal{I} = \Phi$

Property of judgment sets enforced with IC_Φ:

Completeness: $p_\alpha \vee p_{\neg\alpha}$ for all $\alpha \in \Phi$
Consistency: $\neg(\bigwedge_{\alpha \in S} p_\alpha)$ for every mi-set $S \subseteq \Phi$

Complete and consistent JA procedures for Φ $\Leftrightarrow$ Binary aggregation proc. CR with respect to IC_Φ

References: *List and Puppe (2009), Endriss, Grandi and Porello (AAMAS-2010)*

Doctrinal Paradox

	α	β	$\alpha \wedge \beta$
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_\beta \wedge p_{\neg(\alpha\wedge\beta)})$

Median property & co.

Agenda properties can be written as syntactic conditions on IC_Φ

Proposition 3. *An agenda* Φ *does not generate a paradox (*Φ *is safe) for neutral judgment aggregation procedures if and only if* Φ *satisfies the syntactic median property (i.e., only inconsistencies between equivalent formulas).*

New proof method:
Use BA characterisation results to guarantee safety

- Use correspondence between neutral JA and BA procedures
- Use characterisation result: a neutral BA procedure is collectively rational with respect to IC iff it is of the form $p_i \leftrightarrow p_j$.
- IC_Φ is a set of positive bi-implications iff Φ has simplified median property

The Majority Rule

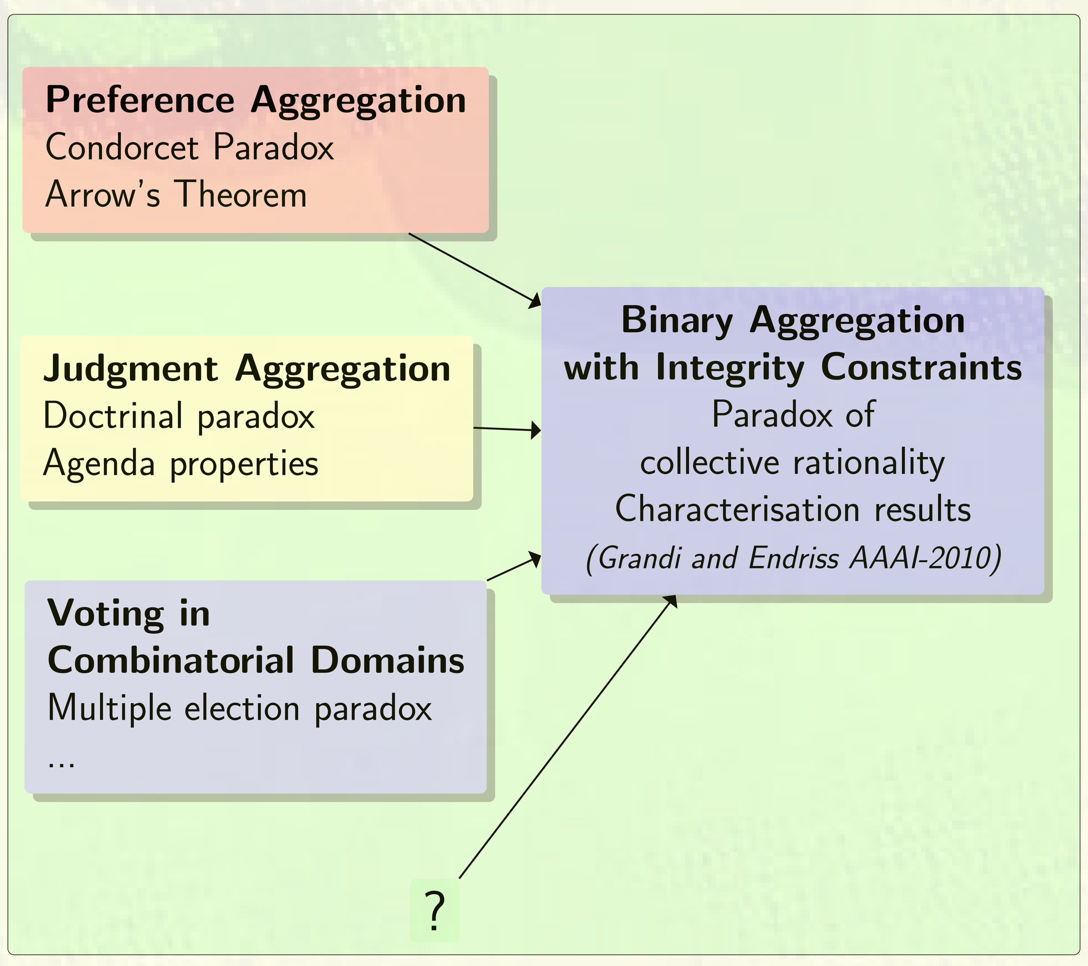
We solved an open problem from Grandi and Endriss (AAAI-2010):

Proposition 4. *The majority rule is CR with respect to IC if and only if IC is equivalent to a conjunction of clauses of size* $\leqslant 2$.

Proof sketch:

- BA problems can be viewed as special JA problems with a universally accepted law represented by the IC
- Import results from JA: Nehring and Puppe (2007) proves that majority rule is consistent iff there are no minimally inconsistent subsets of size less than 2 in the agenda

Common feature of previous paradoxes:
Condorcet: $p_{ab} \wedge p_{bc} \to p_{ac}$
Doctrinal: $\neg(p_\alpha \wedge p_\beta \wedge p_{\neg(\alpha\wedge\beta)})$
Town Council: $(p_S \wedge p_T) \to p_R$
Clauses of size 3 are not lifted by majority



How to avoid all paradoxes?

A generalised dictatorship copies the ballot of a (possibly different) individual in every profile.

Proposition 5 (Grandi and Endriss, AAAI-2010). F *is collectively rational with respect to all IC in* $\mathcal{L}_{PS}$ *if and only if* F *is a generalised dictatorship.*

An interesting definition: choose the individual whose ballot is closest to the ballots of the others:

$$\mathrm{DBGD}(\underline{B}) = \underset{\{B_i \mid i \in \mathcal{N}\}}{\operatorname{argmin}} \sum_{i' \in \mathcal{N}} H(B_i, B_{i'}),$$

where $H(B, B') = \sum_{j \in \mathcal{I}} |b_j - b'_j|$ is the Hamming distance.

A good compromise between paradoxes and complexity?

- Satisfies neutrality, anonimity and monotonicity (adapted)
- Is computationally tractable
- Is collectively rational for any IC

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