Reduction of Economic Inequality in Combinatorial Domains

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Abstract: Criteria for measuring economic inequality, such as the Lorenz curve and the Gini index, are widely used in the social sciences but have hardly been explored in Multiagent Systems, even though the significance of other concepts from fair division is widely accepted in the field. In a departure from the standard model used in Economics, we apply inequality criteria to allocation problems with indivisible goods, i.e., to the kind of problem typically analysed in Multiagent Systems. This gives rise to the combinatorial optimisation problem of computing an allocation that reduces inequality with respect to an initial allocation (and the closely related problem of minimising inequality), for a chosen inequality measure. We define this problem, we discuss the computational complexity of various aspects of it, and we formulate a generic approach to designing modular algorithms for solving it using integer programming.

The Model
We work in the standard model of multiagent resource allocation. Finite sets of agents $\mathcal{N} = \{1, \ldots, n\}$ and of indivisible goods $\mathcal{G}$. Each good needs to be allocated to exactly one agent. Any given allocation $A$ induces a utility of $u_i(A)$ for agent $i \in \mathcal{N}$. So any allocation $A$ induces a utility vector $(u_1(A), \ldots, u_n(A))$.

The Problem
We want fair allocations. How can we measure the inequality inherent in a given utility vector (induced by a given allocation)? Once we have settled on a definition, how can we compute good allocations in view of that definition in practice? And how complex is this task in theory?

Measuring Inequality
Which is more equal, $(1,2,7,7,8)$ or $(1,3,5,6,10)$?

The Pigou-Dalton Principle
A move from allocation $A$ to $A'$ is called a Pigou-Dalton transfer if there are two agents $i, j \in \mathcal{N}$ such that:

- Only the bundles held by $i$ and $j$ change.
- Inequality reduces: $|u_i(A) - u_i(A')| > |u_j(A') - u_j(A)|$
- Total utility does not reduce: $u_i(A) + u_j(A) \leq u_i(A') + u_j(A')$

By the Pigou-Dalton Principle, any measure of fairness should consider Pigou-Dalton transfers (weak) improvements.

The Lorenz Curve
Ideally, all agents enjoy the same utility. The Lorenz curve is a way to visualise how far we are from this ideal.

Let $u^*(A)$ be the ordered utility vector of $A$. Then $L_k(A) = \sum_{i=1}^{k} u_i^*(A)$ is the total utility of the $k$ poorest agents. The vector $(L_1(A), \ldots, L_n(A))$ is called the Lorenz curve of $A$.

Inequality Indices
An inequality index is a quantitative measure of inequality of an allocation (usually $\in [0,1]$). Two popular indices:

- Gini index = area between line of perfect equality and Lorenz curve (divided by a suitable normalisation factor)
- Robin Hood index = maximal distance between line of perfect equality and Lorenz curve (also normalised)

Representation Languages
Before we can design an algorithm to reason about a multiagent resource allocation problem and before we can study its complexity, we need to fix a language for representing utility functions.

Three natural choices (familiar from combinatorial auctions):

- OR-language: compact, reasonably expressive, highly intractable
- XOR-language: not compact, fully expressive
- weighted goals: compact, fully expressive (for certain goals)

Interesting special case: additive utility functions (weighted atoms).

Complexity Results
Checking whether a Pigou-Dalton transfer is possible is

- polynomial for the XOR-language;
- NP-complete for the OR-language [NP-membership is open];
- not polynomial unless $\text{NP} = \text{coNP}$ for additive utilities.

Checking whether a Lorenz improvement is possible is

- NP-complete for the XOR-language;
- NP-complete for the OR-language;
- NP-complete for many weighted goal languages.

Also in the paper: algorithms for computing Lorenz improvements and minimising inequality indices (using integer programming)