Modal Logics of Ordered Trees: A Summary*

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Abstract

We report on a new modal logic that is suitable to model complex systems evolving over time in a modular fashion. This logic may be regarded as the result of extending propositional linear temporal logic by a second dimension that allows us to "zoom" into states and thereby to further refine the specification of events associated with these states. In this sense, our logic may be considered an extended temporal logic. From a more abstract point of view, our logic is best described as a modal logic based on frames that are ordered trees.

1 Zooming in

Despite their success in the area of systems specification, a drawback of standard point-based temporal logics is that they do not support the notion of refinement in a natural manner [4]. There is no simple way to extend a given specification in, say, propositional linear temporal logic by the specification of a new subsystem. To overcome this problem, we propose to *add a zoom to linear temporal logic* by explicitly relating the state to be refined to another time line which represents the course of events taking place *during* that "state" (or rather the time interval associated with that state), at the next lower level of abstraction. The new states may themselves be refined, i.e. we may also zoom into the states on the second level of abstraction, and so forth. This idea is illustrated in Figure 1.

In this paper, we discuss an extended temporal logic that is appropriate to speak about this kind of structure. Figure 1 also shows some of the modal operators included into our new temporal language. The *horizontal* operators are those of linear temporal logic (albeit without an *until*-operator). For instance, we write $\Diamond\!\!\!\rightarrow\varphi$ to say that φ is true at *some future* state (at the same level of abstraction), while $\ominus\varphi$ expresses that φ is true at the *next* state (again at the

*This paper summarises the main results of the author's PhD thesis, written under the supervision of Dov Gabbay at King's College London [3].

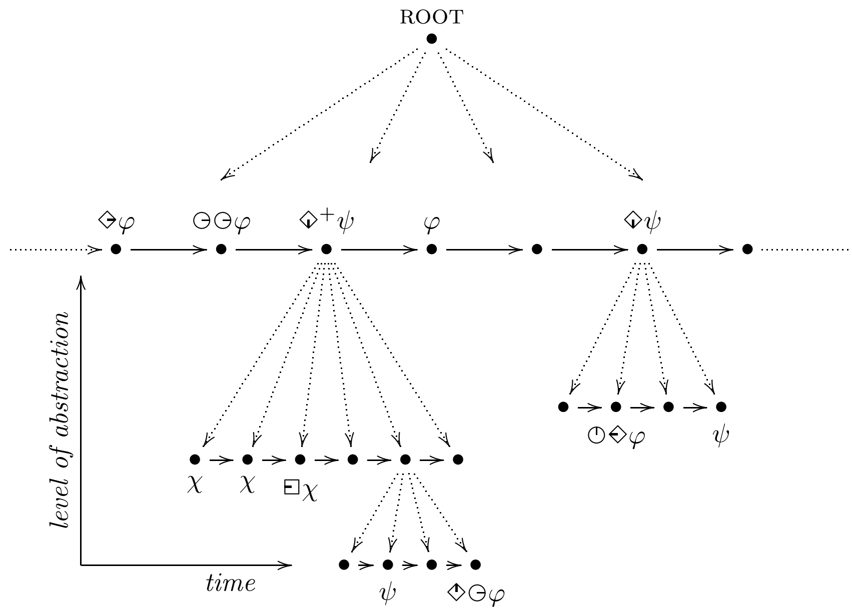

Figure 1: Zooming in

same level of abstraction). There are similar operators available to speak about the past. As for the *vertical* operators, we write $\Diamond_{\downarrow}\varphi$ to say that φ is true at *some* state belonging to the *next lower* level of abstraction, and $\Diamond_{\downarrow}^{+}\varphi$ if φ is true at *some* state of *some lower* level of abstraction. To be able to move back up again in the hierarchy of states, we may use either $\circ_{\uparrow}$ (to refer to the *next higher* level of abstraction) or $\Diamond_{\uparrow}$ (to refer to *some higher* level).

2 Ordered tree logics

If we add a "first level of abstraction" consisting of only a single state (marked as ROOT in Figure 1) to the kind of model we have described before, we end up with a tree-like structure where the children of each node are ordered; that is, our logic can be characterised as a *modal logic of ordered trees*. Here, the horizontal order declared over sibling nodes may be discrete (or even finite) as in Figure 1, but in general we admit any strict linear order over sibling nodes (which distinguishes our logic from that of Blackburn and Meyer-Viol [2]).

We call our logic OTL. The set of well-formed formulas of OTL is defined as follows (P denotes propositional letters):

$$A \ ::= \ P \mid \neg A \mid A \wedge A \mid \circ_{\leftarrow} A \mid \circ_{\rightarrow} A \mid \circ_{\uparrow} A \mid \Diamond_{\leftarrow} A \mid \Diamond_{\rightarrow} A \mid \Diamond_{\uparrow} A \mid \Diamond_{\downarrow} A \mid \Diamond_{\downarrow}^{+} A$$

Further propositional connectives are defined as syntactical abbreviations in the usual way. Also, for each of the diamond-operators in our language we introduce a corresponding box-operator, e.g. $\Box_{\rightarrow}\varphi = \neg\Diamond_{\rightarrow}\neg\varphi$. Each group of modal operators corresponds to one of the four dimensions in an ordered tree. The diamond-operator $\Diamond_{\leftarrow}$ is used to refer to the *lefthand siblings* of a given node. The corresponding next-operator $\bigcirc_{\leftarrow}$ points to the *immediate* lefthand sibling (or *neighbour*), where such a node exists. The operators to refer to *righthand* siblings and neighbours are $\Diamond_{\rightarrow}$ and $\bigcirc_{\rightarrow}$, respectively. The upward diamond-operator $\Diamond_{\uparrow}$ is used to speak about *ancestors*, while $\bigcirc_{\uparrow}$ refers to the *immediate* ancestor, i.e. the *parent* of a node. When moving down a tree there is no concept of a "next" node, which is why we need two kinds of diamond-operators here: $\Diamond_{\downarrow}$ for the *children* of a node and $\Diamond_{\downarrow}^{+}$ for its *descendants*.

OTL is a modal logic with frames that are ordered trees, i.e. a Kripke-style semantics can be given in the usual manner [1]. For instance, given a model $\mathcal{M}$ and a node t in the ordered tree underlying that model, we have $\mathcal{M}, t \models \Diamond_{\rightarrow}\varphi$ iff there exists a righthand sibling t' of t such that $\mathcal{M}, t' \models \varphi$ holds. Similarly, we have $\mathcal{M}, t \models \Box_{\rightarrow}\varphi$ iff φ is true at *all* righthand siblings of t, and $\mathcal{M}, t \models \bigcirc_{\rightarrow}\varphi$ iff t has got a righthand neighbour that satisfies φ.

3 Axiomatisation

An axiom system that is *complete* for the fragment of OTL excluding the transitive descendent modality $\Diamond_{\downarrow}^{+}$ (as well as its dual) is given in [3]. The system includes distributivity axioms (K) for all box- and next-operators, axioms encoding the fact that operators such as $\bigcirc_{\rightarrow}$ and $\bigcirc_{\leftarrow}$ are inverse to each other, "functionality" axioms (for the next-operators), as well as "mixing" axioms such as $\Diamond_{\uparrow}A \leftrightarrow (\bigcirc_{\uparrow}A \vee \bigcirc_{\uparrow}\Diamond_{\uparrow}A)$. Most interestingly, we require only two axioms to characterise the interactions between the vertical and the horizontal modalities:

(X1) $\Diamond_{\rightarrow}A \rightarrow \bigcirc_{\uparrow}\Diamond_{\downarrow}A$

(X2) $\bigcirc_{\uparrow}\Diamond_{\downarrow}A \rightarrow (A \vee \Diamond_{\leftarrow}A \vee \Diamond_{\rightarrow}A)$

We hope to address the issue of extending this axiomatisation to the full logic in our future work. The difficulties with proving completeness for full OTL stem from the fact that this logic incorporates three "problematic" features: (i) the descendant operator $\Diamond_{\downarrow}^{+}$ refers to the *transitive closure* of the child operator $\Diamond_{\downarrow}$; (ii) the child relation itself is required to be *irreflexive* (which means that the usual filtration-based approach is not immediately applicable); and (iii) due to the *interaction* of vertical and horizontal modalities, we cannot freely transform an ordered tree model (via bulldozing, for example) in order to overcome the difficulties associated with the irreflexivity requirement. This is not to say that it is impossible to prove completeness for OTL using some combination of standard techniques, but for the time being this remains an open question.

4 Decidability

OTL is a *decidable* logic. The proof uses a reduction to Rabin's Theorem on the decidability of S2S, the monadic second-order theory of two successor functions [8]. This reduction builds on the well-known construction used to prove the decidability of propositional linear temporal logic over the rationals [5, 8]. An important intermediate lemma establishes that every formula with any ordered tree model at all will also have a model based on an ordered tree with (at most) a countable number of nodes. This allows us to lift the decidability result obtained using reduction (which applies only to OTL over countable trees) to the general case. Using similar arguments, we can show that OTL restricted to certain classes of ordered trees (such as those where sequences of siblings are required to be discrete orders) is also decidable.

One may argue that, given Rabin's Theorem, the decidability of a modal logic based on *some* class of trees is not that surprising a result. On the other hand, two-dimensional modal logics with interacting modalities can often be undecidable [6]. Another reason why we consider the decidability of OTL an important result is the fact that OTL may be considered an (albeit restricted) *interval logic* (rather than just a point-based temporal logic); and modal interval logics, such as the logic of Halpern and Shoham [7] for instance, are also often found to be undecidable.

5 Conclusion

We have introduced OTL, a new modal logic based on ordered trees, and argued that this logic is suitable to model complex systems evolving over time in a modular fashion. For further information on OTL, proofs of the results reported here, and a discussion of related work we refer to [3].

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