Modal Logic and Temporal Constraint Networks

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Introduction. Description logics nowadays are widely regarded as *the* family of formal knowledge representation languages. Temporal description logics aim at extending these languages to allow also for some kind of dynamic knowledge representation, that is, the modelling of situations that are or may be time dependent. Despite the fact that a number of temporal description logics have been proposed in the past, as yet no standards have emerged and application developers are still asking for improvements.

Our research aims to contribute to this objective. The description logic in mind will based on $\mathcal{ML}_{\scriptscriptstyle TCN}$, a new modal logic of time intervals, whose syntax and semantics we are going to briefly introduce here.

Temporal constraint networks. We start by defining the notion of (qualitative) temporal constraint networks. A time interval i could lie *before* another interval j or it could be *during* j, and so on. There are 13 (mutually exclusive) interval relations of this kind, often referred to as *Allen relations*, due to Allen's influential paper [1]. We shall refer to the set of these 13 basic relations as $\mathcal{R} = \{\mathit{equals}, \mathit{before}, \mathit{after}, \ldots\}$.

Given intervals i and j a *temporal constraint* $(i, j) : R$ with $R \subseteq \mathcal{R}$ determines i and j as being related via one of the basic interval relations in R. A *temporal constraint network (TCN)* over a set of intervals I is a set of temporal constraints containing exactly one constraint per pair of intervals over I. Constraints not specifically mentioned are by default assumed to be the full set $\mathcal{R}$. A constraint $(i, j) : R$ is called *satisfied* by a TCN T iff $[(i, j) : R'] \in T$ with $R' \subseteq R$. A *singleton labelling* for a TCN T is a specialisation of T where each and every edge between two intervals is labelled with *exactly one* basic relation.

A TCN T over a set of intervals I is called *consistent* iff it has a singleton labelling for which the endpoints of all intervals in I can be ordered in such a way that every constraint in T is satisfied. Please note that we refer to endpoints of intervals only for this very purpose: the definition of consistency of a TCN. Assertions expressed in the logic that we shall define in the sequel will only refer to intervals, not points.

Syntax. We are going to define the logic $\mathcal{ML}_{\scriptscriptstyle TCN}$ as a multi-modal logic based on frames that are temporal constraint networks. The set of well-formed *formulas* is the smallest set such that:

1. propositional letters, $\top$, and $\bot$ are formulas;
2. if φ is a formula so is $\neg\varphi$;

3. if φ and ψ are formulas so are $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi$; and
4. if φ is a formula and $R \subseteq \mathcal{R}$ then also $[R]\,\varphi$ and $\langle R\rangle\,\varphi$ are formulas.

Semantics. An *interval frame* is a pair $\mathcal{F} = (I, T)$ where I is a non-empty set (of intervals) and T is a temporal constraint network over I that is a consistent singleton labelling.

A *model* is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F} = (I, T)$ is an interval frame and V is a valuation, that is a mapping from propositional letters to subsets of I, with $i \in V(P)$ iff $j \in V(P)$ for every propositional letter P whenever $[(i,j) : \{\mathit{equals}\}] \in T$. Informally, we think of $V(P)$ as the set of intervals at which the atomic proposition P holds.

We extend this notion to general formulas as follows. Let $i \in I$ be an interval in a model $\mathcal{M} = (I, T, V)$. We inductively define a formula φ being *satisfied* in $\mathcal{M}$ at i as follows:

1. $\mathcal{M}, i \models P$ iff $i \in V(P)$ for propositional letters P;
2. $\mathcal{M}, i \models \top$ for all i;
3. $\mathcal{M}, i \models \neg\varphi$ iff not $\mathcal{M}, i \models \varphi$;
4. $\mathcal{M}, i \models \varphi \vee \psi$ iff $\mathcal{M}, i \models \varphi$ or $\mathcal{M}, i \models \psi$;
5. $\mathcal{M}, i \models [R]\,\varphi$ iff $\mathcal{M}, j \models \varphi$ for all j with $(i,j) : R$ being satisfied by T.

The semantics of the other operators can be inferred from the following equivalences: $\bot \equiv \neg\top$, $\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$, $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$, and $\langle R\rangle\,\varphi \equiv \neg[R]\,\neg\varphi$.

An *assertion* $i : \varphi$ is *true* (or *satisfied*) in a model $\mathcal{M}$ iff $\mathcal{M}, i \models \varphi$ holds. A set of constraints and assertions is called *consistent* iff there exists a model in which they are all true. A formula φ is said to be *globally true* in $\mathcal{M}$ (write $\mathcal{M} \models \varphi$) iff it is satisfied at every interval in $\mathcal{M}$. Such formulas are also called *global axioms*. Consistency as defined here corresponds closely to consistency of the assertional component of a description logics based knowledge representation system. If we add global axioms this corresponds to checking consistency in description logics with respect to a terminology. Investigations into a deduction system for our logic are currently underway.

Related work. $\mathcal{ML}_{\scriptscriptstyle TCN}$ is closely related to $\mathcal{HS}$, the modal logic of time intervals proposed by Halpern and Shoham [2], but there are some important differences in the definition of the semantics for these logics. Instead of introducing the notion of temporal constraint networks and constructing an interpretation of $\mathcal{HS}$ formulas of top of it, Halpern and Shoham interpret formulas directly over pairs of interval-endpoints, which are elements of some underlying temporal structure, like, for example, the real numbers.

References

1. J. F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, November 1983.
2. J. Y. Halpern and Y. Shoham. A propositional modal logic of time intervals. *Journal of the ACM*, 38(4):935–962, 1991.