

# Collective Rationality in Graph Aggregation

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**Abstract.** Suppose a number of agents each provide us with a directed graph over a common set of vertices. Graph aggregation is the problem of computing a single “collective” graph that best represents the information inherent in this profile of individual graphs. We consider this aggregation problem from the point of view of social choice theory and ask what properties shared by the individual graphs will transfer to the graph computed by a given aggregation procedure. Our main result is a general impossibility theorem that applies to a wide range of graph properties.

## 1 INTRODUCTION

Suppose a group of agents each supply us with a particular piece of information and we want to aggregate this information into a collective view to obtain a good overall representation of the individual views provided. This may be interpreted as a problem of social choice. In classical social choice theory (SCT) the objects of aggregation have been preference orders on a set of alternatives [2, 21]. More recently, the same methodology has also been applied to other types of information, notably beliefs [16], judgments [17], ontologies [20], taxonomic models used for classification [3], and rankings provided by Internet search engines [10].

Here we consider the problem of *graph aggregation*, i.e., the problem of devising methods to aggregate the information inherent in a profile of individual (directed) graphs, one for each agent, into a single collective graph. Given that a preference order is a special kind of directed graph, graph aggregation may be viewed as a direct generalisation of classical preference aggregation. This is a useful generalisation, because also several other problem domains in which aggregation is relevant are naturally modelled as graphs. For instance, some authors have studied the aggregation of the graphs underlying abstract argumentation frameworks [6, 9, 22]. Others have considered the aggregation of incomplete transitive relations to account for the bounded rationality of agents expressing preferences [19]. Special instances of the graph aggregation problem have also been studied in work on the aggregation of judgments regarding causal relations between variables [5] and the design of voting agendas for multi-issue elections based on individually reported preferential dependencies between issues [1]. Finally, graph aggregation is also at the core of recent work on the aggregation of different logics [23]. However, a general account of the graph aggregation problem *as such* has so far been missing. Our goal here is to bridge this gap.

While graph aggregation is more general than preference aggregation, it is (in some sense) less general than judgment aggregation [17] or binary aggregation [8, 12]: just like classical preference aggregation, graph aggregation can—in principle—be embedded into these

frameworks. For a given problem domain, it is important to find the right level of abstraction, and graphs appear to be a particularly useful level of abstraction for a wide range of problems.

The question we ask in this paper is what properties shared by the individual graphs will transfer to the collective graph returned by a given aggregator. For example, if we aggregate individual graphs by computing their *union* (i.e., if we include an edge from  $x$  to  $y$  in our collective graph if at least one of the individual graphs includes that edge), then it is not difficult to verify that the property of *reflexivity* will transfer while that of *transitivity* will not. Thus, if all individual graphs are reflexive, then so is their union, while there are transitive graphs the union of which is not transitive. We say that the union rule is *collectively rational* w.r.t. reflexivity, but not w.r.t. transitivity.

Collective rationality is an important concept in SCT. In preference aggregation an aggregator is called collectively rational if it ensures transitivity and completeness of the preference structure produced [2]. In judgment aggregation it refers to aggregators that ensure logically consistent outcomes [17]. Work in binary aggregation has generalised beyond these domain-specific uses of the term and considered collective rationality w.r.t. arbitrary properties [12].

Besides introducing a formal framework for graph aggregation, our main contribution in this paper is to prove a very general *impossibility theorem*. Arrow’s classical result for preference aggregation states that no aggregator meeting certain basic axiomatic requirements can possibly be collectively rational w.r.t. both transitivity and completeness [2]. We show that the same kind of result applies to a wide range of other properties of graphs. Rather than proving our result for a specific combination of graph properties, we introduce three meta-properties and show that collectively rational aggregation is impossible for any combination of properties that are instances of these meta-properties. For instance, both transitivity and the so-called Euclidean properties are instances of the meta-property of “implicativeness”, as they all stipulate that the inclusion of one type of edge is *implied* by the inclusion of certain other edges.

The remainder of the paper is organised as follows. The formal framework of graph aggregation is defined in Section 2. Section 3 is devoted to our general impossibility theorem. Finally, Section 4 concludes and suggests possible directions for future work.<sup>3</sup>

## 2 GRAPH AGGREGATION

Fix a finite set of *vertices*  $V$ . A (directed) *graph*  $G = \langle V, E \rangle$  based on  $V$  is defined by a set of *edges*  $E \subseteq V \times V$ . We write  $xEy$  for  $(x, y) \in E$ . As  $V$  is fixed,  $G$  is in fact fully determined by  $E$ . We therefore refer to sets of edges  $E \subseteq V \times V$  simply as graphs. Table 1 lists some well-known *properties* of graphs.<sup>4</sup>

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<sup>3</sup> An early version of this work has been presented at COMSOC-2012 [11].

<sup>4</sup> The rightmost column of Table 1 specifies which properties are *contagious*, *implicative*, and *disjunctive*, respectively (see Section 3.2).

PROPERTY	FIRST-ORDER CONDITION	C/I/D
Reflexivity	$\forall x.xEx$	— — —
Irreflexivity	$\neg\exists x.xEx$	— — —
Transitivity	$\forall xyz.(xEy \wedge yEz \rightarrow xEz)$	+ + —
Right Euclidean	$\forall xyz.(xEy \wedge xEz \rightarrow yEz)$	+ + —
Left Euclidean	$\forall xyz.(xEy \wedge zEy \rightarrow zEx)$	+ + —
Seriality	$\forall x.\exists y.xEy$	— — +
Completeness	$\forall xy.[x \neq y \rightarrow (xEy \vee yEx)]$	— — +
Connectedness	$\forall xyz.[xEy \wedge xEz \rightarrow (yEz \vee zEy)]$	+ + +
Negative Trans.	$\forall xyz.[xEy \rightarrow (xEz \vee zEy)]$	+ — +

**Table 1.** Common properties of directed graphs.

For example, a *weak order* is a directed graph that is reflexive, transitive, and complete.

Let  $\mathcal{N}$  be a finite set of (two or more) *individuals*. Each individual  $i \in \mathcal{N}$  specifies a graph  $E_i \subseteq V \times V$ , giving rise to a *profile*  $\mathbf{E} = (E_1, \dots, E_n)$ .  $N_e^{\mathbf{E}} := \{i \in \mathcal{N} \mid e \in E_i\}$  denotes the set of individuals accepting edge  $e$  under profile  $\mathbf{E}$ . An *aggregator* is a function  $F : (2^{V \times V})^n \rightarrow 2^{V \times V}$ , mapping any such profile into a single collective graph  $E$ . An example is the *majority rule*, accepting a given edge if and only if more than half of the individuals accept it.

**Example 1** (Preferences). *An example for a graph aggregation problem is preference aggregation as classically studied in SCT [2]. In this context, vertices are interpreted as alternatives and the graphs considered are weak orders on these alternatives. Our aggregators then reduce to so-called social welfare functions.*

**Example 2** (Knowledge). *If we think of  $V$  as a set of possible worlds, then a graph on  $V$  that is reflexive and transitive (and possibly also symmetric) can be used to model an agent’s knowledge:  $(x, y)$  being an edge means that, if  $x$  is the true current world, then our agent will consider  $y$  a possible world [14]. If we aggregate the graphs of several agents by taking their intersection, then the resulting collective graph represents the distributed knowledge of the group. If, on the other hand, we aggregate by taking the transitive closure of the union of the individual graphs, then we obtain a model of the group’s common knowledge.*

Recall that we have assumed that every individual specifies a graph on the same set of vertices  $V$ . For both of our examples above this is a natural assumption to make, but in general we might also be interested in aggregating graphs defined on different sets of vertices. Observe that in this case our framework is still applicable, as we may think of  $V$  as the union of all the individual sets of vertices (with each individual only providing edges involving “their” vertices).

## 2.1 Axioms: Properties of graph aggregators

Adopting the *axiomatic method* familiar from SCT [21], we can characterise certain classes of aggregators in terms of *axioms*, i.e., intuitively appealing properties. The first such axiom is an independence condition that requires that the decision of whether or not a given edge  $e$  should be part of the collective graph should only depend on which of the individual graphs include  $e$ . This corresponds to well-known axioms in preference and judgment aggregation [2, 17].

**Definition 1.**  $F$  is *independent of irrelevant edges (IIE)* if  $N_e^{\mathbf{E}} = N_e^{\mathbf{E}'}$  implies  $e \in F(\mathbf{E}) \Leftrightarrow e \in F(\mathbf{E}')$ .

That is, if exactly the same individuals accept  $e$  under profiles  $\mathbf{E}$  and  $\mathbf{E}'$ , then  $e$  should be part of either both or none of the corresponding

collective graphs. The definition above applies to *all* edges  $e \in V \times V$  and *all* pairs of profiles  $\mathbf{E}, \mathbf{E}' \in (2^{V \times V})^n$ . We shall leave this kind of universal quantification implicit also in later definitions.

The fundamental economic principle of *unanimity* requires that an edge should be accepted by the group if all individuals in it accept it.

**Definition 2.**  $F$  is *unanimous* if  $F(\mathbf{E}) \supseteq E_1 \cap \dots \cap E_n$ .

A requirement that, in some sense, is dual to unanimity is to ask that the collective graph should only include edges that are part of at least one of the individual graphs. In the context of ontology aggregation this axiom has been called *groundedness* [20].

**Definition 3.**  $F$  is *grounded* if  $F(\mathbf{E}) \subseteq E_1 \cup \dots \cup E_n$ .

The remaining two axioms are standard desiderata and closely modelled on their counterparts in the field of judgment aggregation [17].

**Definition 4.**  $F$  is *anonymous* if  $F(\mathbf{E}) = F(E_{\pi(1)}, \dots, E_{\pi(n)})$  for any permutation  $\pi : \mathcal{N} \rightarrow \mathcal{N}$ .

**Definition 5.**  $F$  is *neutral* if  $N_e^{\mathbf{E}} = N_{e'}^{\mathbf{E}}$  implies  $e \in F(\mathbf{E}) \Leftrightarrow e' \in F(\mathbf{E})$ .

Anonymity and neutrality are basic symmetry requirements w.r.t. individuals and edges, respectively. An extreme form of violating anonymity is to use an aggregator that is *dictatorial* in the sense that a single individual can determine the shape of the collective graph.

**Definition 6.**  $F$  is *dictatorial* if there exists an individual  $i^* \in \mathcal{N}$  (the dictator) such that  $e \in F(\mathbf{E}) \Leftrightarrow e \in E_{i^*}$  for every edge  $e \in V \times V$ .

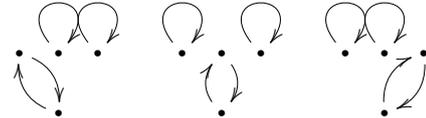
An aggregator  $F$  that is not a dictatorship for any of the individuals is called *nondictatorial*. Sometimes we are only interested in the properties of an aggregator as far as the *nonreflexive* edges  $e = (x, y)$  with  $x \neq y$  are concerned. Specifically, we call  $F$  *NR-neutral* if  $N_{(x,y)}^{\mathbf{E}} = N_{(x',y')}^{\mathbf{E}'}$  implies  $(x, y) \in F(\mathbf{E}) \Leftrightarrow (x', y') \in F(\mathbf{E}')$  for all  $x \neq y$  and  $x' \neq y'$ ; and we call  $F$  *NR-nondictatorial* if there exists no  $i^* \in \mathcal{N}$  such that  $(x, y) \in F(\mathbf{E}) \Leftrightarrow (x, y) \in E_{i^*}$  for all  $x \neq y$ . That is, NR-neutrality is slightly weaker than neutrality and NR-nondictatoriality is slightly stronger than nondictatoriality.

## 2.2 Collective rationality

In this paper, we want to analyse to what extent aggregators can ensure that a given property that is satisfied by each of the individual graphs is preserved during aggregation.

**Definition 7.**  $F$  is *collectively rational (CR)* w.r.t. graph property  $P$  if  $F(\mathbf{E})$  satisfies  $P$  whenever all of the individual graphs in  $\mathbf{E}$  do.

**Example 3** (Collective rationality). *Suppose three individuals each provide us with a graph over the same set of four vertices:*



*If we apply the majority rule, then we obtain a graph where the only edges are those connecting the upper three vertices with themselves. That is, the majority rule is not CR w.r.t. seriality, as each individual graph is serial, but the collective graph is not.*

For some graph properties, collective rationality is easy to achieve, as the following simple possibility results demonstrate.

**Proposition 1.** *Any unanimous aggregator is CR w.r.t. reflexivity.*

*Proof.* If every individual graph includes all edges of the form  $(x, x)$ , then unanimity ensures the same for the collective graph.  $\square$

**Proposition 2.** *Any grounded aggregator is CR w.r.t. irreflexivity.*

*Proof.* If no individual graph includes the edge  $(x, x)$ , then groundedness ensures the same for the collective graph.  $\square$

### 3 A GENERAL IMPOSSIBILITY THEOREM

An *impossibility result* states that it is not possible to devise an aggregator that satisfies certain axioms and that is also CR w.r.t. a certain combination of properties of the structures being aggregated (which in our case are graphs). In this section, we will prove a powerful impossibility result for graph aggregation.

#### 3.1 Arrow's Theorem

The prime example of an impossibility result is Arrow's Theorem for preference aggregation [2]. It states that there exists no nondictatorial, Paretian, and independent aggregator mapping profiles of weak orders over three or more alternatives to collective weak orders. We can reformulate this result in our framework for graph aggregation:

*For  $|V| \geq 3$ , there exists no nondictatorial, unanimous, grounded, and IIE aggregator that is CR w.r.t. reflexivity, transitivity, and completeness.*

Note that we have translated Arrow's (weak) Pareto condition (if every individual ranks  $x$  strictly above  $y$ , then so should the collective) to a combination of unanimity and groundedness. In fact, in the context of the other requirements, Pareto efficiency *implies* both of these properties (so our version is at least as strong as Arrow's Theorem). In the sequel, we will sometimes refer to aggregators that are unanimous, grounded, and IIE as *Arrowian aggregators*.

#### 3.2 Winning coalitions

As is well understood in SCT, impossibility theorems in preference aggregation heavily feed on the notion of independence (in our case IIE). Observe that an aggregator  $F$  satisfies IIE if and only if for each edge  $e \in V \times V$  there exists a set of *winning coalitions*  $\mathcal{W}_e \subseteq 2^N$  such that  $e \in F(\mathbf{E}) \Leftrightarrow N_e^{\mathbf{E}} \in \mathcal{W}_e$ . That is,  $F$  accepts  $e$  if and only if exactly the individuals in one of the winning coalitions for  $e$  do. Imposing additional axioms on  $F$  corresponds to restrictions on the associated family of winning coalitions  $\{\mathcal{W}_e\}_{e \in V \times V}$ :

- If  $F$  is unanimous, then  $\mathcal{N} \in \mathcal{W}_e$  for any edge  $e$ .
- If  $F$  is grounded, then  $\emptyset \notin \mathcal{W}_e$  for any edge  $e$ .
- If  $F$  is (NR-)neutral, then  $\mathcal{W}_e = \mathcal{W}_{e'}$  for any two (nonreflexive) edges  $e$  and  $e'$ .

Recall that neutrality does not feature in Arrow's Theorem. As we shall see soon, the reason is that the same restriction on winning coalitions is already enforced by collective rationality w.r.t. transitivity. This is an interesting link between a specific collective rationality requirement and a specific axiom. In the literature, this fact is often called the *Contagion Lemma* [21], although the connection to neutrality is not usually made explicit. The same kind of result can also be obtained for other graph properties with a similar structure. Let us now develop a definition for a class of graph properties that will allow us to derive neutrality.

#### 3.3 Contagious properties

It will be useful to think of a graph property  $P$ , such as transitivity or reflexivity, as a subset of  $2^{V \times V}$  (the set of all graphs over the set of vertices  $V$ ). For two disjoint sets of edges  $S^+$  and  $S^-$  and a graph property  $P$ , let  $P[S^+, S^-] = \{E \in P \mid S^+ \subseteq E \text{ and } S^- \cap E = \emptyset\}$  denote the set of graphs in  $P$  that include all of the edges in  $S^+$  and none of those in  $S^-$ . We start with a technical definition.

**Definition 8.** *Let  $x, y, z, w \in V$ . A graph property  $P \subseteq 2^{V \times V}$  is *xy/zw-contagious* if there exist two disjoint sets  $S^+, S^- \subseteq V \times V$  such that (i) for every graph  $E \in P[S^+, S^-]$  it is the case that  $(x, y) \in E$  implies  $(z, w) \in E$  and (ii) there exist graphs  $E_0, E_1 \in P[S^+, S^-]$  with  $(z, w) \notin E_0$  and  $(x, y) \in E_1$ .*

Part (i) of Definition 8 says that, if you accept edge  $(x, y)$ , then you must also accept edge  $(z, w)$ —at least if the side condition of you also accepting all the edges in  $S^+$  but none of those in  $S^-$  is met. That is, the property of contagiousness may be paraphrased as the formula  $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [xEy \rightarrow zEw]$ . Part (ii) is a richness condition that says that you have the option of accepting neither or both of  $(x, y)$  and  $(z, w)$ . It requires the existence of a graph  $E_0$  where neither  $(x, y)$  nor  $(z, w)$  are accepted, and the existence of a graph  $E_1$  where both  $(x, y)$  and  $(z, w)$  are accepted.

Contagiousness w.r.t. two given edges will be useful for our purposes if those two edges stand in a specific relationship to each other.

**Definition 9.** *A graph property  $P \subseteq 2^{V \times V}$  is *contagious* if it satisfies at least one of the three conditions below:*

- (i)  *$P$  is *xy/yz-contagious* for all triples of vertices  $x, y, z \in V$ .*
- (ii)  *$P$  is *xy/zx-contagious* for all triples of vertices  $x, y, z \in V$ .*
- (iii)  *$P$  is *xy/xz-contagious* and *xy/zy-contagious* for all  $x, y, z \in V$ .*

That is, Definition 9 covers pairs of edges where (i) the second edge is a successor of the first edge, where (ii) the second edge is a predecessor of the first edge, and where (iii) the two edges share either a starting point or an end point. This covers all cases of two edges meeting in one point. As will become clear in the proof of Lemma 4, case (iii) differs from the other two, as only one of these two types of connections would not be sufficient to “traverse” the full graph.

**Fact 3.** *For  $|V| \geq 3$ , transitivity, the two Euclidean properties, negative transitivity, and connectedness are contagious graph properties.*

*Proof.* Let us first consider the property of being right Euclidean. It satisfies condition (i) of Definition 9. To prove this, we will show that the right-Euclidean property is *xy/yz-contagious* for all triples  $x, y, z \in V$ . Let  $S^+ = \{(x, z)\}$  and  $S^- = \emptyset$ , i.e.,  $P[S^+, S^-]$  is the set of all graphs containing  $(x, z)$ . Condition (i) of Definition 8 is met: any graph in  $P[S^+, S^-]$  contains  $(x, z)$ ; therefore, by the right-Euclidean property  $(y, z)$  needs to be accepted whenever  $(x, y)$  is. Condition (ii) is also satisfied. Let  $E_0$  be the graph only containing the single edge  $(x, z)$ , and let  $E_1$  be the graph containing exactly the three edges  $(x, y)$ ,  $(y, z)$ , and  $(x, z)$ . Both graphs are right-Euclidean and, since they include  $(x, z)$ , they also belong to  $P[S^+, S^-]$ .

An alternative way to see that the right-Euclidean property is contagious is to observe that it is equivalent to the formula  $[xEz] \rightarrow [xEy \rightarrow yEz]$ , with all variables universally quantified. Similarly, the left-Euclidean property, which can be rewritten as  $[zEy] \rightarrow [xEy \rightarrow zEx]$ , is contagious by condition (ii). Connectedness can be rewritten as  $[xEz \wedge \neg zEy] \rightarrow [xEy \rightarrow yEz]$  and thus satisfies condition (i). Transitivity satisfies condition (iii), as we can rewrite it as either  $[yEz] \rightarrow [xEy \rightarrow xEz]$  or  $[zEx] \rightarrow [xEy \rightarrow zEy]$ .

Negative transitivity, finally, can be rewritten as either  $[\neg(zEy)] \rightarrow [xEy \rightarrow xEz]$  or  $[\neg(xEz)] \rightarrow [xEy \rightarrow zEy]$  and thus also satisfies condition (iii). For all these cases, the richness conditions are easily verified to hold as well.  $\square$

We are now ready to prove a powerful lemma showing that any Arrowian aggregator that is CR w.r.t. a contagious graph property must be neutral (at least as far as nonreflexive edges are concerned).

**Lemma 4.** *For  $|V| \geq 3$ , any unanimous, grounded, and IIE aggregator that is CR w.r.t. a contagious graph property is NR-neutral.*

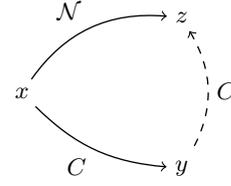
*Proof.* We will first establish a generic result for collective rationality w.r.t.  $xy/zw$ -contagiousness. Let  $x, y, z, w \in V$ . Take any graph property  $P$  that is  $xy/zw$ -contagious and take any aggregator  $F$  that is unanimous, grounded, IIE, and CR w.r.t.  $P$ . Let  $\{\mathcal{W}_e\}_{e \in V \times V}$  be the family of winning coalitions associated with  $F$ . We want to show that  $\mathcal{W}_{(x,y)} \subseteq \mathcal{W}_{(z,w)}$ . So let  $C \in \mathcal{W}_{(x,y)}$ . Let  $S^+, S^- \subseteq V \times V$  and  $E_0, E_1 \in P[S^+, S^-]$  be defined as in Definition 8. Consider a profile  $\mathbf{E}$  in which the individuals in  $C$  propose graph  $E_1$  and all others propose  $E_0$ . That is, all individuals accept the edges in  $S^+$ , none accept any of those in  $S^-$ , exactly the individuals in  $C$  accept edge  $(x, y)$ , and exactly those in  $C$  also accept  $(z, w)$ . Now consider the collective graph  $F(\mathbf{E})$ . By unanimity  $S^+ \subseteq F(\mathbf{E})$ , by groundedness  $S^- \cap F(\mathbf{E}) = \emptyset$ , and finally  $(x, y) \in F(\mathbf{E})$  due to  $C$  being a winning coalition for  $(x, y)$ . By collective rationality,  $F(\mathbf{E}) \in P$  and thus also  $F(\mathbf{E}) \in P[S^+, S^-]$ . But then, due to  $xy/zw$ -contagiousness of  $F(\mathbf{E})$ , we get  $(z, w) \in F(\mathbf{E})$ . As it was exactly the individuals in  $C$  who accepted  $(z, w)$ , coalition  $C$  must be winning for  $(z, w)$ , i.e.,  $C \in \mathcal{W}_{(z,w)}$ , and we are done.

We are now ready to prove the lemma. Take any graph property  $P$  that is contagious and take any aggregator  $F$  that is unanimous, grounded, IIE, and CR w.r.t.  $P$ . Let  $\{\mathcal{W}_e\}_{e \in V \times V}$  be the family of winning coalitions associated with  $F$ . We need to show that there exists a unique  $\mathcal{W} \subseteq 2^N$  such that  $\mathcal{W} = \mathcal{W}_e$  for every nonreflexive edge  $e$ . By unanimity, the sets  $\mathcal{W}_e$  are not empty. Consider any three vertices  $x, y, z \in V$  and any coalition  $C \in \mathcal{W}_{(x,y)}$ . We will show that  $C$  is also winning for both  $(y, z)$  and  $(y, x)$ . If we can show this for any  $x, y, z$ , then we are done, as we can then repeat the same method several times until all nonreflexive edges are covered.

For each of the three possible ways in which  $P$  can be contagious (see Definition 9), we will use different instances of our generic result for  $xy/zw$ -contagiousness above. First, if  $P$  is contagious by virtue of condition (i), then we can use  $xy/yz$ -contagiousness to get  $C \in \mathcal{W}_{(y,z)}$  and its instance  $xy/yx$ -contagiousness (with  $z := x$ ) to obtain also  $C \in \mathcal{W}_{(y,x)}$ . Second, if  $P$  is contagious due to condition (ii), we use  $xy/yx$ -contagiousness to get  $C \in \mathcal{W}_{(y,x)}$ , and then  $yx/zy$ -contagiousness to get  $C \in \mathcal{W}_{(z,y)}$  and  $zy/yz$ -contagiousness to get  $C \in \mathcal{W}_{(y,z)}$ .

Third, suppose  $P$  is contagious by virtue of condition (iii). We first use  $xy/zy$ -contagiousness to obtain  $C \in \mathcal{W}_{(z,y)}$  and then  $zy/zx$ -contagiousness to get  $C \in \mathcal{W}_{(z,x)}$ . From the latter, via  $zx/yx$ -contagiousness we get  $C \in \mathcal{W}_{(y,x)}$ . Finally,  $yx/yz$ -contagiousness then entails  $C \in \mathcal{W}_{(y,z)}$ . Hence, we obtain the required transfer from one edge  $(x, y)$  to both its successor  $(y, z)$  and its inverse  $(y, x)$  in all three cases, and our proof is complete.  $\square$

Figure 1 provides an illustration of a specific instance of the main argument in the proof of Lemma 4 when the right-Euclidean property is considered, which is  $xy/yz$ -contagious by Fact 3. We have  $S^+ = \{(x, z)\}$  and  $S^- = \emptyset$ .  $E_1$  is the graph that accepts all three edges  $(x, y)$ ,  $(y, z)$  and  $(x, z)$ , and  $E_0$  accepts only edge  $(x, z)$ . Consider profile  $\mathbf{E}$ , in which the individuals in  $C$  choose  $E_1$  and all



**Figure 1.** CR w.r.t. the right-Euclidean property implies neutrality.

others choose  $E_0$ . That is, the individuals in  $C$  accept  $(x, y)$  and  $(y, z)$ , while  $(x, z)$  is accepted by all individuals in  $\mathcal{N}$ . By unanimity,  $(x, z)$  must be accepted, and due to  $C \in \mathcal{W}_{(x,y)}$  also  $(x, y)$  should be accepted. We can now conclude, since  $F$  is CR w.r.t. the right-Euclidean property, that  $(y, z)$  should also be accepted, and hence that  $C \in \mathcal{W}_{(y,z)}$ . It is then sufficient to consider all triples to obtain neutrality over all (nonreflexive) edges.

### 3.4 Implicative and disjunctive properties

Our next goal is to prove a modular result that derives an impossibility from a combination of two types of graph properties. We first introduce these new meta-properties.

**Definition 10.** *A graph property  $P \subseteq 2^{V \times V}$  is **implicative** if there exist two disjoint sets  $S^+, S^- \subseteq V \times V$  and three distinct edges  $e_1, e_2, e_3 \in V \times V \setminus (S^+ \cup S^-)$  such that (i) for every graph  $E \in P[S^+, S^-]$  it is the case that  $e_1, e_2 \in E$  implies  $e_3 \in E$  and (ii) there exist graphs  $E_0, E_1, E_2, E_3 \in P[S^+, S^-]$  with  $E_0 \cap \{e_1, e_2, e_3\} = \emptyset$ ,  $E_1 \cap \{e_1, e_2, e_3\} = \{e_1\}$ ,  $E_2 \cap \{e_1, e_2, e_3\} = \{e_2\}$ , and  $\{e_1, e_2, e_3\} \subseteq E_3$ .*

Part (i) expresses that all graphs with property  $P$  (that also include all edges in  $S^+$  and none from  $S^-$ ) must satisfy the formula  $e_1 \wedge e_2 \rightarrow e_3$ . Part (ii) is a richness condition saying that accepting/rejecting any combination of  $e_1$  and  $e_2$  is possible and that  $e_3$  need not be accepted unless both  $e_1$  and  $e_2$  are. Observe that Definition 10 has an existential form, i.e., we simply need to find two subsets  $S^+$  and  $S^-$  for the precondition, and three edges  $e_1, e_2$  and  $e_3$  that satisfy the two requirements (i) and (ii). This meta-condition for graph properties may be paraphrased as the formula  $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \wedge e_2 \rightarrow e_3]$ .

**Fact 5.** *For  $|V| \geq 3$ , transitivity, the two Euclidean properties, and connectedness are implicative graph properties.*

*Proof (sketch).* Let  $V = \{v_1, v_2, v_3, \dots\}$ . To see that transitivity satisfies Definition 10, choose  $S^+ = S^- = \emptyset$ ,  $e_1 = (v_1, v_2)$ ,  $e_2 = (v_2, v_3)$ , and  $e_3 = (v_1, v_3)$ . Transitivity implies that if both  $e_1$  and  $e_2$  are accepted, then also  $e_3$  should be accepted. All remaining acceptance/rejection patterns of  $e_1, e_2$ , and  $e_3$  are possible, in accordance with condition (ii). The proofs for the Euclidean properties are similar. Rewriting connectedness as  $[\neg yEz] \rightarrow [xEy \wedge xEz \rightarrow zEy]$  shows that it is implicative as well.  $\square$

Note that implicativeness is a very weak requirement: even transitivity restricted to a single triple of edges is sufficient to satisfy it.

**Definition 11.** *A graph property  $P \subseteq 2^{V \times V}$  is **disjunctive** if there exist two disjoint sets  $S^+, S^- \subseteq V \times V$  and two distinct edges  $e_1, e_2 \in V \times V \setminus (S^+ \cup S^-)$  such that (i) for every graph  $E \in P[S^+, S^-]$  we have  $e_1 \in E$  or  $e_2 \in E$  and (ii) there exist two graphs  $E_1, E_2 \in P[S^+, S^-]$  with  $E_1 \cap \{e_1, e_2\} = \{e_1\}$  and  $E_2 \cap \{e_1, e_2\} = \{e_2\}$ .*

Part (i) ensures that all graphs with property  $P$  (that meet the precondition of including all edges in  $S^+$  and none from  $S^-$ ) satisfy the formula  $e_1 \vee e_2$ . Part (ii) is a richness condition ensuring that there are at least two graphs that each include only one of  $e_1$  and  $e_2$ . Observe that Definition 11 also has an existential form, and that it may be paraphrased as the formula  $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \vee e_2]$ .

**Fact 6.** For  $|V| \geq 3$ , completeness, connectedness, seriality, and negative transitivity are disjunctive graph properties.

*Proof.* Let  $V = \{v_1, \dots, v_m\}$ . For completeness, choose  $S^+ = S^- = \emptyset$ ,  $e_1 = (v_1, v_2)$ , and  $e_2 = (v_2, v_1)$  to see that the conditions are satisfied. For connectedness, choose  $S^+ = \{(v_1, v_2), (v_1, v_3)\}$ ,  $S^- = \emptyset$ ,  $e_1 = (v_2, v_3)$ , and  $e_2 = (v_3, v_2)$ . For seriality, choose  $S^+ = \emptyset$ ,  $S^- = \{(v_1, v_1), (v_1, v_2), \dots, (v_1, v_{m-2})\}$ ,  $e_1 = (v_1, v_{m-1})$ , and  $e_2 = (v_1, v_m)$ . For negative transitivity, choose  $S^+ = \emptyset$ ,  $S^- = \{(v_1, v_3)\}$ ,  $e_1 = (v_1, v_2)$ , and  $e_2 = (v_3, v_2)$ .  $\square$

Table 1 summarises which of our standard graph properties are contagious, implicative, and disjunctive, respectively.

### 3.5 Impossibility of graph aggregation

We now prove a first version of our main impossibility result, initially under the additional assumption of neutrality. We do this by proving that the set of winning coalitions corresponding to any aggregator that meets certain conditions is an *ultrafilter* [7].

**Definition 12.** An *ultrafilter*  $\mathcal{W}$  on a set  $\mathcal{N}$  is a collection of subsets of  $\mathcal{N}$  satisfying the following three conditions:

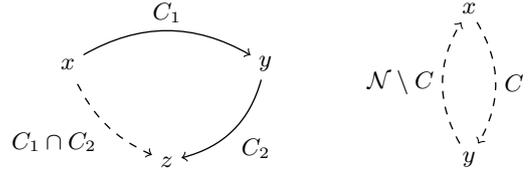
- (i)  $\emptyset \notin \mathcal{W}$
- (ii)  $C_1, C_2 \in \mathcal{W}$  implies  $C_1 \cap C_2 \in \mathcal{W}$  (closure under intersection)
- (iii)  $C$  or  $\mathcal{N} \setminus C$  is in  $\mathcal{W}$  for any  $C \subseteq \mathcal{N}$  (maximality)

In SCT, the ultrafilter method has first been used by Kirman and Sondermann [15] to prove Arrow's Theorem, and it has also found applications in judgment aggregation [13]. The following result applies to graph properties that are both implicative and disjunctive, e.g., the property of being both transitive and complete.

**Proposition 7.** For  $|V| \geq 3$ , there exists no NR-nondictatorial, unanimous, grounded, NR-neutral, and IIE aggregator that is CR w.r.t. any graph property that is both implicative and disjunctive.

*Proof.* Take any graph property  $P$  that is implicative and disjunctive, and any aggregator  $F$  that is unanimous, grounded, NR-neutral, IIE, and CR w.r.t.  $P$ . Due to IIE and NR-neutrality, there exists a set of winning coalitions  $\mathcal{W} \subseteq 2^{\mathcal{N}}$  with  $e \in F(\mathbf{E}) \Leftrightarrow N_e^{\mathbf{E}} \in \mathcal{W}$  for any nonreflexive edge  $e$ . We shall prove that the set of winning coalitions  $\mathcal{W}$  is an ultrafilter. Condition (i) holds, as  $F$  is grounded.

For condition (ii) we will make use of the assumption that  $P$  is implicative. Let  $S^+, S^- \subseteq V \times V$ ;  $e_1, e_2, e_3 \in V \times V$ ; and  $E_0, E_1, E_2, E_3 \in P[S^+, S^-]$  be defined as in Definition 10. Now take any two winning coalitions  $C_1, C_2 \in \mathcal{W}$ . Consider a profile of graphs  $\mathbf{E}$  meeting  $P$  in which exactly the individuals in  $C_1 \cap C_2$  propose  $E_3$ , those in  $C_1 \setminus C_2$  propose  $E_1$ , those in  $C_2 \setminus C_1$  propose  $E_2$ , and all others propose  $E_0$ . Thus, exactly the individuals in  $C_1$  accept  $e_1$ , exactly those in  $C_2$  accept  $e_2$ , and exactly those in  $C_1 \cap C_2$  accept  $e_3$ . Furthermore, all individuals accept  $S^+$  and all of them reject  $S^-$ . Hence, due to unanimity, all edges in  $S^+$  must be part of the collective graph  $F(\mathbf{E})$ , while due to groundedness, none of the edges in  $S^-$  can be part of  $F(\mathbf{E})$ . As  $F$  is CR w.r.t.  $P$ , we get  $F(\mathbf{E}) \in P[S^+, S^-]$ . Now, since  $C_1$  and  $C_2$  are winning coalitions,



**Figure 2.** The set of winning coalitions is an ultrafilter.

$e_1$  and  $e_2$  must be part of  $F(\mathbf{E})$ . As  $P$  is implicative, this means that  $e_3 \in F(\mathbf{E})$ . Hence,  $C_1 \cap C_2 \in \mathcal{W}$ .

For condition (iii) we will make use of the assumption that  $P$  is disjunctive. Let  $S^+, S^- \subseteq V \times V$ ;  $e_1, e_2 \in V \times V$ ; and  $E_1, E_2 \in P[S^+, S^-]$  be defined as in Definition 11. Now take any winning coalition  $C \in \mathcal{W}$ . Consider a profile  $\mathbf{E}$  meeting  $P$  in which exactly the individuals in  $C$  propose  $E_1$  and exactly those in  $\mathcal{N} \setminus C$  propose  $E_2$ . Recall that  $S^+ \subseteq E_1$  and  $S^+ \subseteq E_2$ , i.e., all individuals accept  $S^+$ . Thus, due to unanimity, all of the edges in  $S^+$  must be part of the collective graph  $F(\mathbf{E})$ . Analogously, due to groundedness, none of the edges in  $S^-$  can be part of  $F(\mathbf{E})$ . Thus, as  $F$  is CR w.r.t.  $P$  we get  $F(\mathbf{E}) \in P[S^+, S^-]$ . As  $P$  is disjunctive, this means that one of  $e_1$  and  $e_2$  has to be part of  $F(\mathbf{E})$ . Hence,  $C \in \mathcal{W}$  or  $(\mathcal{N} \setminus C) \in \mathcal{W}$ .

Recall that  $\mathcal{N}$  is required to be finite. An ultrafilter  $\mathcal{W}$  on a set  $\mathcal{N}$  is called *principal* if it is of the form  $\mathcal{W} = \{C \in 2^{\mathcal{N}} \mid i^* \in C\}$  for some fixed  $i^* \in \mathcal{N}$ . In our setting, principality of  $\mathcal{W}$  corresponds to  $F$  being dictatorial (with dictator  $i^*$ ) on nonreflexive edges. Now, it is a well-known fact that every ultrafilter on a finite set must be principal [7], which shows that  $F$  cannot be NR-nondictatorial.  $\square$

Figure 2 provides an illustration of a specific instance of the main argument in the proof of Proposition 7. To show that the set of winning coalitions is an ultrafilter we need to prove that conditions (ii) and (iii) of Definition 12 hold—condition (i) is implied by groundedness. For condition (ii), i.e., closure under intersection, we can use transitivity, which is an implicative property by Fact 5. Let  $e_1 = (x, y)$ ,  $e_2 = (y, z)$ , and  $e_3 = (x, z)$ . Consider the profile depicted in the left part of Figure 2, in which exactly the individuals in  $C_1$  accept  $(x, y)$ , exactly those in  $C_2$  accept  $e_2$ , and exactly those in  $C_1 \cap C_2$  accept  $(x, z)$ . As both  $C_1$  and  $C_2$  are winning coalitions, we obtain that both  $(x, y)$  and  $(y, z)$  need to be collectively accepted. We can now conclude, since  $F$  is CR w.r.t. transitivity, that the edge  $(x, z)$  should also be accepted, and hence also  $C_1 \cap C_2 \in \mathcal{W}$ . To show that  $\mathcal{W}$  is maximal, i.e., that condition (iii) holds, we can use completeness, which is a disjunctive property by Fact 6. Let  $e_1 = (x, y)$  and  $e_2 = (y, x)$ , and consider the profile in the right part of Figure 2, in which exactly the individuals in  $C$  accept the edge  $(x, y)$  and exactly those in  $\mathcal{N} \setminus C$  accept  $(y, x)$ . As  $F$  is CR w.r.t. completeness, one of the two edges needs to be accepted, showing that either  $C \in \mathcal{W}$  or  $\mathcal{N} \setminus C \in \mathcal{W}$ .

We are now ready to state and prove our main result:

**Theorem 8.** For  $|V| \geq 3$ , there exists no NR-nondictatorial, unanimous, grounded, and IIE aggregator that is CR w.r.t. any graph property that is contagious, implicative, and disjunctive.

*Proof.* Immediate from Proposition 7, after an initial application of Lemma 4 to obtain NR-neutrality.  $\square$

Implicativeness and disjunctiveness are much less demanding properties than contagiousness. Thus, beyond Theorem 8, Proposition 7 is of some interest in its own right, as it applies to an even wider range of aggregation problems—provided we are willing to accept neutrality as an *a priori* requirement rather than a derived property.

### 3.6 Applications

Theorem 8 applies to many different classes of graphs. Let us briefly discuss some examples.

Arrow's Theorem, as stated in Section 3.1, is an immediate consequence of Theorem 8: First, recall that transitivity is both contagious and implicative, and completeness is disjunctive. Second, the additional requirement of being CR w.r.t. reflexivity does not affect the logical strength of the theorem, as we know from Proposition 1 that every unanimous aggregator is CR w.r.t. reflexivity. Finally, requiring NR-nondictatoriality rather than just nondictatoriality is not a restriction, as the two properties coincide when all input graphs are reflexive. Observe that by the same kind of argument, and using Proposition 2, we also immediately obtain a variant of Arrow's Theorem for strict (i.e., irreflexive) linear orders.

By exchanging transitivity and completeness for other graph properties that together cover contagiousness, implicativeness, and disjunctiveness (see Table 1), we can generate any number of variations of Arrow's Theorem (e.g., there exists no NR-nondictatorial Arrovian aggregator that is CR w.r.t. the right Euclidean property and seriality). One property, connectedness, stands out for having all three meta-properties. Thus, there exists no NR-nondictatorial Arrovian aggregator that is CR w.r.t. connectedness.

Theorem 8 also implies a recent result by Pini et al. [19], a variant of Arrow's Theorem for preorders, i.e., for preferences that need not be complete. Besides transitivity, they require the collective preference order to have one element that is weakly preferred (or dispreferred) to all other elements. This is a clear example for a (very minimalist) disjunctive property, i.e., Theorem 8 applies directly.

In work on belief merging, Maynard-Zhang and Lehman [18] suggest an approach to circumvent Arrow's Theorem by (a) replacing completeness by negative transitivity (which they call *modularity*) and (b) weakening the independence axiom. In the discussion of their result, they stress the significance of both of these changes. However, our analysis clearly shows that replacing completeness by negative transitivity alone has no effect on Arrow's impossibility, as negative transitivity is also a disjunctive property. Hence, the crucial source for the possibility result of Maynard-Zhang and Lehman must be their modification of the independence axiom.

## 4 CONCLUSION

We have argued that graph aggregation is an important problem with several potential applications and we have introduced a simple formal framework to study this problem. We have then proved a general impossibility theorem that generalises Arrow's Theorem for preference aggregation to the case of graph aggregation and that can be instantiated for many different combinations of properties of graphs—rather than just transitivity and completeness (the properties featuring in Arrow's Theorem). In doing so, we have refined the ultrafilter method, clearly relating how certain features of graph properties correspond to, first, the neutrality axiom and, second, the two ultrafilter properties of closure under intersection and maximality.

In the long run, we believe that such fundamental results have the potential to contribute to the solution of practical problems arising in some of the application domains mentioned in the introduction, e.g., argumentation studies, as well as others, e.g., social networks or multiagent systems. But, as we have been able to demonstrate, already now our results can be applied to better understand phenomena in certain areas, including in particular recent work in AI.

An interesting direction for future work that we have begun to

explore is to study collective rationality w.r.t. graph properties expressed in terms of formulas of *modal logic* [4]. The semantics of modal logic suggests two novel notions of collective rationality. While the standard notion of collective rationality corresponds to the *frame-validity* of formulas being preserved under aggregation, we may also consider the cases of preservation of *truth in a model* or truth at a specific *world* in a model.

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