

# Simulating Multiwinner Voting Rules in Judgment Aggregation

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## ABSTRACT

We simulate voting rules for multiwinner elections in a model of judgment aggregation that distinguishes between rationality and feasibility constraints. These constraints restrict the structure of the individual judgments and of the collective outcome computed by the rule, respectively. We extend known results regarding the simulation of single-winner voting rules to the multiwinner setting, both for elections with ordinal preferences and for elections with approval-based preferences. This not only provides us with a new tool to analyse multiwinner elections, but it also suggests the definition of new judgment aggregation rules, by generalising some of the principles at the core of well-known multiwinner voting rules to this richer setting. We explore this opportunity with regards to the principle of proportionality. Finally, in view of the computational difficulty associated with many judgment aggregation rules, we investigate the computational complexity of our embeddings and of the new judgment aggregation rules we put forward.

## KEYWORDS

Multiwinner Voting; Judgment Aggregation; Proportional Representation; Computational Complexity

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## 1 INTRODUCTION

A wide variety of formal frameworks have been proposed in the literature to both perform and analyse collective decision making. Here several agents report an individual view and we need to determine a single collective view that presents a reasonable compromise. Examples for collective decision making by groups of human agents include voting [41], participatory budgeting [5], and judgment aggregation in a court of law [28]. But also many problems long studied in AI—such as belief merging [13], collective argumentation [7], and consensus clustering [24]—can be seen in this vein [17]. To improve our understanding of the mechanisms that have been proposed for different settings and to enable us to transfer some of the knowledge gained in one domain to another domain of collective decision making, it is important to isolate fundamental building blocks that are common to different solutions.

In this paper, we contribute to this broad research agenda by investigating the extent to which it is possible to simulate so-called *multiwinner voting rules*, i.e., voting rules to elect a committee of representatives based on the preferences expressed by a group of

voters [19], within the framework of logic-based *judgment aggregation* (JA) [15, 25, 30]. Regarding the input to an election, we consider both rules in which voters express *ordinal preferences* (by providing a ranking of the candidates) and rules in which they express *approval-based preferences* (by indicating which candidates they do and do not approve of). Regarding the output, we focus on the standard scenario where a *committee of fixed size* is to be elected. For our embeddings of multiwinner voting rules, we use the framework of JA with *rationality* and *feasibility constraints* [16]. The former constrains the range of admissible inputs to an aggregation problem, and the latter constrains its possible outputs. This leads to particularly simple and natural embeddings.

Embedding a multiwinner voting rule into JA, which is much more expressive than most voting frameworks, makes it very natural to study refinements of standard rules in a principled manner, e.g., by imposing additional constraints on outcomes or by varying the types of preferences voters can report. This not only permits us to clarify the commonalities (and differences) between rules originally developed for different purposes, but also allows for the development of new rules with particular properties. Finally, our approach can also be used to import ideas from multiwinner voting into JA, and we shall do so for the concept of *proportionality*, which has received significant attention in multiwinner voting but not JA. Importantly though, the increased expressive power of JA does not come for free: JA is a framework that, generally speaking, is computationally much more demanding than voting. To address this challenge, we analyse the extent to which the feasibility constraints featuring in our embeddings can be encoded as Boolean circuits in *decomposable negation normal form*, which by a recent result allows for the design of tractable aggregation rules [26].

**Related work.** The idea of modelling problems of preference aggregation within the framework of JA by means of a so-called preference agenda goes back to, at least, the work of Dietrich and List [12]. While a number of authors, such as Miller and Osherson [32] and Dietrich [11], have discussed parallels between specific voting rules and specific JA rules, Lang and Slavkovik [29] were the first to systematically investigate the question of how to translate common voting rules into JA. Endriss [16] refined their approach and showed that explicitly distinguishing between rationality and feasibility constraints in JA greatly simplifies the task of arriving at principled embeddings. We use the same basic approach also here. Our results regarding the tractability of JA rules build on the approach for identifying tractable fragments of JA developed by de Haan [26]. Recently, this technique has also been used to obtain tractable embeddings of participatory budgeting into JA [38].

**Contribution.** We simulate the  $k$ -Borda,  $k$ -Copeland, and simple Approval Voting multiwinner voting rules in JA and show how to use some of the additive majority rules to produce outcomes that correspond to Gehrlein-stable committees. We propose JA rules aimed at transferring the proportionality notion found in

multiwinner voting. We also analyse the complexity of the relevant JA rules, obtaining both tractability and intractability results.

**Outline.** The remainder of this paper is structured as follows. In Section 2, we recall basic definitions regarding both multiwinner voting and JA. Section 3 presents our results for both the simulation of multiwinner voting rules in JA, and a complexity analysis of the resulting embeddings. In Section 4, we study proportionality in and analyse the complexity of our new JA rules for proportionality.

## 2 PRELIMINARIES

In this section, we recall the standard models of both multiwinner voting and JA. Throughout, for any given set  $U$ , we use  $\mathcal{P}(U)$  to represent its powerset,  $\mathcal{P}_+(U)$  the set of all of its nonempty subsets, and  $\mathcal{P}_k(U)$  the set of all of its subsets of size  $k$ .

### 2.1 Multiwinner Voting

Let  $X$  be a finite set of *alternatives* and let  $N = \{1, \dots, n\}$  be a set of *agents*. Each agent  $i \in N$  has a weak preference order  $\succsim_i$  on  $X$ , with  $\succ_i$  denoting the strict part of  $\succsim_i$  and  $R$  being the set of all possible weak preference orders. We use weak preferences as this allows us to consider the two scenarios considered in much of the literature on multiwinner voting—namely (strict) *ordinal* preferences and *approval-based* preferences [19]—as special cases. In the former, each voter  $i$  provides a strict ranking of alternatives, such as  $a \succ_i b \succ_i c$ . The latter requires a voter  $i$  to simply provide a set of alternatives  $A_i \subseteq X$  she approves of. This can be modelled with a *dichotomous* preference order.

A *profile* is a vector  $(\succ_1, \dots, \succ_n)$  of preferences, one for each voter. A *voting rule* is a function  $F : R^n \rightarrow \mathcal{P}_+(X)$  mapping any such profile to a set of winning committees. Ideally, there will be a single winning committee, but in general  $F$  may be irresolute. So in practice, a tie-breaking rule may have to be used post-election. When we refer to a fixed target committee size, we denote it as  $k$ .

First, let us recall two rules for ordinal ballots  $\succ_i$ . The *k-Borda* rule elects the committee/s of the  $k$  alternatives with the highest Borda scores, defined as  $B(x) = \sum_{i \in N} |\{y \in X \mid x \succ_i y\}|$  for alternative  $x$ . The *k-Copeland* rule selects the committee/s that consist of the  $k$  alternatives with the highest Copeland scores. The Copeland score of a single alternative  $x$  is defined as  $C(x) = |\{y \in X \mid x \succ_M y\}| - |\{y \in X \mid y \succ_M x\}|$ , where  $\succ_M$  is the strict majority relation ( $x \succ_M y$  if and only if  $|\{i \in N \mid x \succ_i y\}| > n/2$ ).

Next, let us review some of the most important proposals for approval-based rules from the literature on multiwinner voting. Several of these rules belong to the family of *Thiele rules* [40]. Suppose we want to elect a committee of size  $k$ . For a given vector of numbers  $\mathbf{w}^{(k)} = (w_1, \dots, w_k)$ , called the *scoring vector*, the corresponding Thiele rule elects the following committee/s when presented with a profile of approval ballots  $(A_1, \dots, A_n)$ :  $\text{argmax}_{C \in \mathcal{P}_k(X)} \sum_{i \in N} \sum_{j=1}^{|C \cap A_i|} w_j$ . Given this general template, we adjust the scoring vector to vary the rules. For instance, (simple) *Approval Voting* (AV) uses  $(1, 1, 1, \dots, 1)$  as the scoring vector. Thus, AV outputs the  $k$  alternatives that appear in the approval ballots most often. Two important rules often advocated when we require some form of *proportional representation* are *Proportional Approval Voting* (PAV) and the *Approval-Based Chamberlin-Courant* rule  $\alpha$ -CC, with the vectors  $(1, 1/2, 1/3, \dots, 1/k)$  and  $(1, 0, 0, \dots, 0)$ , respectively.

### 2.2 Judgment Aggregation

Again, let  $N = \{1, \dots, n\}$  be a set of *agents*. For the purposes of this paper, we shall assume by default that  $n$  is odd, unless explicitly stated otherwise, so as to avoid considerations of tied majorities. We ask each agent to either accept or reject each of the issues in the *agenda*  $\Phi$ , a finite set of propositional atoms which we refer to as *propositions*. A judgment is a function  $J : \Phi \rightarrow \{0, 1\}$ , where acceptance of an agenda item is represented by 1 and rejection by 0. For any two judgments  $J$  and  $J'$ , we use  $\text{Agr}(J, J') = \{\varphi \in \Phi \mid J(\varphi) = J'(\varphi)\}$  to refer to the set of agenda items that they agree on and  $\text{Dis}(J, J') = \{\varphi \in \Phi \mid J(\varphi) \neq J'(\varphi)\}$  to refer to the set of those they disagree on. We extend both of these definitions to versions where we only count (dis)agreements in case the first judgment accepts the proposition in question:  $\text{Agr}(J, J')^+ = \text{Agr}(J, J') \cap \{\varphi \in \Phi \mid J(\varphi) = 1\}$  and  $\text{Dis}(J, J')^+ = \text{Dis}(J, J') \cap \{\varphi \in \Phi \mid J(\varphi) = 1\}$ . A *profile* is a vector  $\mathbf{J} = (J_1, \dots, J_n) \in (\{0, 1\}^\Phi)^n$  of judgments, one for each agent. The intensity of the *support* of an issue  $\varphi \in \Phi$  in a profile  $\mathbf{J}$  is denoted as  $n_{(\mathbf{J}, \varphi)} = |\{i \in N \mid J_i(\varphi) = 1\}|$ . Given  $\mathbf{J}$ , we can define the (strict) *majority judgment* for each  $\varphi \in \Phi$  as follows:  $\text{Maj}(\mathbf{J})(\varphi) = 1$  if  $n_{(\mathbf{J}, \varphi)} > n/2$  and  $\text{Maj}(\mathbf{J})(\varphi) = 0$  otherwise.

An *aggregation rule* is a function  $F$  that takes as input a profile and outputs a judgment that is meant to represent a reasonable choice for a collective judgment. Although, ideally, this rule returns a single judgment, most rules allow for a tie between judgments, thereby possibly returning a set of judgments. Thus, formally, an aggregation rule is a function  $F : (\{0, 1\}^\Phi)^n \rightarrow \mathcal{P}_+(\{0, 1\}^\Phi)$  that maps any given profile to a nonempty set of judgments. An example of such a function is the majority rule  $F : \mathbf{J} \mapsto \{\text{Maj}(\mathbf{J})\}$ . Let  $\mathcal{L}(\Phi)$  denote the propositional language with the set of agenda items in  $\Phi$  taking the role of propositional variables. We use formulas in this language to express constraints  $\Gamma$  regarding judgments:  $J \models \Gamma$  holds if  $\Gamma$  is true under the truth assignment corresponding to  $J$ . The set of all judgments that satisfy a given constraint  $\Gamma$  is  $\text{Mod}(\Gamma) = \{J \in \{0, 1\}^\Phi \mid J \models \Gamma\}$ . Following Endriss [16], we use such constraints both to express *rationality* constraints, i.e., constraints indicating an acceptable input to an aggregation rule, and *feasibility* constraints, i.e., constraints indicating an acceptable output. We then say that an aggregation rule  $F$  *guarantees*  $\Gamma'$ -feasible outcomes on  $\Gamma$ -rational profiles, if  $F(\mathbf{J}) \subseteq \text{Mod}(\Gamma')$  holds for every profile  $\mathbf{J} \in \text{Mod}(\Gamma)^n$ .

### 2.3 Majoritarian Rules

We will make use of well-known JA rules, all of which guarantee by definition that the output will satisfy a given feasibility constraint  $\Gamma'$ . We adopt the naming conventions used by Endriss [16] but also mention alternatives used in the literature.

The *max-num* rule, also known as the *endpoint rule* [32] and the *generalised Slater rule* [34], selects judgments for which the number of agreements with the majority outcome is maximal:

$$\text{max-num}(\mathbf{J}, \Gamma') = \text{argmax}_{J \in \text{Mod}(\Gamma')} |\text{Agr}(J, \text{Maj}(\mathbf{J}))|$$

The *max-sum* rule, also known as the *prototype rule* [32], the *median rule* [34], and the *generalised Kemeny rule* [11], maximises the sum of agreements with the profile:

$$\text{max-sum}(\mathbf{J}, \Gamma') = \text{argmax}_{J \in \text{Mod}(\Gamma')} \sum_{i \in N} |\text{Agr}(J, J_i)|$$

Note that both of these two rules really constitutes an entire family of aggregation rules, one for each feasibility constraint  $\Gamma'$ . To refer to the aggregation rule from a family of aggregation rules  $F$ , for a given feasibility constraint  $\Gamma'$ , we write  $F(\cdot, \Gamma')$ .

### 3 SIMULATION OF MULTIWINNER RULES

We begin this section by detailing a preference agenda for our JA model. We then define our method of simulation, before presenting simulations for specific multiwinner voting rules. Finally, we investigate the computational efficiency of the outcome determination problem in JA for the constraints featuring in these simulations.

#### 3.1 Preference Agenda and Constraints

When voting, every agent reports a preference, as either a strict ranking or an approval set. The outcome of an election can also be viewed as a preference: all of the alternatives elected are (collectively) preferred to all those not elected. Next, we prepare the grounds for our embeddings of multiwinner voting rules into JA by defining a number of constraints that can be used to model relevant properties of such preferences.

Given a set of alternatives  $X$ , we let  $\Phi_{\succsim}^X = \{p_{x \succsim y} \mid x, y \in X\}$  be the *preference agenda* [12, 15]. Now we can think of accepting the proposition  $p_{x \succ y}$  as expressing a (weak) preference of  $x$  over  $y$ . We furthermore write  $p_{x > y}$  as a shorthand for  $p_{x \succ y} \wedge \neg p_{y \succ x}$ .

We now can express properties of binary relations as constraints in our logical language. Each constraint is defined for some set  $A \subseteq X$ . This includes, in particular, common properties of preference relations such as completeness, antisymmetry, and transitivity:

$$\text{COMPLETE}_A = \bigwedge_{x, y \in A} (p_{x \succ y} \vee p_{y \succ x})$$

$$\text{ANTI-SYM}_A = \bigwedge_{x, y \in A \text{ s.t. } x \neq y} \neg(p_{x \succ y} \wedge p_{y \succ x})$$

$$\text{TRANSITIVE}_A = \bigwedge_{x, y, z \in A} (p_{x \succ y} \wedge p_{y \succ z} \rightarrow p_{x \succ z})$$

We can now formulate a constraint that is satisfied by a judgment on  $\Phi_{\succsim}^X$  that corresponds to a strict ranking of all alternatives in  $X$ :

$$\text{RANKING} = \text{COMPLETE}_X \wedge \text{ANTI-SYM}_X \wedge \text{TRANSITIVE}_X$$

For two alternatives, for which it is not the case that you strictly prefer one over the other, there are two possibilities: you either are *indifferent* between them or you consider them *incomparable*. The next two constraints express indifference and incomparability, respectively, between all the alternatives in  $A$ :

$$\text{INDIFF}_A = \bigwedge_{x, y \in A} (p_{x \succ y} \wedge p_{y \succ x})$$

$$\text{INCOMP}_A = \bigwedge_{x, y \in A} \neg(p_{x \succ y} \vee p_{y \succ x})$$

Again for a given set  $A \subseteq X$ , the following constraint expresses a strict preference for all alternatives in  $A$  over all those not in  $A$ :

$$\text{TOP}_A = \bigwedge_{x \in A} \bigwedge_{y \in X \setminus A} (p_{x > y})$$

This property is satisfied, for instance, by an agent's approval ballot in case the agent approves of exactly the alternatives in  $A$ . But it also holds for the collective preference returned by a multiwinner voting rule in case  $A$  is the set of winning alternatives.

We are going to require constraints to describe both that (i) there exists such a set  $A$  of most preferred alternatives and that (ii) there exists such a set and that this set has size  $k$ . In both cases, we may assume either indifference between all the alternatives within the same set, or all of these alternatives being incomparable. We focus on five such constraints. For the first two of them, the first part of

the name indicates the structure of the top set, while the second part indicates that of the bottom set. The remaining three constraints follow the same naming convention, while also being prefixed with a number  $k$  to indicate that the top set has size  $k$ .

$$\text{INDIFF-INDIFF} = \bigvee_{A \in \mathcal{P}_+(X)} (\text{TOP}_A \wedge \text{INDIFF}_A \wedge \text{INDIFF}_{X \setminus A})$$

$$\text{INDIFF-INCOMP} = \bigvee_{A \in \mathcal{P}_+(X)} (\text{TOP}_A \wedge \text{INDIFF}_A \wedge \text{INCOMP}_{X \setminus A})$$

$$k\text{-INDIFF-INDIFF} = \bigvee_{A \in \mathcal{P}_k(X)} (\text{TOP}_A \wedge \text{INDIFF}_A \wedge \text{INDIFF}_{X \setminus A})$$

$$k\text{-INDIFF-INCOMP} = \bigvee_{A \in \mathcal{P}_k(X)} (\text{TOP}_A \wedge \text{INDIFF}_A \wedge \text{INCOMP}_{X \setminus A})$$

$$k\text{-INCOMP-INCOMP} = \bigvee_{A \in \mathcal{P}_k(X)} (\text{TOP}_A \wedge \text{INCOMP}_A \wedge \text{INCOMP}_{X \setminus A})$$

#### 3.2 Extracting Election Winners

To simulate a multiwinner voting rule, the agents' preferences are turned into judgments that satisfy a suitable rationality constraint  $\Gamma$ . In the case of ordinal preferences, this is RANKING. In the case of approval-based preferences, we are going to use INDIFF-INCOMP, i.e., we are going to assume that an agent who approves of the set  $A$  does not share any views regarding the relative desirability of the alternatives she does not approve of (rather than to declare indifference between them). We consider this the most natural interpretation of an approval ballot (and it is an interpretation that will turn out to be technically convenient as well).

We can then apply a JA rule to the preferences thus encoded, obtaining a collective judgment. In case that collective judgment satisfies the constraint  $\text{TOP}_A$  for some set  $A \subseteq X$ , we can declare the alternatives in  $A$  the winners of the original election. With this in mind, we are now ready to present our central definition relating multiwinner voting rules and JA rules, which is similar to the definition given by Endriss [16] for single-winner voting rules.

**DEFINITION 1 (SIMULATION).** *Given a set  $X$  of alternatives, a JA rule  $F$  for the preference agenda  $\Phi_{\succsim}^X$ , and a multiwinner voting rule  $F'$  for  $X$ , let  $\Gamma = \text{RANKING}$  in case  $F'$  uses ordinal preferences and  $\Gamma = \text{INDIFF-INCOMP}$  in case it uses approval-based preferences. Then we say that  $F$  simulates  $F'$  if, for every preference profile  $(\succsim_1, \dots, \succsim_n) \in \text{Mod}(\Gamma)^n$  and corresponding judgment profile  $\mathbf{J} = (J_1, \dots, J_n)$ , we have that  $F'(\succsim_1, \dots, \succsim_n) = \bigcup_{J \in F(\mathbf{J})} \{A \subseteq X \mid J \models \text{TOP}_A\}$ .*

Observe that  $F$  can simulate  $F'$  only if  $F$  satisfies a feasibility constraint  $\Gamma'$  that ensures that all outcomes  $J \in F(\mathbf{J})$  satisfy  $\text{TOP}_A$  for some  $A \subseteq X$ . Furthermore, for  $F$  to simulate a rule that returns committees of size  $k$ , the constraint  $\Gamma'$  needs to ensure that these sets  $A$  indeed always have size  $k$ . So constraints such as  $k\text{-INDIFF-INDIFF}$  and  $k\text{-INDIFF-INCOMP}$  are natural candidates for  $\Gamma'$ .

#### 3.3 Simulation Results

With all the relevant definitions in place, we now present our simulation results for specific multiwinner voting rules. We start with results for two ordinal-based rules that may be regarded as the multiwinner counterparts of corresponding results by Endriss [16] for single-winner voting rules. The first concerns the  $k$ -Borda rule.

**THEOREM 1.** *When restricted to RANKING-rational profiles, the max-sum( $\cdot, k\text{-INDIFF-INCOMP}$ ) rule simulates  $k\text{-Borda}$ .*

**PROOF.** For any given set  $A \subseteq X$ , let  $J^A$  be defined as the unique judgment such that, for any given  $x, y \in X$ , we have  $J^A(p_{x > y}) = 1$  if and only if  $x \in A$ . Observe that the judgments  $J^A$  for sets  $A$  with

$|A| = k$ , are precisely the judgments that satisfy  $k$ -INDIFF-INCOMP:  $\text{Mod}(k\text{-INDIFF-INCOMP}) = \{J^A \mid A \in \mathcal{P}_k(X)\}$ .

Now consider any profile  $J \in \text{Mod}(\text{RANKING})^n$ . The max-sum rule induced by  $k$ -INDIFF-INCOMP, acting on the profile  $J$ , returns  $\text{argmax}_{J \in \text{Mod}(k\text{-INDIFF-INCOMP})} \sum_{i \in N} |\text{Agr}(J, J_i)|$ . This is equal to  $\text{argmax}_{J^A \text{ s.t. } A \in \mathcal{P}_k(X)} \sum_{i \in N} |\{\varphi \in \Phi_{\geq}^X \mid J^A(\varphi) = J_i(\varphi)\}|$ . To determine the outcome, we need to compute a score that is obtained by summing over the elements  $\varphi$  of the preference agenda  $\Phi_{\geq}^X$ . For any given  $A \subseteq X$ , we can separate this agenda into the following five disjoint parts:  $\{p_{x \succ y} \mid x = y\}$ ,  $\{p_{x \succ y} \mid x \neq y \text{ and } x, y \notin A\}$ ,  $\{p_{x \succ y} \mid x \neq y \text{ and } x, y \in A\}$ ,  $\{p_{x \succ y} \mid x \in A, y \notin A\}$ , and  $\{p_{x \succ y} \mid x \notin A, y \in A\}$ . We need to count the propositions  $\varphi$  in  $\Phi_{\geq}^X$  on which the two judgments, namely  $J^A$  and  $J_i \in \text{Mod}(\text{RANKING})$  representing voter  $i$ 's judgment, agree.

First, for propositions in  $\{p_{x \succ y} \mid x = y\}$ , the judgments agree when  $x \in A$ . This number remains the same regardless on the selection of  $A$  so omit it from the final count. Second,  $J^A$  rejects all propositions in  $\{p_{x \succ y} \mid x \neq y \text{ and } x, y \notin A\}$ , while  $J_i$ , by virtue of denoting a strict ranking, accepts exactly half of them (independently of the specific preference reported by  $i$ ). Since this number also does not depend on the choice of  $A$ , we omit it going forward. The same is true for  $\{p_{x \succ y} \mid x \neq y \text{ and } x, y \in A\}$ :  $J^A$  again accepts all propositions, while  $J_i$  accepts exactly half of them, namely those propositions  $p_{x \succ y}$  for which  $x >_i y$ . However, as will become clear shortly, in this case it will be convenient to explicitly include this number in our count. For the remaining two parts of the agenda,  $\{p_{x \succ y} \mid x \in A, y \notin A\}$  and  $\{p_{x \succ y} \mid x \notin A, y \in A\}$ , the judgment  $J^A$  accepts all propositions in the former and rejects all those in the latter. If  $J_i$  accepts a proposition  $p_{x \succ y}$  in the former, it rejects  $p_{y \succ x}$  in the latter. So  $J_i$  is in agreement with  $J^A$ , in both parts of the agenda, for exactly those pairs  $(x, y)$  for which  $x >_i y$ . So these judgments agree on the same number for both parts of the agenda, meaning that we need to consider only one.

To summarise, omitting the terms we can ignore, we obtain that the max-sum rule induced by  $k$ -INDIFF-INCOMP maps any given preference profile  $(\succ_1, \dots, \succ_n)$  to the outcome  $\text{argmax}_{J^A \text{ s.t. } A \in \mathcal{P}_k(X)} \sum_{i \in N} |\{(x, y) \mid x >_i y \text{ and } x \in A, y \notin A\}| + |\{(x, y) \mid x >_i y \text{ and } x, y \in A\}|$ . The latter can be further simplified to  $\text{argmax}_{J^A \text{ s.t. } A \in \mathcal{P}_k(X)} \sum_{i \in N} |\{y \in X \mid x >_i y \text{ and } x \in A\}|$ . Hence, the elected  $k$ -sized committee/s clearly maximise the Borda scores of the winning alternatives; so this is the  $k$ -Borda rule.  $\square$

As feasibility constraint we used  $k$ -INDIFF-INCOMP to capture the intuition of collective incomparability within the losing set. However, there is also another intuitive route, namely the one where we assume indifference between the non-winners. Indeed, when counting agreements, as agents provide strict relations between alternative pairs, one can freely choose between indifference and incomparability amongst alternatives grouped together in the outcome. We demonstrate this with the following result using  $k$ -INDIFF-INDIFF.

**PROPOSITION 2.** *When restricted to RANKING-rational profiles, the max-sum( $\cdot$ ,  $k$ -INDIFF-INDIFF) rule simulates  $k$ -Borda.*

**PROOF (SKETCH).** We define the judgment  $J^A$  as accepting a proposition  $p_{x \succ y}$  for  $x, y \in X$ , if and only if one of the following three conditions are satisfied: (i)  $x, y \in A$ , (ii)  $x \in A$  but  $y \notin A$ ,

or (iii)  $x, y \notin A$ . This ensures that  $\text{Mod}(k\text{-INDIFF-INDIFF}) = \{J^A \mid A \in \mathcal{P}_k(X)\}$ . The proof now proceeds along the same lines as that of Theorem 1. Regarding agreements, both judgments accept all of  $\{p_{x \succ y} \mid x = y\}$  and agree on half of  $\{p_{x \succ y} \mid x \neq y \text{ and } x, y \notin A\}$  with  $J^A$  accepting all them. The other agreements are as in Theorem 1. The same holds for the final count.  $\square$

We return to  $k$ -INDIFF-INCOMP and apply the constraint to max-sum instead of max-sum as we transition to simulating  $k$ -Copeland.

**THEOREM 3.** *When restricted to RANKING-rational profiles, the max-num( $\cdot$ ,  $k$ -INDIFF-INCOMP) rule simulates  $k$ -Copeland.*

**PROOF (SKETCH).** Take the same judgment  $J^A$  and agenda decomposition from Theorem 1. We proceed to assess the agreements between  $J^A$  and the  $\text{Maj}(J)$  judgment. Notice that when checking for the agreements, for the case with an odd number of agents, that  $\text{Maj}(J)$  sets, for any pair of distinct alternatives  $x, y \in X$ , exactly one of  $p_{x \succ y}$  and  $p_{y \succ x}$  to true. This is much like the considered judgment  $J_i$  from Theorem 1. Hence, it is clear that the rule is equivalent to  $\text{argmax}_{J^A \text{ s.t. } A \in \mathcal{P}_k(X)} |\{p_{x \succ y} \mid \text{Maj}(J)(p_{x \succ y}) = 1 \text{ and } x \in A\}|$ . The returned  $k$ -sized committee/s maximise the pairwise majority wins of the committee members and thus, this is  $k$ -Copeland.  $\square$

As with  $k$ -Borda,  $k$ -Copeland has an alternative simulation using  $k$ -INDIFF-INDIFF. The (omitted) proof of this fact is analogous to that of Proposition 2.

**PROPOSITION 4.** *When restricted to RANKING-rational profiles, the max-num( $\cdot$ ,  $k$ -INDIFF-INDIFF) rule simulates  $k$ -Copeland.*

Having simulated  $k$ -Borda and  $k$ -Copeland with varying feasibility constraints, let us now consider the simulation of rules that exhibit different qualities to the two aforementioned rules. In the multiwinner voting literature,  $k$ -Borda and  $k$ -Copeland have been proposed as suitable candidates to perform tasks such as shortlisting, as they satisfy the axiom of *committee monotonicity*. This property ensures that winning alternatives in a  $k$ -sized committee will remain winners if the target committee size is increased [3, 6, 14, 19]. Barberá and Coelho [6] showed this property to be incompatible with another well-studied property, namely the *Condorcet*-related notion of *Gehrlein stability* [3, 6, 23, 37]. We recall its definition below.

**DEFINITION 2 (GEHRLEIN STABILITY).** *Take a set of alternatives  $X$ , a target committee size  $k$ , and a set of agents  $N$  with each  $i \in N$  providing a strict ranking  $\succ_i$ . A committee  $A \subseteq \mathcal{P}_k(X)$  is (weakly) Gehrlein-stable if for any  $x \in A$  and  $y \in X \setminus A$ , it is the case that  $|\{i \in N \mid x >_i y\}| \geq |\{i \in N \mid y >_i x\}|$ .*

Since we work with weak Gehrlein stability, we can allow for evenly-sized  $N$ . And note that for some  $X, N, k$ , and strict preference profiles, a Gehrlein-stable committee may not exist.

As we aim at simulating rules that output stable committees, the incompatibility with committee monotonicity leads us away from the  $k$ -INDIFF-INCOMP and  $k$ -INDIFF-INDIFF constraints.<sup>1</sup> We now show that, when certain members of the class of *additive majority rules* (AMRs) are induced by  $k$ -INCOMP-INCOMP on RANKING-rational profiles, the resultant JA rule returns judgments that correspond to  $k$ -sized Gehrlein-stable committees given such committees

<sup>1</sup>Aziz et al. [3] showed this incompatibility does not occur for strict Gehrlein stability.

exist for the given profiles. We now recall the definition of AMRs, which include max-sum and max-num [8, 33].<sup>2</sup>

**DEFINITION 3 (ADDITIVE MAJORITY RULE).** A JA rule  $F$  is an **additive majority rule (AMR)** if there exists a non-decreasing gain function  $g : [0, n] \rightarrow \mathbb{R}$  such that  $g(t) < g(t')$  for  $t < \frac{n}{2} \leq t'$ , and for every feasibility constraint  $\Gamma'$  and JA profile  $J$ , it holds that:

$$F(J, \Gamma') = \operatorname{argmax}_{J \in \operatorname{Mod}(\Gamma')} \sum_{\varphi \in J} g(n_{(J, \varphi)})$$

We obtain the following simulation result:

**THEOREM 5.** *When restricted to RANKING-rational profiles, every JA outcome  $J \in F(\cdot, k\text{-INCOMP-INCOMP})$  for an AMR  $F$  that is based on a gain function  $g$  with the property that  $g(t) = g(t')$  for any two  $t, t' \geq \frac{n}{2}$  corresponds to a **weakly Gehrlein-stable committee**, provided such a stable committee exists at all.*

**PROOF.** Take an  $m$ -sized set of alternatives  $X$  and suppose that a weakly Gehrlein-stable committee  $S \subseteq \mathcal{P}_k(X)$  exists. Moreover, to derive a contradiction, suppose that there is some judgment  $J^A \in F(J, k\text{-INCOMP-INCOMP})$  that corresponds to a committee  $A \subseteq \mathcal{P}_k(X)$  that is not weakly Gehrlein-stable.

Since  $A$  is not weakly Gehrlein-stable, there must be some  $x \in A$  and some  $y \in X \setminus A$  such that  $|\{i \in N \mid x \succ_i y\}| < |\{i \in N \mid y \succ_i x\}|$ . Then the score  $\sum_{\varphi \in J^A} g(n_{(J, \varphi)})$  achieved by  $J^A$  is strictly less than  $k(m - k)g_{\max}$ , where  $g_{\max} = g(\lceil n/2 \rceil) = \dots = g(n)$ .

However, the judgment  $J^S$  that corresponds to the committee  $S$  achieves the score  $\sum_{\varphi \in J^S} g(n_{(J, \varphi)}) = k(m - k)g_{\max}$ , and thus achieves a strictly higher score than  $J^A$ . This is a contradiction with our assumption that  $J^A \in F(J, k\text{-INCOMP-INCOMP})$ . Therefore we can conclude that all judgments in  $F(J, k\text{-INCOMP-INCOMP})$  correspond to a weakly Gehrlein-stable committee.  $\square$

This result makes a large selection from the AMR class available to those interested in committee stability with tools to easily define novel Gehrlein-stable rules. In fact, the subclass of AMRs to which Theorem 5 applies includes some rules that correspond to multiwinner voting rules that have been studied in the literature.

For example, consider the AMR based on the gain function  $g$  with  $g(t) = 1$  when  $t \geq \frac{n}{2}$  and  $g(t) = 0$  otherwise—which coincides with the max-num rule. When restricted to RANKING-rational profiles, using  $k\text{-INCOMP-INCOMP}$  as the feasibility constraint, this rule simulates the rule known as *Number of External Defeats* [3, 19].

Another example is the AMR based on the gain function  $g$  with  $g(t) = 0$  when  $t \geq \frac{n}{2}$  and  $g(t) = 2t - n$  otherwise, or put differently,  $g(t) = \max\{0, 2t - n\}$ . When restricted to RANKING-rational profiles, using  $k\text{-INCOMP-INCOMP}$  as constraint, this rule simulates the  $k$ -Kemeny multiwinner voting rule [6, 37]. This correspondence revolves around the fact that for each pair  $(x, y)$ , if  $x$  is selected in the outcome and  $y$  is not, a score of  $\max\{0, |\{i \in N \mid y \succ_i x\}| - |\{i \in N \mid x \succ_i y\}|\}$  is added for  $p_{x \succ y}$  to the total score.

We now transition to approval-based rules, with a natural starting point being the simple AV rule. As previously mentioned, the rationality constraint for these rules will be  $\text{INDIFF-INCOMP}$ , which allows agents to have approval ballots of arbitrary size.

**THEOREM 6.** *When restricted to  $\text{INDIFF-INCOMP}$ -rational profiles, the max-sum( $\cdot, k\text{-INDIFF-INCOMP}$ ) rule simulates AV.*

**PROOF (SKETCH).** Take  $J^A$  from Proposition 2 and the usual agenda decomposition. With  $\text{INDIFF-INCOMP}$ -rational profiles, each voter sets indifference between her most-preferred alternatives. We fix an approval set  $P_i$  for voter  $i$  such that  $J_i(p_{x \succ y}) = 1$  if and only if  $x \in P_i$ . It is easy to verify through counting relevant agreements that the max-sum rule induced by  $k\text{-INDIFF-INCOMP}$  on  $\text{INDIFF-INCOMP}$ -rational profiles is  $\operatorname{argmax}_{J^A} \text{s.t. } A \in \mathcal{P}_k(X) \sum_{i \in N} |\{x \mid x \in A \cap P_i\}|$ . The elected committee/s of size  $k$  maximise the approval of the committee members which gives us a simulation of AV.  $\square$

We now extend the AV simulation to other Thiele rules. Let us adjust max-sum by incorporating a scoring vector. For any scoring vector  $\mathbf{w}^{(k)}$  and number  $\ell \geq 0$ , let  $f_{\mathbf{w}^{(k)}}(\ell) = \sum_{i=0}^{\ell} w_i$ . This function allows us to refine max-sum to  $f\text{-max-sum}(J, \Gamma', \mathbf{w}^{(k)}) = \operatorname{argmax}_{J \in \operatorname{Mod}(\Gamma')} \sum_{i \in N} f_{\mathbf{w}^{(k)}}(|\operatorname{Agr}(J, J_i)|)$ . That the  $f\text{-max-sum}$  rule facilitates the simulation of PAV and  $\alpha\text{-CC}$  is immediate from its definition and our proof sketch for Theorem 6, so we present the next result without proof.

**PROPOSITION 7.** *When restricted to  $\text{INDIFF-INCOMP}$ -rational profiles, the  $f\text{-max-sum}$  rule induced by  $k\text{-INDIFF-INCOMP}$  simulates PAV and  $\alpha\text{-CC}$  when using the scoring vectors  $(1, 1/2, 1/3, \dots, 1/k)$  and  $(1, 0, \dots, 0)$ , respectively.*

### 3.4 Constraints as Circuits

Worst-case intractability has been shown for many JA rules. Specifically, computing outcomes under max-sum and max-num is  $\Theta_2^P$ -hard [18]. Thus, when simulating multiwinner rules in JA, we encounter the paradox that ordinarily easy-to-compute rules, such as  $k$ -Borda and AV [4, 14], now seem computationally difficult to implement. To address this mismatch, we employ the approach proposed by de Haan [26] who showed that JA rules can be used efficiently when the integrity constraint is represented as a circuit in *decomposable negation normal form*, or a DNNF circuit. We begin with the circuit definition given by Darwiche and Marquis [10].

**DEFINITION 4 (DNNF CIRCUITS).** *A Boolean circuit in negation normal form (NNF) is a directed acyclic graph with a single root where each internal node is labelled with  $\vee$  or  $\wedge$ , and every leaf is labelled with  $\top$ ,  $\perp$ ,  $x$  or  $\neg x$  for a propositional variable  $x$ . A **DNNF circuit** is an NNF circuit that satisfies *decomposability*: for each conjunction in the circuit, no two conjuncts share a propositional variable.*

The results of de Haan [26] cover certain members in the class of scoring rules [11], including max-sum and max-num. Scoring rules select those constraint-satisfying JA outcomes that maximise the score of an associated scoring function. Such a function attaches a score to each issue with respect to an agent's judgment. Before restating the relevant result in Theorem 8, we define the *outcome determination problem*  $\text{OUTCOME}(F)$  for a given JA rule  $F$ .

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**OUTCOME**( $F$ )

**Given:** A judgment profile  $J$  for an agenda  $\Phi$ , an integrity constraint  $\Gamma'$ , and a partial judgment  $d$  on  $\Phi$ .

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**Question:** Is there a  $J \in F(J, \Gamma')$  that agrees with  $d$ ?

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<sup>2</sup>We obtain the max-sum rule for  $g(t) = t$ , while max-num has  $g(t) = 1$  when  $t \geq \frac{n}{2}$  and  $g(t) = 0$  otherwise.

THEOREM 8 (DE HAAN, 2018). *When the integrity constraint  $\Gamma'$  is represented as a DNNF circuit, then both  $\text{OUTCOME}(\text{max-sum})$  and  $\text{OUTCOME}(\text{max-num})$  are polynomial-time solvable.*<sup>3</sup>

### 3.5 Encoding and Complexity Results

We now show that the  $k$ -INDIFF-INCOMP constraint can be represented as a DNNF circuit. Recall that this constraint sets indifference amongst the top alternatives and incomparability between those in the bottom set. Observe that for any alternative  $x$  in the top set, the proposition  $p_{x \succcurlyeq y}$  is true for all  $y \in X$ . On the other hand, if  $x$  is in the bottom set, the proposition  $p_{x \succcurlyeq y}$  is false for all  $y \in X$ .

THEOREM 9. *Given a finite set  $X$  of alternatives and a corresponding preference agenda  $\Phi_{\succcurlyeq}^X$ , the  $k$ -INDIFF-INCOMP constraint can be encoded into a DNNF circuit in polynomial time.*

PROOF. Given the set of alternatives  $X = \{x_1, \dots, x_m\}$ , we construct the circuit according to the (arbitrary) ordering  $x_1, \dots, x_m$  of  $X$ . An ordering of propositions such as  $p_{x \succcurlyeq x}$  for each  $x \in X$  is then set. We say  $x_i$  is the  $i$ th alternative in  $X$  while  $p_{x_i \succcurlyeq x_i}$  is the corresponding proposition in  $\Phi_{\succcurlyeq}^X$ . We also use the counting variables  $i$  and  $j$  during the circuit construction, both starting at 0.

The circuit contains nodes  $N_{i,j}$ , each of which denoting that we have assessed the propositions in the sequence up to (and including) index  $i - 1$  with  $j$  the current size of the winning set. We set  $N_{0,0}$  to be the root of the circuit. If  $i = |X| + 1$  and  $j = k$ , then  $N_{i,j} = \top$ . If  $i = |X| + 1$  and  $j \neq k$ , then  $N_{i,j} = \perp$ . Now if  $i < |X| + 1$ , we either have (i)  $p_{x_i \succcurlyeq x_i}$  is true or (ii)  $p_{x_i \succcurlyeq x_i}$  is false. We set the node  $N_{i,j}$  to be the disjunction  $\alpha \vee \beta$ , where  $\alpha = (N(i + 1, j + 1) \wedge \bigwedge_{y \in X} p_{x_i \succcurlyeq y})$  and  $\beta = (N(i + 1, j) \wedge \bigwedge_{y \in X} \neg p_{x_i \succcurlyeq y})$ . We have that every leaf is either  $\top$ ,  $\perp$  or  $p_{x \succcurlyeq y}$  for some  $x$  and  $y$ . Thus, we have a NNF circuit. Each proposition appears exactly once in the circuit so we also have that it is decomposable. The circuit is only satisfied by a preference agenda  $\Phi_{\succcurlyeq}^X$  if the agenda has a  $k$ -sized top set of alternatives with indifference between them while the bottom set's alternatives are incomparable. So we have that the circuit corresponds to our constraint. This circuit can also be constructed in polynomial time as the process terminates once each alternative in  $X$  has been assessed exactly once.  $\square$

COROLLARY 10. *Given a finite set  $X$  of alternatives and a corresponding preference agenda  $\Phi_{\succcurlyeq}^X$ , the  $\text{INDIFF-INCOMP}$  constraint can be encoded into a DNNF circuit in polynomial time.*<sup>4</sup>

These results ensure that for max-sum and max-num when using  $k$ -INDIFF-INCOMP as the feasibility constraint, such as in our simulations, computing the outcomes can still be done in polynomial time. We continue with this approach to analyse  $k$ -INDIFF-INDIFF.

Recall that  $k$ -INDIFF-INDIFF sets indifference within both the  $k$ -sized top set and the bottom set. We now show that this constraint cannot be constructed as a DNNF circuit in polynomial time. The claim is that, given a set of alternatives  $X$ , computing the max-sum rule induced by  $k$ -INDIFF-INDIFF with the rationality constraint  $\top$ , i.e., when the ballots are unconstrained, is a computationally

difficult problem. This implies that we cannot construct a DNNF circuit representing  $k$ -INDIFF-INDIFF in polynomial time (assuming that  $P \neq \text{NP}$ ). We show that this problem is NP-hard by giving a reduction from the following problem.

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NAE-3SAT

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**Given:** A formula  $\varphi$  in 3CNF.

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**Question:** Is there a truth assignment satisfying  $\varphi$  that falsifies at least one literal in each clause of  $\varphi$ ?

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THEOREM 11. *Given a finite set  $X$  of alternatives and a corresponding preference agenda  $\Phi_{\succcurlyeq}^X$ , computing any outcome of the max-sum rule induced by  $k$ -INDIFF-INDIFF on  $\top$ -restricted ballots is NP-hard.*

PROOF (SKETCH). We reduce from NAE-3SAT. Let  $\varphi$  be an arbitrary instance of NAE-3SAT, where  $x_1, \dots, x_m$  are the variables in  $\varphi$  and  $c_1, \dots, c_u$  are the clauses in  $\varphi$ . For each variable  $x_i$  in  $\varphi$ , we create an alternative  $a_{x_i}$  for each literal of the variable  $x_i$ . This produces  $2m$  alternatives. The profile on  $\Phi_{\succcurlyeq}^X$ , is as follows: for each variable  $x_i$ , we add  $10u$  agents, each of which has a judgment set that sets every preference issue to true except for  $p_{a_{x_i} \succcurlyeq a_{\neg x_i}}$  and  $p_{a_{\neg x_i} \succcurlyeq a_{x_i}}$ . For each clause  $c_j$ , and each pair of literals  $(\ell_1, \ell_2)$  within  $c_j$ , we create an individual that has the judgment set that sets every issue in the agenda to true except for  $p_{a_{\ell_1} \succcurlyeq a_{\ell_2}}$  and  $p_{a_{\ell_2} \succcurlyeq a_{\ell_1}}$ . We claim that there exists a truth assignment that both satisfies and falsifies at least one literal in each clause of  $\varphi$  if and only if the translated outcome, given by max-sum induced by  $m$ -INDIFF-INDIFF, has a score of at least  $(10mu + 3u) \cdot (4 \binom{m}{2} + m^2) - 10mu - 4u$ . Verifying the claim is straightforward. For space reasons, we omit the details.  $\square$

So in general, we cannot efficiently use the  $k$ -INDIFF-INDIFF-induced max-sum rule. However, when used on RANKING-restricted profiles, it simulates  $k$ -Borda (Proposition 2). So in this particular case, we obtain computational efficiency. This highlights the care required in JA constraint selection.

## 4 AIMING FOR PROPORTIONAL JA RULES

So far, we have shown simulations of existing multiwinner voting rules using existing JA rules (or, in one case, rules that are very close to existing JA rules). But our embedding approach also suggests itself as a tool for porting ideas from multiwinner voting to JA that have so far not been considered in the latter field. An example is the notion of *proportional representation*, which plays a central role in multiwinner voting [2, 39].<sup>5</sup> While the simulation of PAV and  $\alpha$ -CC represents an initial step towards introducing this notion into JA, in this section we explore this direction more systematically.

We start by introducing a proportionality axiom for JA as well as two concrete aggregation rules designed to satisfy this axiom (at least under certain conditions). We then proceed to studying the computational complexity of our rules.

### 4.1 Proportional JA Rules

While PAV and  $\alpha$ -CC ensure some form of proportionality when electing a committee of fixed size, it is much less clear how to design

<sup>3</sup>Rey et al. [38] extend this result to the general class of *additive rules*, which includes both the scoring rules and the AMRs.

<sup>4</sup>The proof of the circuit encoding for  $\text{INDIFF-INCOMP}$  works as in Theorem 9 except the tracking of the winning set's size is omitted.

<sup>5</sup>Recent work by Haret et al. [27] on importing proportionality into belief merging, a formalism that is conceptually similar to JA, has similar motivations and underlines the relevance of this idea. We note that despite this conceptual similarity, the technical results obtained by the authors are technically unrelated to ours.

such a proportional rule for electing variable-sized committees (because the trivial solution of electing *all* alternatives to maximise voter approval is clearly not attractive). But for more general JA applications, a rule that does not force us to accept a fixed number of propositions seems more relevant. Also, such a rule must be measured against some criteria to determine the extent to which its outcomes are proportional. To this end, we adapt the multiwinner axiom of *proportional justified representation* (PJR) [39].

**DEFINITION 5** (SÁNCHEZ-FERNÁNDEZ ET AL., 2017). *Given an approval ballot profile  $A = (A_1, \dots, A_n)$  over a set of alternatives  $X$  and a fixed committee size  $k \leq |X|$ , a group of voters is  $\ell$ -cohesive for some  $\ell \in [k]$ , if  $|N^*| \geq \ell \cdot \frac{n}{k}$  and  $|\bigcap_{i \in N^*} A_i| \geq \ell$ . A committee  $C \in \mathcal{P}_k(X)$  satisfies **proportional justified representation** (PJR) for  $A$  and  $k$ , if for every  $\ell \in [k]$  and every  $\ell$ -cohesive group of voters  $N^* \subseteq N$ , it is the case that  $|C \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$ .<sup>6</sup> An approval-based voting rule satisfies PJR if for every profile  $A$  and every committee size  $k$ , it outputs a committee that satisfies PJR for  $A$  and  $k$ .*

For our JA axiom, we cannot rely on a fixed size  $k$  to identify cohesive groups, as a variable number of issues may be accepted by a JA outcome. So we use this number of accepted issues as if it were the committee target size to begin with, which differs from other approaches to variable-sized multiwinner proportionality [22]. Also, to account for logical dependencies between JA issues, we focus on the issues that groups agree on that are, in a sense, logically independent of the agenda. This approach may lead to a weaker notion that is less compatible with restrictive constraints, but it is clear that complex constraints may rule out proportionality.

We provide extra notation for our axiom. For an integrity constraint  $\Gamma$ , an issue  $\varphi$  is logically independent of another issue  $\psi$  if both  $\Gamma \not\models \psi \rightarrow \varphi$  and  $\Gamma \not\models \psi \rightarrow \neg\varphi$  are the case. We say an issue is logically independent of some set  $S$  if  $\Gamma \not\models (\bigwedge_{\psi \in S} \psi) \rightarrow \varphi$  and  $\Gamma \not\models (\bigwedge_{\psi \in S} \psi) \rightarrow \neg\varphi$ . The set of issues accepted by a judgment  $J$  is denoted as  $J^+ = \{\varphi \in \Phi \mid J(\varphi) = 1\}$ . For an agent group  $N^* \subseteq N$ , we define a judgment  $U_J(N^*)$  such that, for every  $\varphi \in \Phi$ , it is the case that  $U_J(N^*)(\varphi) = 1$  if  $J_i(\varphi) = 1$  for some agent  $i$  in  $N^*$ ; otherwise,  $U_J(N^*)(\varphi) = 0$ . The judgment  $I_J(N^*)$  requires every agent  $i$  in  $N^*$  to be accepting of  $\varphi$  for  $I_J(N^*)(\varphi) = 1$  to hold; otherwise,  $I_J(N^*)(\varphi) = 0$ . We now define our JA proportionality axiom.

**DEFINITION 6** ( $\ell$ -JA-PJR). *Consider some  $\ell \in [|J^+|]$  for a judgment  $J$  over an agenda  $\Phi$  accepting  $|J^+|$  issues. We say a group of agents  $N^* \subseteq N$  is  $(J, \Gamma, \ell)$ -cohesive if  $|N^*| \geq \ell \cdot \frac{n}{|J^+|}$  and  $|\{\varphi \in I_J(N^*)^+ \mid \varphi \text{ is logically independent of } \Phi \setminus \{\varphi\}\}| \geq \ell$ . Given a judgment profile  $J$  and an integrity constraint  $\Gamma$ , we say that an outcome  $J$  provides  **$\ell$ -JA proportional justified representation** ( $\ell$ -JA-PJR), if for every  $(J, \Gamma, \ell)$ -cohesive group of agents  $N^* \subseteq N$ , it is the case that  $|\text{Agr}(J, U_J(N^*))^+| \geq \ell$ . We say a JA rule  $F$  satisfies  $\ell$ -JA-PJR if every JA outcome  $J \in F(J, \Gamma)$  provides  $\ell$ -JA-PJR.*

Next, we are going to propose new JA rules geared towards proportionality. In the variable-sized multiwinner literature, there is a class of rules that take both approvals and disapprovals into account when scoring a committee [9, 20]. We adopt this approval-disapproval dynamic and apply it to a JA outcome's accepted issues.

<sup>6</sup>For  $\ell = 1$ , this axiom reduces to *justified representation*, as defined by Aziz et al. [2].

Let  $f_{\mathbf{w}^{(m)}}(\ell) = \sum_{i=0}^{\ell} w_i$  for any given scoring vector  $\mathbf{w}^{(m)}$ . The general form of our rules is the following variant of max-sum, but now using two separate scoring vectors,  $\mathbf{u}^{(m)}$  and  $\mathbf{v}^{(m)}$ , for approvals and disapprovals, respectively:

$$\begin{aligned} & (a-d)\text{-max-sum}(J, \Gamma) \\ &= \operatorname{argmax}_{J \in \text{Mod}(\Gamma')} \sum_{i \in N} f_{\mathbf{u}^{(m)}}(|\text{Agr}(J, J_i)^+|) - f_{\mathbf{v}^{(m)}}(|\text{Dis}(J, J_i)^+|) \end{aligned}$$

Through varying the scoring vectors, we now define candidates for new JA rules. The first attaches standard AV scoring to agreed-upon accepted issues, and 'penalises' disagreed-with accepted issues with an 'inverted' PAV scoring.

**DEFINITION 7** (PAV-JA). *Given an agenda  $\Phi$  with  $m$  issues and an integrity constraint  $\Gamma$ , the **PAV-JA rule** is defined as the  $(a-d)$ -max-sum rule with the scoring functions induced by following scoring vectors  $\mathbf{u}^{(m)} = (1, 1, \dots, 1)$  and  $\mathbf{v}^{(m)} = (1/m, \dots, 1/2, 1)$ .*

For the second rule, an agent awards points to an outcome as with  $\alpha$ -CC scoring, but subtracts a point if the majority threshold, a commonly-used threshold [1, 20, 21], of rejected issues is crossed.

**DEFINITION 8** (CC-JA). *Given an agenda  $\Phi$  with  $m$  issues and an integrity constraint  $\Gamma$ , the **CC-JA rule** is defined as the  $(a-d)$ -max-sum rule with the scoring functions induced by the scoring vectors  $\mathbf{u}^{(m)} = (1, 0, \dots, 0)$  and  $\mathbf{v}^{(m)} = (0, \dots, 0, 1, 0, \dots, 0)$ . In  $\mathbf{v}^{(m)}$  we set the scoring vector's 'threshold' at position  $\lfloor m/2 \rfloor + 1$ .*

Having proposed two JA rules based on well-known proportional approaches, we assess whether these rules satisfy our  $\ell$ -JA-PJR axiom, beginning with PAV-JA.

**THEOREM 12.** *The rule PAV-JA( $J, \Gamma$ ) satisfies the  $\ell$ -JA-PJR axiom for every value  $\ell \geq |J^+| / (m - |J^+| + 1)$ .*

**PROOF (SKETCH).** Take a  $J \in \text{PAV-JA}(J, \Gamma)$  and assume there is a  $(J, \Gamma, \ell)$ -cohesive group  $N^*$  such that  $|\text{Agr}(J, U_J(N^*))^+| < \ell$ . We can show that a judgment  $J'$  that accepts a currently-rejected issue in  $I_J(N^*)^+$ , all else being equal, yields a strictly higher PAV-JA score. We now detail the change in score which occurs with the acceptance of  $\varphi$ . The group  $N^*$  adds at least  $\ell \cdot (n/|J^+|)$  to the score. At least one agent is already satisfied by  $J$  and this agent deducts at most  $1/(m - |J^+| + 1)$  while at most  $n - |N^*| - 1$  agents will each deduct at most  $1/(m - |J^+|)$ . So the score strictly increases when we have:  $|N^*| \geq \ell \cdot (n/|J^+|) > (n - |N^*| - 1)/(m - |J^+|) + 1/(m - |J^+| + 1) < (n(|J^+| - \ell))/(|J^+|(m - |J^+|))$ . To conclude, observe that the score is strictly positive when  $\ell \geq |J^+|/(m - |J^+| + 1)$ .  $\square$

Next, we establish that CC-JA generally fails  $\ell$ -JA-PJR; but with a stronger independence assumption, CC-JA satisfies 1-JA-PJR.

**THEOREM 13.** *Assuming logical independence between all agenda items, CC-JA satisfies  $\ell$ -JA-PJR for  $\ell = 1$  and fails it for every  $\ell > 1$ .*

**PROOF (SKETCH).** Our claim is that, for any  $\ell > 1$ , we can construct an agenda  $\Phi$  and a profile  $J$  such that, for  $\Gamma = \top$ , there is a  $J \in \text{CC-JA}(J, \Gamma)$  that does not provide  $\ell$ -JA-PJR. Consider an arbitrary  $\ell > 1$ . We choose an agenda with an odd number  $m \geq 5$  of issues such that  $\ell = \lfloor m/2 \rfloor$ . We fix a set of agents  $N$  with  $|N| = m + 1$  and an agent subset  $N^\ell \subset N$  such that  $|N^\ell| = (|N| \cdot \lfloor m/2 \rfloor) / \lfloor m/2 \rfloor = |N| - 2$ . This ensures the existence of two agents  $i, j \notin N^\ell$ . Given this agent population, we can define the

judgment profile  $J$ . In this profile, the agents in  $N^\ell$  uniformly accept the same  $\ell$  issues, so we have  $|I_J(N^\ell)^+| = |U_J(N^\ell)^+| = \ell$ . For the judgments of agents  $i, j \notin N^\ell$ , we have two issues  $\varphi, \psi \notin I_J(N^\ell)^+$ , i.e., neither  $\varphi$  nor  $\psi$  are accepted by agents in  $N^\ell$ , such that  $U_J(\{i\})^+ = \{\varphi\}$  and  $U_J(\{j\})^+ = \{\psi\}$ . This completes the construction of the profile. Note that  $|U_J(N)^+| = \lceil m/2 \rceil + 1$ . Now we assess the issues accepted by CC-JA for  $J$ . Once  $N^\ell$  is represented by  $\ell - 1$  issues, observe that (due to  $\ell > 1$ ) accepting the issue  $\varphi$  for agent  $i \notin N^\ell$  gives a greater score than accepting an  $\ell$ -th issue in  $I_J(N^\ell)^+$ . The same holds for  $\psi$  and agent  $j \notin N^\ell$ . After  $\varphi$  and  $\psi$  are accepted, accepting an  $\ell$ -th issue in  $I_J(N^\ell)^+$  decreases the score as the majority threshold is crossed. Notice that CC-JA accepts exactly  $\lceil m/2 \rceil$  issues and thus, from the definition of  $N^\ell$ 's size, it follows that  $N^\ell$  is a  $(J, \Gamma, \ell)$ -cohesive group. So we have an outcome where a  $(J, \Gamma, \ell)$ -cohesive group is only represented by  $\ell - 1$  issues.

Next, we show that every  $J \in \text{CC-JA}(J, \Gamma)$  provides  $\ell$ -JA-PJR for  $\ell = 1$ , assuming logical independence throughout the agenda. The argument is that any rejected issue in  $I_J(N^*)^+$  for an unrepresented  $(J, \Gamma, \ell)$ -cohesive group  $N^*$  can be 'swapped' for some accepted issue in an outcome  $J$ . This only decreases the CC-JA score if every already-accepted issue represents a unique group that is at least as large as  $N^*$ . Thus, any such issue would at least match the contribution to the score of an issue accepted by  $N^*$ . However, this cannot be the case for all  $|J^+|$  issues as this would imply that at least  $|J^+| \cdot \frac{n}{|J^+|} = n$  agents have been represented thus far, contradicting our assumption of the existence of this unrepresented group  $N^*$ .  $\square$

Beyond defining  $\ell$ -JA-PJR, we established some values of  $\ell$  for which our new rules satisfy the JA axiom. Hence, despite  $\ell$ -JA-PJR being restrictive, the axiom can be used to study JA rules, as those that fail it outright may be ill-suited for JA proportionality.

## 4.2 Hardness of JA Rules for Proportionality

We end by showing computational intractability of the PAV-JA and CC-JA rules, beginning with an NP-hardness result for the former.

**THEOREM 14.** *OUTCOME(PAV-JA) is NP-hard.*

**PROOF (SKETCH).** We show NP-hardness by reducing from the following problem. Take a positive integer  $t \in \mathbb{N}$  and an approval-based multiwinner election. Moreover, we know that there exists some  $b \in \mathbb{N}$  such that: (i) the maximum PAV score of any committee of size  $t$  equals  $b \cdot t$ , and this can only be achieved by getting a score of 1 from  $b \cdot t$  different voters, (ii) there exists at least one such committee of size  $t$ , and (iii) each voter approves of exactly 2 candidates. The problem is to decide, for a given candidate  $c$ , whether it is part of a committee of size  $t$  with maximum PAV score.

This problem can straightforwardly be shown to be NP-hard, by adapting the proof of a known result for multiwinner PAV voting [4, Theorem 1], which uses a reduction from the classical problem of independent set. By instead considering a suitable variant of independent set—where the size of the maximum independent set is known in advance, and the question is whether there is a maximum independent set that contains a given vertex—this proof directly yields NP-hardness of the problem that we will reduce from.

In the restricted setting where conditions (i)–(iii) hold, the simulation of multiwinner PAV that we used to establish Proposition 7

also works if we consider PAV-JA instead. Therefore, we can use this simulation to construct a reduction to  $\text{OUTCOME}(\text{PAV-JA})$ .  $\square$

Moving on to CC-JA, we see that, not only is  $\alpha$ -CC NP-hard [31, 36], but computing outcomes for CC-JA is also hard. We give a reduction from a variant of the well-known problem  $\text{MAXSAT}$  [35]—one can straightforwardly prove that this variant is  $\Theta_2^P$ -complete as well. In this variant of the problem, we are given a set  $\mathcal{L}$  of literals, two sets  $\varphi_1$  and  $\varphi_2$  of clauses, with clauses in both being of size at most 3, and some variable  $x^*$  occurring in  $\varphi_2$ . The question is to decide whether—among the truth assignments that satisfy all clauses in  $\varphi_1$  and that satisfy a maximum number of clauses in  $\varphi_2$ —there is a truth assignment that sets  $x^*$  to true.

**THEOREM 15.** *OUTCOME(CC-JA) is  $\Theta_2^P$ -complete.*

**PROOF (SKETCH).** We describe  $\Theta_2^P$ -hardness by reducing from the  $\text{MAXSAT}$ -variant described above. For space reasons, we omit the straightforward proof of membership in  $\Theta_2^P$ .

We introduce an agenda item  $y_\ell$  for each literal  $\ell$  over the variables in  $\varphi_1$  and  $\varphi_2$ . Hence, we have issues such as  $y_{x_1}$  and  $y_{\neg x_1}$  in the agenda  $\Phi$ . For each clause  $v_i$  appearing in  $\varphi_2$ , we create a voter  $i$ . These voters accept those issues which correspond to literals appearing in their associated clause. We construct an integrity constraint that expresses that one of two cases must hold: (i) at most three issues are set to true, or (ii) exactly one of each pairs of issues,  $y_{x_i}$  or  $y_{\neg x_i}$ , is set to true in a way that satisfies all clauses in  $\varphi_1$ . Finally, we construct a partial ballot  $d$  that only sets  $y_{x^*}$  to true. We claim that the outcomes of the CC-JA rule over the constructed profile correspond to the truth assignments that satisfy all of  $\varphi_1$  and that maximise the number of satisfied clauses in  $\varphi_2$ . Therefore, there is an outcome that agrees with the partial ballot  $d$  if and only if the original instance is a yes-instance. For reasons of space, we omit a detailed proof of this claim.  $\square$

## 5 CONCLUSION

We illustrated how the JA model with rationality and feasibility constraints enables us to simulate important multiwinner voting rules in JA. We subsequently showed, by encoding the constraints as DNNF circuits, that some of these simulations retain the computational efficiency of their multiwinner counterparts. On the other hand, for one specific feasibility constraint, this efficiency cannot be retained in general. Also, we demonstrated how a class of JA rules can help produce Gehrlein-stable multiwinner voting rules. Finally, we suggested an axiom and two aggregation rules for JA intended to reflect and satisfy as suitable notion of proportionality, and we briefly analysed the complexity of using these rules.

For future research, the JA simulation of more sophisticated multiwinner voting rules, such as sequential rules, should be explored. Our brief excursion into JA proportionality suggests multiple research paths, such as adapting  $\ell$ -JA-PJR to handle complex logical constraints. And for the newly-proposed JA rules, the development of approximate versions of the rules can be pursued. Also, JA rules could aid the enrichment of multiwinner voting with new rules satisfying notions other than committee stability.

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