Effort-Based Fairness for Participatory Budgeting

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Abstract

We introduce a new family of normative principles for fairness in participatory budgeting. These principles are based on the fundamental idea that budget allocations should be fair in terms of the effort or resources invested into meeting the wishes of each voter. This is in contrast to earlier proposals that are based on specific assumptions about the satisfaction of voters with a given budget allocation. We analyse these new principles in axiomatic, algorithmic, and experimental terms.

1 Introduction

Budgeting, i.e., the allocation of money or other sparse resources to specific projects, is one of the key decisions any political body or organisation has to take. Participatory budgeting (PB) was developed in the 1990s as a method for making such decisions in more democratic a way, by putting the selection of projects to be funded to a vote [Cabannes, 2004; Shah, 2007]. It has found rapid adoption worldwide, in particular at the municipal level [Wampler *et al.*, 2021].

The most common form of eliciting the views of voters is to ask which projects they approve of [Goel *et al.*, 2019], but the question of how one should then select the projects to be funded is not yet settled. In this paper, we advocate the use of measures based on the *effort* or *resources* spent on behalf of the voters. Specifically, we focus on a measure of effort called the *share*, which has recently been introduced by Lackner *et al.* [2021]. It is computed by equally dividing the cost of each funded project amongst the supporters of that project.

So why is that an appropriate approach to taking a budgeting decision? Suppose 40% of the citizens of a city support funding more cycling infrastructure, while 60% prefer more car infrastructure. Then under the kind of voting rule usually employed in practice, where the projects with the most support get selected, only car-centric projects would get funded. This clearly is not desirable. Therefore, in recent years, researchers have started to look for voting rules that make the PB process fairer by producing *proportional* outcomes [Fain *et al.*, 2016; Aziz *et al.*, 2018b; Peters *et al.*, 2021; Aziz and Lee, 2021; Los *et al.*, 2022]. Such a rule would fund a mixture of cycling and car infrastructure projects, so as to accurately reflect the proportion of supporters for both types of projects.

But how should one define proportionality? So far, the literature has focused on generalising from approval-based multiwinner voting, where we often aim for a proportional distribution of *satisfaction* amongst voters, assuming that the election of each approved candidate provides the same satisfaction to all of her supporters [Faliszewski *et al.*, 2017; Lackner and Skowron, 2022]. But lifting this assumption to the richer framework of PB is questionable, as projects vary in cost.

So, given her approval ballot, how should one infer a voter's satisfaction for a set of selected projects?[1] Most researchers assume that the satisfaction of a voter is either equal for all approved projects [Peters *et al.*, 2021; Talmon and Faliszewski, 2019; Los *et al.*, 2022] or proportional to the cost of a project [Fain *et al.*, 2016; Aziz *et al.*, 2018b; Lackner *et al.*, 2021; Sreedurga *et al.*, 2022]. Both assumptions are problematic. Regarding the former, for example, in the 2021 Paris participatory budgeting process a project costing 3 million euros ("Act towards a cleaner city") and a project costing 10,000 euros ("Enabling houseless people to charge their phones") were funded [City of Paris, 2022]. It seems unlikely that both projects offer the same utility to their supporters. This is particularly striking when we are concerned with fairness: Funding a new highway for cars for millions of dollars as well as a new bicycle stand for a few thousand dollars hardly seems like a fair outcome for the supporters of cycling infrastructure, even though the same number of projects were funded for both groups At the same time, full proportionality of utility and cost seems implausible because the cost effectiveness of different projects can vary widely. Consider, for example, a scenario where two parks of equal size could be built in different neighbourhoods. Now, it might be more expensive to build the park in one neighbourhood due to higher property prices. In that case, there is no reason to assume that the more expensive park offers more utility to its supporters. Crucially, these two examples show that a higher cost sometimes implies a higher utility, while sometimes it does not. That makes it hard to imagine an alternative way of estimating utilities in a principled way that works for both examples.

To circumvent these difficulties we propose to develop fair-

[1]In principle, there is also another possibility, namely to directly ask the voters for their satisfaction (or utility). But this imposes a large cognitive burden on voters, and it is debatable whether it is even possible to elicit utilities in a way that allows for interpersonal comparisons [Hicks and Allen, 1934; Blackorby *et al.*, 2002].

ness measures which are not based on *equality of welfare* but aim for *equality of resources*, an idea first proposed by Ronald Dworkin [Dworkin, 1981a,b]. In other words, we do not aim for a fair distribution of satisfaction, but instead we strive to put the same effort into satisfying each voter. The advantage is that effort, i.e., invested budget, is a quantity we can measure objectively. We formalise effort with the notion of *share*, which is the total budget invested in a voter assuming that costs of projects are divided equally among approving voters. Ideally, we want to find a budget allocation where each voter has the same share. This is in contrast to the satisfaction-based notions of fairness considered in multiwinner voting, where it is usually assumed that voters that are part of a larger cohesive group deserve higher satisfaction [Aziz *et al.*, 2017]. Finally, fairness notions based on share provide an explanation on how each voter's part of the budget was spent. In contrast to the related property of *priceability* [Peters and Skowron, 2020], here all supporters of a project "contribute" the same amount.

In this paper, we investigate the viability of the share as a basis of fairness notions in PB in three different ways. First, we propose several axioms that formalise what it means for an outcome to be fair in terms of share. We observe that it is not always possible to guarantee each voter their *fair share*, which we define as the budget divided by the numbers of agents. For this reason, we consider several relaxations, such as the *justified share*, where we spend on a voter only the effort they deserve by virtue of being part of a coherent group. Moreover, we identify a voting rule, a version of Rule X [Peters and Skowron, 2020], that satisfies all axioms that are known to be satisfiable by a tractable voting rule. Secondly, we investigate the *price of fairness* for our share-based fairness axioms, by comparing the maximally achievable social welfare to the maximal welfare achieved by a fair outcome. In this context, we focus on the special case where all projects have the same cost, so we can use the number of approved projects in the outcome as a good proxy for the welfare of a voter. Finally, using data from a large number of real-life PB exercises [Stolicki *et al.*, 2020], we analyse to what extent it is possible to provide voters with their fair share in practice.

Roadmap. After presenting the model in Section 2, we investigate the fair share in Section 3 and the justified share in Section 4. We discuss how the different concepts relate to one another in Section 5, analyse their price of fairness in Section 6, and report on an experimental study in Section 7. Full proofs of all results are available in Maly *et al.* [2022].

2 The Model

A PB problem is described by an *instance* $I = \langle \mathcal{P}, c, b \rangle$ where $\mathcal{P}$ is the set of available *projects*, $c : \mathcal{P} \to \mathbb{N}$ is the *cost function*—mapping any given project $p \in \mathcal{P}$ to its cost $c(p) \in \mathbb{N}$—and $b \in \mathbb{N}$ is the *budget limit*. We write $c(P)$ instead of $\sum_{p \in P} c(p)$ for sets of projects $P \subseteq \mathcal{P}$. If $c(p) = 1$ for all $p \in \mathcal{P}$, then we say that I belongs to the *unit-cost setting*.

Let $\mathcal{N} = \{1, \ldots, n\}$ be a set of *agents*. When facing a PB instance, each agent is asked to submit an (not necessarily feasible) *approval ballot* representing the subset of projects they approve of. The approval ballot of agent $i \in \mathcal{N}$ is denoted by $A_i \subseteq \mathcal{P}$, and the resulting vector $\boldsymbol{A} = (A_1, \ldots, A_n)$ of approval ballots is called a *profile*. We assume w.l.o.g. that every project is approved by at least one agent.

Given an instance $I = \langle \mathcal{P}, c, b \rangle$, we need to select a set of projects $\pi \subseteq \mathcal{P}$ to implement. Such a *budget allocation* π has to be *feasible*, i.e., we require $c(\pi) \leq b$. Let $\mathcal{A}(I) = \{\pi \subseteq \mathcal{P} \mid c(\pi) \leq b\}$ be the set of feasible budget allocations for I.

Computing allocations is done by means of (resolute) *PB rules*. Such a rule F is a function that maps an instance I and a profile $\boldsymbol{A}$ over I to a single feasible budget allocation $F(I, \boldsymbol{A}) \in \mathcal{A}(I)$. We assume that ties are broken in a fixed and consistent manner (e.g. lexicographically).

We are going to propose several fairness properties we might want a rule to satisfy. All of these properties will be defined in terms of the fundamental notion of an agent's *share*.

Definition 1 (Share)**.** *Given an instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$, the share of an agent i for a subset of projects $P \subseteq \mathcal{P}$ is defined as follows:*

$$share(I, \boldsymbol{A}, P, i) = \sum_{p \in P \cap A_i} \frac{c(p)}{|\{A \in \boldsymbol{A} \mid p \in A\}|}.$$

When clear from context, we will omit the arguments of I and $\boldsymbol{A}$. We interpret an agent's share as the effort spent by the decision maker on satisfying the needs of that agent. It is important to note that the share cannot be captured via independent cardinal utility functions as the share of an agent depends on the ballots submitted by the other agents.

In the sequel, we shall extend the definition of every property of budget allocations we define to a property of rules in the natural manner: F is said to satisfy property $\mathcal{F}$ defined for budget allocations if, for every I and $\boldsymbol{A}$, $F(I, \boldsymbol{A})$ satisfies $\mathcal{F}$.

3 Fair Share

The first fairness property we study is based on the idea that every voter deserves $1/n$ of the budget—a fundamental idea familiar, for instance, from the classical fair division ("cake cutting") literature [Robertson and Webb, 1998]. So a perfect allocation would give every voter a share of b/n (unless they do not approve of enough projects for this to be possible).

Definition 2 (Fair Share)**.** *Given an instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy fair share (FS) if for every agent we have:*

$$share(\pi, i) \geq \min\{b/n, share(A_i, i)\}.$$

It is easy to see that for some instances, no budget allocation would provide a fair share, and thus no rule can possibly satisfy FS. Take for instance two projects of cost 1, a budget limit of 1 and two agents each approving of a different project. Then, both agents deserve a share of $\min\{1/2, 1\} = 1/2$. However, whichever project is selected (at most one can be selected), the share of one agent would be 0.

Even more, we show that no polynomial-time computable rule can return an FS allocation whenever one exists. Indeed, checking whether an FS allocation exists is NP-complete.

Proposition 1. *Given an instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$, checking whether there exists a feasible budget allocation $\pi \in \mathcal{A}(I)$ that satisfies fair share is an* NP*-complete problem, even in the unit-cost setting.*

Because of these shortcomings of FS, we introduce two relaxations that are inspired by Extended Justified Representation (EJR) up to one project [Peters *et al.*, 2021] and Local-Proportional Justified Representation (PJR) [Aziz *et al.*, 2018b]. Let us first investigate FS up to one project.

Definition 3 (FS up to one project)**.** *Given an instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy fair share up to one project (FS-1) if, for every agent i, there is a project $p \in \mathcal{P}$ such that:*

$$share(\pi \cup \{p\}, i) \geq \min \{{}^{b}\!/_{n}, share(A_i, i)\}.$$

FS-1 requires that every agent is only one project away from their fair share. Unfortunately, FS-1 is not always satisfiable.

Proposition 2. *There exist instances I such that no budget allocation $\pi \in \mathcal{A}(I)$ provides FS-1.*

Proof. Consider an instance with three projects of cost 3 and a budget limit $b = 5$. Consider three agents, with approval ballots $\{p_1, p_2\}$, $\{p_1, p_3\}$ and $\{p_2, p_3\}$ respectively.

Here the fair share of each agent is $5/3 \approx 1.67$. But as a single project only yields a share of 1.5 to an agent who approves of it, for any agent to reach their fair share threshold, two projects must be selected. However, a feasible budget allocation can select at most one project, meaning that for one agent none of the projects they approve of will be selected. So even if we were to select an extra project, that agent would still not obtain their fair share. □

Alternatively, we can require that every project that is not part of the winning budget allocation should give some voter at least their fair share when that project is added.[2]

Definition 4 (Local-FS)**.** *Given an instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy local fair share (Local-FS) if there is no project $p \in \mathcal{P} \setminus \pi$ such that, for all agents $i \in \mathcal{N}$ with $p \in A_i$, we have:*

$$share(\pi \cup \{p\}, i) < \min\{{}^{b}\!/_{n}, share(A_i, i)\}.$$

Intuitively, if there exists a project p that could be added to the budget allocation π without any supporter of p receiving at least their fair share, then every supporter of p receives strictly less than their fair share and one of the following holds:

- p can be selected without exceeding the budget limit b;
- some voter $i^\star$ receives more than their fair share.

In the first case, it is clear that p should be selected and then π is deemed unfair. In the second case, it might be considered fairer to exchange one project supported by $i^\star$ with p. In this sense, the property can be seen as an "upper quota" property, as we have to add projects such that no voter receives more than their fair share as long as possible.

In contrast to FS-1, we can always find an allocation that satisfies Local-FS. Indeed, an adaption of rule *Rule X* [Peters *et al.*, 2021][3] satisfies Local-FS. Our definition closely resembles the definition of Rule X for PB with additive utilities [Peters *et al.*, 2021]. We adapt it by plugging the share as the utility function in the definition of Peters *et al.* [2021]. Note that this rule can be executed in polynomial time.

Definition 5 (Effort-based Rule X, Rule X^e)**.** *Given an instance I and a profile $\boldsymbol{A}$, Rule X^e constructs a budget allocation π, initially empty, iteratively as follows. A load $\ell_i : 2^{\mathcal{P}} \to \mathbb{R}_{\geq 0}$, is associated with every agent $i \in \mathcal{N}$, initialised as $\ell_i(\emptyset) = 0$ for all $i \in \mathcal{N}$. Given π and a scalar $\alpha \geq 0$, the contribution of agent $i \in \mathcal{N}$ for project $p \in \mathcal{P} \setminus \pi$ is defined by:*

$$\gamma_i(\pi, \alpha, p) = \min\left({}^{b}\!/_{n} - \ell_i(\pi), \alpha \cdot share(\{p\}, i)\right).$$

Given a budget allocation π, a project $p \in \mathcal{P} \setminus \pi$ is said to be α-affordable, for $\alpha \geq 0$, if $\sum_{i \in \mathcal{N}} \gamma_i(\pi, \alpha, p) \cdot \mathbb{1}_{p \in A_i} = c(p)$.

At a given round with current budget allocation π, if no project is α-affordable for any α, Rule X^e terminates. Otherwise, it selects a project $p \in \mathcal{P}$ that is α^*-affordable where α^* is the smallest α such that one project is α-affordable (π is updated to $\pi \cup \{p\}$). The agents' loads are then updated: If $p \notin A_i$, then $\ell_i(\pi \cup \{p\}) = \ell_i(\pi)$, and otherwise $\ell_i(\pi \cup \{p\}) = \ell_i(\pi) + \gamma_i(\pi, \alpha, p)$.

Theorem 3. *Rule X^e satisfies Local-FS.*

Proof. Given a budget allocation π and a scalar $\alpha > 0$, we say that agent $i \in \mathcal{N}$ contributes in full to project $p \in A_i$ if we have: $\gamma_i(\pi, \alpha, p) = \alpha \cdot share(\{p\}, i)$.

During a run of Rule X^e, all the supporters of a project $p \in \mathcal{P}$ contribute in full to p if and only if p is 1-affordable. In this case, for all supporters i of p, we have $\ell_i(\{p\}) = share(\{p\}, i)$. Moreover, if a project p is α-affordable but at least one voter cannot contribute in full to p, then $\alpha > 1$. Rule X^e only terminates when no project is α-affordable for any α. Therefore, there is a round where no project p is 1-affordable. Let k be the first such round and let π_k be the budget allocation before round k. It follows that every project in π_k was 1-affordable and hence $\ell_i(\pi_k) = share(\pi_k, i)$ for all $i \in \mathcal{N}$. As no project p is 1-affordable in round k, for no projects in $\mathcal{P} \setminus \pi_k$ can all the supporters contribute in full to. Thus, for every $p \in \mathcal{P} \setminus \pi_k$, there is a voter $i \in \mathcal{N}$ such that ${}^{b}\!/_{n} - \ell_i(\pi_k) < share(\{p\}, i)$. Using the fact that $\ell_i(\pi_k) = share(\pi_k, i)$ and the additivity of share, it follows that $(\pi_k \cup \{p\}, i) > {}^{b}\!/_{n}$. So π_k satisfies Local-FS. As Rule X^e returns an allocation π with $\pi_k \subseteq \pi$, it satisfies Local-FS. □

Remark 1. *The proof of Theorem 3 shows actually a slightly stronger statement: there is no project $p \in \mathcal{P} \setminus \pi$ such that for all agents $i \in \mathcal{N}$ with $p \in A_i$ we have $share(\pi \cup \{p\}, i) \leq {}^{b}\!/_{n}$. In other words, any project added to π gives at least one voter* more *share than their fair share (${}^{b}\!/_{n}$).*

4 Justified Share

Local-FS and FS-1 require the outcome to be, in some sense, close to being FS. Another idea for weakening FS is to spend on a voter only the effort they can claim to deserve by virtue of being part of a cohesive group. This idea is inspired by the well-known axioms of justified representation that are extensively studied in approval-based committee elections [Aziz

[2]We stress that this formulation of Local-FS relies on our assumption that every project p is approved by at least one agent.

[3]Rule X has recently been renamed to *method of equal share* [Peters *et al.*, 2021], but this new name is not related to our definition of share.

et al., 2017, 2018a; Peters and Skowron, 2020; Lackner and Skowron, 2022].

Before exploring this idea further, let us define what we mean by cohesive groups.

Definition 6 (P-cohesive groups)**.** *Given an instance $I = \langle \mathcal{P}, c, b\rangle$ and a profile $\boldsymbol{A}$, for a set of projects $P \subseteq \mathcal{P}$ we say that a non-empty group of agents $N \subseteq \mathcal{N}$ is P-cohesive, if $P \subseteq \bigcap_{i\in N} A_i$ and $\frac{|N|}{n} \geq \frac{c(P)}{b}$.*

That is, a group N is cohesive relative to a set P of projects if, first, everyone in N approves of all the projects in P and, second, N is large enough—relative to the size n of the society and the budget b available—so as to "deserve" the effort for funding the projects in P.

In the unit-cost setting, one of the strongest proportionality property that is known to always be satisfiable by a polynomial time computable rule is called Extended Justified Representation (EJR) [Aziz *et al.*, 2018a; Peters and Skowron, 2020]. Peters *et al.* [2021] generalised EJR to the setting of PB with additive utilities. This generalisation will be our blue-print for modifying EJR to deal with share. Ideally, we would want to satisfy the following property:

Definition 7 (Strong Extended Justified Share)**.** *Given an instance $I = \langle \mathcal{P}, c, b\rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy strong extended justified share (Strong-EJS) if for all $P \subseteq \mathcal{P}$ and all P-cohesive groups N, we have $share(\pi, i) \geq share(P, i)$ for all $i \in N$.*

The idea behind Strong-EJS is the following: since every P-cohesive group S controls enough budget to fund P, every agent in S deserves to enjoy at least as much share as what she would have gotten if P had been the outcome. Intuitively, this is very similar to Strong-EJR, a property which is known to not be always satisfiable [Aziz *et al.*, 2017]. The same holds for Strong-EJS: there exist instances for which no budget allocation can satisfy this axiom.

Example 1. *Consider the following instance and profile with three projects p_1, p_2 and p_3 of cost 1, a budget limit $b = 2$, and four agents $1, \dots, 4$ such that 1 approves project p_1, 2 approves project p_1 and p_2, 3 approves p_1 and p_3 and 4 approves p_2 and p_3. Note that $\{1, 2, 3\}$ is $\{p_1\}$-cohesive, $\{2, 4\}$ is $\{p_2\}$-cohesive and $\{3, 4\}$ is $\{p_3\}$-cohesive. Hence, to satisfy Strong-EJS, one needs to select all three projects which is not possible within the budget limit.*

One can observe that in the previous example, it is not even possible to guarantee each P-cohesive group the same *average* share as they receive from P. We thus weaken Strong-EJS and introduce (simple) EJS.

Definition 8 (Extended Justified Share)**.** *Given an instance $I = \langle \mathcal{P}, c, b\rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy extended justified share (EJS), if for all $P \subseteq \mathcal{P}$ and all P-cohesive groups N, there is an agent $i \in N$ such that $share(\pi, i) \geq share(P, i)$.*

The difference between Strong-EJS and EJS is the switch from a universal to an existential quantifier: for the former, we impose a lower bound on the share of *every* agent in a cohesive group, while for the latter this lower bound only applies to one agent of each cohesive group. Therefore, in Example 1 both $\{p_1, p_3\}$ and $\{p_2, p_3\}$ satisfy EJS, as either 3 or 4 satisfy the share requirement for every cohesive group.

We observe that EJR and EJS, while similar in spirit, do not coincide, not even in the unit-cost case.

Example 2. *Consider the following instances with four voters and six projects with unit cost and $b = 4$, where the approvals are as follows: $A_1 = \{p_1, p_2, p_3\}$, $A_2 = \{p_1, p_2, p_4\}$, $A_3 = A_4 = \{p_4, p_5, p_6\}$. It is now easy to check that $\{p_3, p_4, p_5, p_6\}$ satisfies EJS but not EJR, while $\{p_1, p_4, p_5, p_6\}$ satisfies EJR but not EJS.*

The first question that presents itself is whether EJS is always achievable. This is indeed the case. To see this, one just needs to adapt the well known greedy procedure for satisfying EJR, which was first introduced by Aziz *et al.* [2017] and extended to PB by Peters *et al.* [2021], to the share setting.

Proposition 4. *For every instance $I = \langle \mathcal{P}, c, b\rangle$ and every profile $\boldsymbol{A}$, there exists a budget allocation $\pi \in \mathcal{A}(I)$ that satisfied EJS.*

However, the greedy approach in general needs exponential time. This turns out to be unavoidable, unless $\texttt{P} = \texttt{NP}$, as can be shown by a standard reduction from SUBSET SUM.

Theorem 5. *There is no polynomial-time algorithm that, given an instance I and a profile $\boldsymbol{A}$ as input, always computes a budget allocation satisfying EJS, unless $\texttt{P} = \texttt{NP}$.*

On the other hand, we observe that the greedy approach generally runs in FPT-time, when parameterized by the number of projects [Aziz *et al.*, 2017]. This is also the case in the share setting.

Proposition 6. *For every instance $I = \langle \mathcal{P}, c, b\rangle$ and every profile $\boldsymbol{A}$, we can compute a budget allocation $\pi \in \mathcal{A}(I)$ that satisfied EJS in time $\mathcal{O}(n \cdot 2^{|\mathcal{P}|})$.*

We have seen that EJS can always be satisfied. However, this is not entirely satisfactory, given that no tractable rule can satisfy it. Unfortunately, in many applications of PB, the use of intractable rules is not practical due to the large instance sizes. Therefore, we try to find fairness notions that can be satisfied in polynomial time by relaxing EJS.

Similar to *EJR up to one project (EJR-1)* proposed by Peters *et al.* [2021], we can define EJS up to one project, which states that at least one agent in every cohesive group is at most one project away from being satisfied.[4]

Definition 9 (EJS-1)**.** *Given an instance $I = \langle \mathcal{P}, c, b\rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy extended justified share up to one project (EJS-1) if for all $P \subseteq \mathcal{P}$ and all P-cohesive groups N there is an agent $i \in N$ for which there exists a project $p \in \mathcal{P}$ such that $share(\pi \cup \{p\}, i) \geq share(P, i)$.*

It is possible to adapt the proof of Peters *et al.* [2021] that Rule X satisfies EJR up to one project to our setting to prove that Rule X^e satisfies EJS up to one project.

Proposition 7. *Rule X^e satisfies EJS-1.*

[4] We note that in Definition 9 we require that $share(\pi \cup \{p\}, i) \geq share(P, i)$ instead of a strict inequality as used in the definition of EJR-1. Our rationale is that adding one project guarantees to satisfy the EJS condition (but not more than that).

In particular, this implies toghether with Example 2, that Rule X and Rule X^e are indeed different rules. Finally, note that we can define a local variant of EJS, based on a similar motivation as Local-FS.

Definition 10 (Local-EJS). *Given an instance $I = \langle \mathcal{P}, c, b\rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy local extended justified share (Local-EJS), if there is no P-cohesive group N, where $P \subseteq \mathcal{P}$, for which there exists a project $p \in P \setminus \pi$ for which it holds for all agents $i \in N$ that $share(\pi \cup \{p\}, i) < share(P, i)$.*

The idea behind Local-EJS is that there is no P-cohesive group N that can claim that they could "afford" another project p without a single voter in N receiving more share than they deserve due to their P-cohesiveness. In this sense, any allocation that satisfies Local-EJS is a local optimum for any P-cohesive group. Now, in our setting we observe that Local-EJS is equivalent to a notion that could be called "EJS up to any project".

Proposition 8. *Let $I = \langle \mathcal{P}, c, b\rangle$ be an instance and $\boldsymbol{A}$ a profile. An allocation π satisfies Local-EJS if and only if for every $P \subseteq \mathcal{P}$ and P-cohesive group N there exists an agent i such that for all projects $p \in P \setminus \pi$ we have $share(\pi \cup \{p\}, i) \geq share(P, i)$.*

Proof. It is clear that the statement above implies Local-EJS. Now, let π be an allocation that satisfies Local-EJS, let $P \subseteq \mathcal{P}$ be a set of projects and N a P-cohesive group. Let $i^* \in N$ be an agent with maximal share from π in N. Consider $p \in P \setminus \pi$. By Local-EJS there is an agent i_p such that $share(\pi \cup \{p\}, i_p) > share(P, i_p)$. By the choice of i^* we have $share(\pi, i^*) \geq share(\pi, i_p)$. By the definition of share, it follows that $share(\pi \cup \{p\}, i^*) > share(P, i^*)$. □

From this equivalence, it is easy to see that Local-EJS implies EJS-1. Unfortunately, Rule X^e fails Local-EJS, as the following example shows.

Example 3. *Consider an instance with five projects, a budget limit $b = 20$, and four agents where the costs are as follows:*

$$c(p_1) = 8, c(p_2) = 5, c(p_3) = c(p_4) = 2, c(p_5).$$

Moreover, voters 1 and 2 approve projects p_1, p_2, p_3 and p_4 and voters 3 and 4 prove p_3, p_4 and p_5.

With a suitable tie-breaking, Rule X^e can return the budget allocation $\pi = \{p_2, p_3, p_5\}$. Note that voters 1 and 2 are $\{p_1, p_4\}$-cohesive and would thus deserve to enjoy a share of 4.5. However, if we add p_4 to π, voters 1 and 2 would only have a share of 3.5, showing that π fails Local-EJS.

Whether Local-EJS can always be satisfied in polynomial time remains an important open question.

Finally, we observe a crucial difference between EJR and EJS: Rule X^e does not satisfy EJS in the unit cost setting!

Example 4. *Assume that there are two voters 1 and 2, and three projects p_1, p_2 and p_3, all of cost 1. The budget limit is $b = 2$. Voter 1 approves of p_1 and p_3 and voter 2 of p_2 and p_3. Then voter 1 is $\{p_1\}$-cohesive and hence deserves a share of 1, the same applies to voter 2 and $\{p_2\}$. Nevertheless, with a suitable tie-breaking, Rule X^e would first select p_3. In that case. neither $\{p_1, p_2\}$, nor $\{p_2, p_3\}$ would satisfy EJS, as at least one voter will have only share $1/2$.*

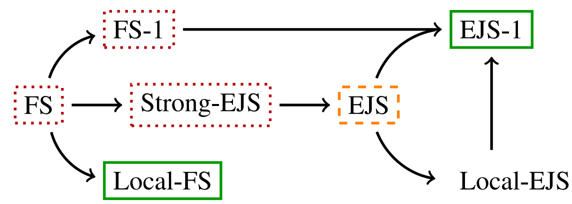

Figure 1: Taxonomy of criteria. An arrow from one criterion to another indicates that any budget allocation satisfying the former also satisfies the latter. For criteria boxed in green solid lines, there are polynomial-time algorithms to compute allocations satisfying them. For the criterion boxed in orange dashed lines, no such algorithm exists (unless P = NP). Criteria boxed in red dotted lines are not always satisfiable. The status is unknown for unboxed criteria.

However, it does satisfy Local-EJS in the unit-cost setting.

Theorem 9. *Rule X^e satisfies Local-EJS in the unit-cost case.*

5 A Taxonomy of the Fairness Criteria

We now investigate links between the different fairness notions we introduced and draw a taxonomy.

Proposition 10. *Given an instance I and a profile $\boldsymbol{A}$, every budget allocation $\pi \in \mathcal{A}(I)$ that satisfies FS also satisfies Strong-EJS and every allocation that satisfies FS-1 also satisfies EJS-1.*

Figure 1 summarises the implications between the different properties. It can be shown that no other implications hold. For example EJS-1 does not imply Local-EJS, even in the unit-cost setting.

Example 5. *Consider an instance with two voters, 1 and 2, and six projects p_1, ..., p_6 all of cost 1. Voter 1 approves of p_1, p_2, p_3, p_4, and p_5; and voter 2 approves of p_4, p_5 and p_6. The budget limit is $b = 4$. It can be checked that $\pi = \{p_1, p_2, p_3, p_4\}$ satisfies EJS-1 but not Local-EJS.*

Moreover, the following example shows that Local-FS does not even imply EJS-1.

Example 6. *Consider an instance with three projects, a budget limit of $b = 6$, and two agents where $c(p_1) = 6$ and $c(p_2) = c(p_3)$. Moreover, 1 approves all projects, 2 only p_2 and p_3. Allocation $\pi = \{p_1\}$ satisfies Local-FS: for both p_2 and p_3, if we were to add them to π, agent 1 would have a fair share. However, it does not satisfy EJS-1: $\{2\}$ is a $\{p_2, p_3\}$-cohesive group but neither project is selected.*

Additional proofs can be found in Maly *et al.* [2022].

6 Fairness versus Social Welfare

It is well-known that enforcing fairness criteria often decreases utilitarian social welfare, i.e., the sum of voters' utilities. In the context of multi-winner voting and participatory budgeting, this phenomenon has been studied by Lackner and Skowron [2020], Elkind *et al.* [2022], and Fairstein *et al.* [2022]. For our purposes, the work of Elkind *et al.* [2022] is particularly relevant as they study the "price" of representation axioms, measured as the relative loss of utilitarian social welfare. Our goal here is to quantify the price of asserting share-based fairness concepts.

Let us first discuss a conceptual difficulty. To be able to define social welfare in participatory budgeting, we have to rely on an understanding of what satisfies voters. However, avoiding assumptions about voters' satisfaction was exactly the motivation of our investigation of share. Thus, we do not consider the general PB setting here, but instead restrict our analysis to the unit-cost setting[5]. In this setting, it is a standard assumption to measure a voter's satisfaction as the number of approved and funded projects (all of which have the same cost). That is, we define the social welfare of a budget allocation π given $\boldsymbol{A}$ as $\mathrm{sw}(\boldsymbol{A}, \pi) = \sum_{i \in \mathcal{N}} |A_i \cap \pi|$. Based on this simplification, we will explore that the link between fairness based on share and social welfare.

We first make a simple observation: EJS and fair share may induce a high cost with respect to social welfare as we can show it is not compatible with a unanimity axiom.

Definition 11 (Unanimity)**.** *Given an instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$, a budget allocation $\pi \in \mathcal{A}(I)$ is said to satisfy unanimity if for every two projects $p, p' \in \mathcal{P}$ such that $|\{A \in \boldsymbol{A} \mid p' \in A\}| < |\{A \in \boldsymbol{A} \mid p \in A\}| = n$, it is never the case that $p' \in \pi$ but $p \notin \pi$.*

Proposition 11. *For some instances I, no budget allocation $\pi \in \mathcal{A}(I)$ satisfying EJS also satisfies unanimity, even in the unit-cost setting. The same holds for fair share.*

We will now formalise the idea that share-based concepts induce a large cost on social welfare. For that we adopt the definition of the *social welfare price* by Elkind *et al.* [2022], which measures the worst case ratio between the social welfare of a budget allocation satisfying a property $\mathcal{F}$ and that of the budget allocation that maximises the social welfare. In the following, let c_1 denote the unit-cost cost function.

Definition 12 (Social Welfare Price)**.** *The social welfare price of a property $\mathcal{F}$ is defined as*

$$P^{\mathcal{F}}_{\mathrm{sw}}(b) = \sup_{\substack{I = \langle \mathcal{P}, c_1, b \rangle \\ \boldsymbol{A}}} \frac{\max_{\pi \in \mathcal{A}(I)} \mathrm{sw}(\boldsymbol{A}, \pi)}{\max_{\substack{\pi \in \mathcal{A}(I) \\ \pi \text{ sat. } \mathcal{F}}} \mathrm{sw}(\boldsymbol{A}, \pi)}.$$

Proposition 12. *It holds that $P^{EJS}_{\mathrm{sw}} \in o(b)$.*

Proof. Let us consider the following instance and profile with budget limit $b \in \mathbb{N}_{>0}$. We introduce $2b - 1$ projects $\mathcal{P} = \{p_1, \ldots, p_{2b-1}\}$, each costing 1. There are b agents. The profile $\boldsymbol{A}$ is such that the approval ballot of any agent $1 \leq i \leq b$ is $A_i = \{p_i\} \cup \{p_{b+1}, \ldots, p_{2b-1}\}$.

Now, for every agent $i \in \mathcal{N}$, it is the case that $\{i\}$ is $\{p_i\}$-cohesive. Hence all agents must have a share of 1 in the final outcome to satisfy EJS. The only way to achieve that is by selecting $\pi = \{p_1, \ldots, p_n\}$. This budget allocation provides a social welfare of b. The budget allocation $\pi^\star = \{p_b, \ldots, p_{2b-1}\}$ maximises the social welfare, reaching a social welfare of $b \cdot (2b - 1) + 1$. □

This result shows that EJS has a relatively high cost compared to proportionality axioms such as EJR, for which the price is only $\Theta(\sqrt{b})$ [Elkind *et al.*, 2022].

[5]The unit-cost setting is equivalent to approval-based committee voting, which is the setting used by Elkind *et al.* [2022].

7 Experimental Analysis

As we saw in Section 3, there exist PB instances for which it is impossible to give every agent their fair share. In this section we report on an experimental study aimed at understanding how serious this problem is for real-life PB instances. We analysed all instances with up to 65 projects found on Pabulib [Stolicki *et al.*, 2020], an online collection of real-world PB instances—except for instances that are trivial (either not a single or the set of all projects are affordable) or that raised parsing errors (unknown projects appearing in the ballots). Five instances have been additionally omitted due to very high compute time. A total of 350 PB instances are covered by our analysis.[6]

For a given budget allocation π, we can compute the *capped fair share ratio* of agent i with approval ballot A_i by dividing her actual share by her fair share (and capping that number at 1 in case she gets more than her fair share):

$$\min\{[\mathit{share}(\pi, i) \,/\, \min\{{}^{b}\!/_{n}, \mathit{share}(A_i, i)\}], 1\}.$$

For each PB instance we searched for an allocation that is as close as possible to the ideal of an FS allocation. We did so in terms of three different optimality criteria:

- the average capped fair share ratio of the agents;
- the minimum capped fair share ratio across all agents;
- the number of agents who got their fair share.

To better understand what might cause an instance to not admit a good solution, we also considered different ways to preprocess the instances by removing 'problematic' projects:

- Threshold: any project not approved by at least x% of agents is removed. We considered 1%, 5%, and 10%.
- Cohesiveness: any project the approvers of which do not control enough money (share of the budget they represent) to actually buy the project is removed.

Threshold preprocessing removes under 10% of projects for a threshold of 1%, around 10–20% for a threshold of 5%, and around 20–30% for a threshold of 10%. Cohesiveness preprocessing removes between 30% (for the largest instances) and 70% of projects (for the smallest instances).

Let us now turn to our results. The first observation is that it is almost always impossible to provide a non-zero share to all agents, so the minimum capped fair share almost always is 0. For the other two criteria, the results are shown in Figure 2.

We draw the following conclusions. Without preprocessing, we can provide agents on average between 45% (for small instances) and 75% (for larger instances) of their fair share, albeit with a lot of variation. Furthermore, we can typically guarantee a fair share to 50–60% of the agents. Preprocessing helps when using the cohesiveness condition, but not with the threshold condition. Importantly, we do not wish to advocate preprocessing as a method to make budget decisions in practice, but rather as a way of checking whether the failure to guarantee fair share is due to the specific structure of real-life PB instances and whether similar instances 'nearby'

[6]Our experiments are implemented using integer linear programs solved with Gurobi 9.5.1 ran on a Debian machine with 16 cores and 16GB RAM. Running the full set takes around 150 hours.

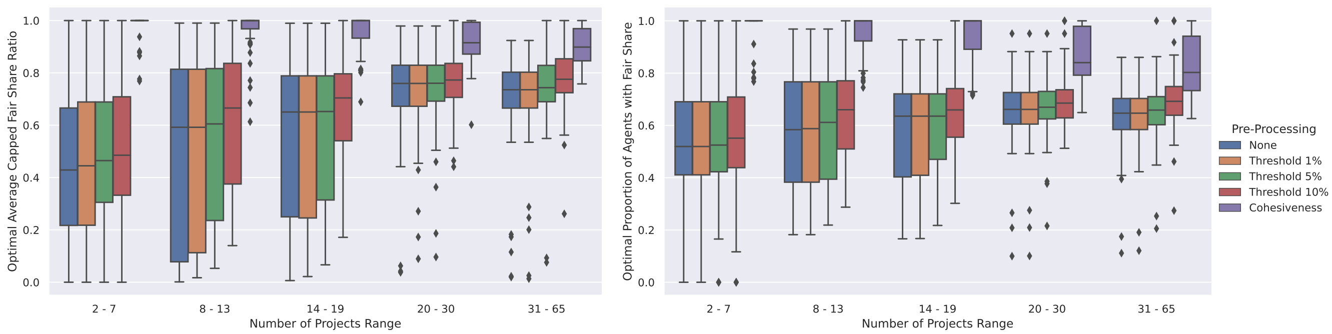

Figure 2: Optimal average capped fair share ratio (left) and optimal number of agents with fair share (right) for Pabulib instances. Each range (for a number of projects) shown on the x-axis contains roughly 70 instances.

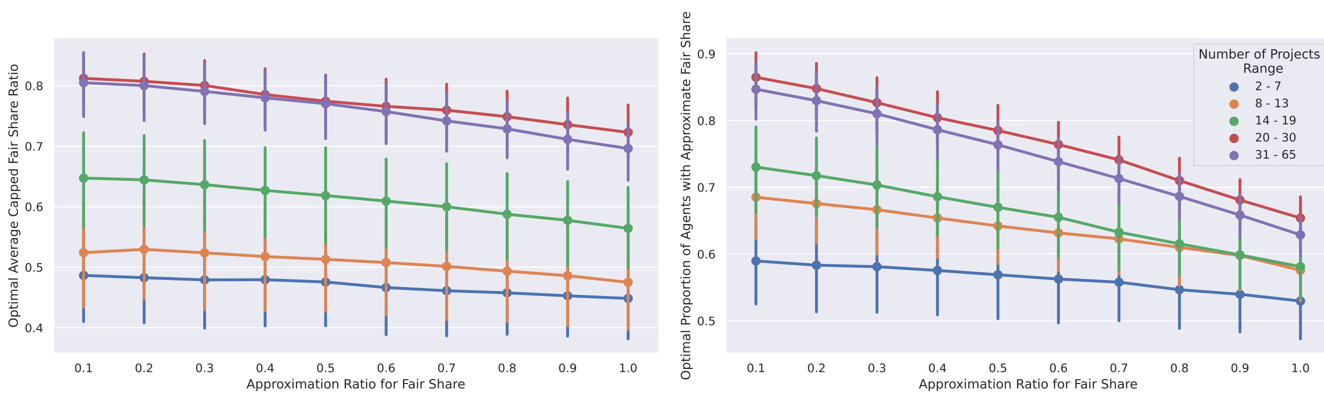

Figure 3: Impact of approximation factor α on average approx. fair share ratio (left) and number of agents with approx. fair share (right).

might be significantly better behaved. Our experimental findings suggest that this is not the case, and that guaranteeing fair share simply is very hard across a wide range of instances.

We then repeated the same set of experiments, but now instead of employing preprocessing we relaxed our objective by trying to approximate the fair share. Specifically, for a number of different given approximation ratios $\alpha \in (0, 1]$, in our formula for the capped fair share ratio we replaced the fair share by $\alpha \cdot \min\{{}^{b}/_{n}, share(A_i, i)\}$. For the second optimality criterion, where we track the worst-off agent, also this relaxation does not improve significantly on the overall picture: we never get beyond 2%, even for $\alpha = 0.1$. For the other two criteria, our results are shown in Figure 3. They further underline the general take-away message that providing fair share is hard. In particular, even for $\alpha = 0.1$, the average ratio does not improve by much. For the number of agents who can be given their approximate fair share, however, the picture is more encouraging. For example, if we are satisfied with providing agents with 50% of their fair share, then, for a large PB instance, we can expect to be able to do this for a large part of the population (for around 75%). A positive finding across our experiments is the insight that the situation tends to improve as we move to larger PB instances.

In general, we conclude that guaranteeing all or most voters a certain share is difficult in real-world PB elections. This strengthens the motivation of share-based notions relying on cohesive groups, which are always satisfiable.

8 Conclusion

We have proposed to use fairness criteria based on the share of a voter as a means of basing budget decisions in PB on the *effort* spent on satisfying the needs of voters rather than on the (assumed) *satisfaction* each voter might derive from an allocation. Our results suggest that these are interesting criteria that deserve further attention. On the one hand, the most demanding criteria are impossible to satisfy in general, computationally hard to satisfy when doing so is possible, and only approximately satisfiable in practice. On the other, for the less demanding criteria our findings are much more encouraging. In addition, our share concepts can be be used to validate outcomes of PB elections, since they can help to explain voters how their "share" of the budget was spent.

An open question we would like to raise is whether a (meaningful) compromise between effort-based and satisfaction-based fairness is possible. A notion of proportionality that guarantees both effort and satisfaction to voters would be very desirable. However, such a combined axiom would have to be significantly weaker than EJS and EJR (in view of, e.g., Proposition 11).

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A Full Proofs

A.1 Proof of Proposition 1

Proof. It is clear that checking if a fair share allocation exists is in NP. We show NP-hardness by a reduction from 3-SET-COVER [Fürer and Yu, 2011], which is the problem of deciding, given a universe $U = \{u_1, \dots, u_{|U|}\}$, a set S of 3-element subsets of U, and an integer k, whether there exists a subset S' of S such that $\bigcup S' = U$ and $|S| \leq k$. Let (U, S, k) be an instance of 3-SET-COVER. We can assume w.l.o.g. that $k \leq |U|$.

We build a PB instance as follows: For every element in U there is a voter $1, \dots, |U|$. Moreover, there are $2|U| + 3$ many auxiliary voters $|U| + 1, \dots, 3|U| + 3$, i.e., $\mathcal{N} = \{1, \dots, 3|U| + 3\}$. Furthermore, $\mathcal{P} = \{p_1, \dots, p_{|S|}, p^*\}$, i.e., for every set $s_j \in S$ there is a project p_j, and there is one auxiliary project p^*. We assume unit costs and $b = k + 1$. The ballot for each voter $i \leq |U|$ is given by $p_j \in A_i$ if and only if $u_i \in s_j$, i.e., i approves the project representing the set s_j if and only if u_i is in s_j. The auxiliary voters all approve only of p^*. We claim that there is an allocation π that satisfies FS if and only if (U, S, k) is a positive instance of 3-SET-COVER.

Assume first that there is no set cover of size k, so for any set $S' \subseteq S$ of size k there is an element u_i that is not contained in any set in S'. It follows that for every allocation π of k or fewer projects there is one voter $i \leq |U|$ with $share(\pi, i) = 0$. Moreover, for any allocation that does not contain p^*, all voters $i > |U|$ have share 0. Hence, no allocation with at most $k + 1$ projects can satisfy FS.

Now assume that S' is a set cover of size k. We claim that $\pi := \{p_j \mid s_j \in S'\} \cup \{p^*\}$ satisfies FS. By assumption,

$$\frac{b}{|\mathcal{N}|} = \frac{k+1}{3|U|+3} \leq \frac{|U|+1}{3|U|+3} = \frac{1}{3}.$$

Moreover, for every project p_j we have $|\{i \mid p_j \in A_i\}| = 3$ because $|s_j| = 3$ for all $s_j \in S$. Now, because S' is a set cover, for each voter $i \leq |U|$ there is a project $p_j \in \pi$ such that $p_j \in A_i$. It follows that $share(\pi, i) \geq 1/3$ for all $i \leq |U|$. For every $i > |U|$ we have $A_i = \{p^*\}$. Now as $p^* \in \pi$ we have $share(\pi, i) = share(A_i, i)$ for all $i > |U|$. It follows that π satisfies FS. ☐

A.2 Proof of Proposition 4

It should be noted that Algorithm 1 runs in exponential time.

Proof. We show that Algorithm 1 computes a feasible budget allocation that satisfies EJS. Let us consider an arbitrary instance $I = \langle \mathcal{P}, c, b \rangle$ and profile $\boldsymbol{A}$.

We first show that the budget allocation returned by the algorithm indeed is feasible.

Claim 1. *The budget allocation π returned by Algorithm 1 on I and $\boldsymbol{A}$ is feasible.*

Proof: Consider the run of the algorithm on I and $\boldsymbol{A}$ and assume that the while-loop is run k times. Let us call (N_j, P_j) the sets of agents and projects that are selected during the j-ith run of the while-loop, for all $j \in \{1, \dots, k\}$. We then

---

Algorithm 1: Greedy EJS

Input: An instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$
Output: A budget allocation $\pi \in \mathcal{A}(I)$ satisfying EJS
Intialise π and $N^\star$ as the empty set: $\pi \leftarrow \emptyset$, $N^\star \leftarrow \emptyset$
while *there exists an $N \subseteq \mathcal{N} \setminus N^\star$ with $N \neq \emptyset$ and a $P \subseteq \mathcal{P} \setminus \pi$ with $P \neq \emptyset$, such that N is P-cohesive* **do**
  Let $N \subseteq \mathcal{N} \setminus N^\star$ and $P \subseteq \mathcal{P} \setminus \pi$ be such that:

$$(N, P) \in \underset{\substack{(N', P') \in 2^{\mathcal{N} \setminus N^\star} \times 2^{\mathcal{P} \setminus \pi} \\ N' \text{ is } P'\text{-cohesive}}}{\arg\max} \max_{i \in N'} share(P', i)$$

  Select the projects in P: $\pi \leftarrow \pi \cup P$
  Agents in N have been satisfied: $N^\star \leftarrow N^\star \cup N$
return *the budget allocation π*

---

have:

$$c(\pi) = \sum_{j=1}^{k} c(P_j) \leq \sum_{j=1}^{k} \frac{|N_j| \times b}{n} = b.$$

The first equality comes from the fact that $P_1, \dots, P_k$ is a partition of π. The inequality is derived from the fact that N_j is a P_j-cohesive group, for all $j \in \{1, \dots, k\}$ (it is an inequality because for any of the projects $p \in P_j$, some agents outside of N_j may approve of it; $c(p)$ can thus be split among more than $|N_j|$ agents). The final equality is linked to the fact that $N_1, \dots, N_k$ is a partition of $\mathcal{N}$. Overall, the outcome of Algorithm 1 is a feasible budget allocation. ■

Let us now prove that the algorithm does compute an EJS budget allocation.

Claim 2. *The budget allocation π returned by Algorithm 1 on I and $\boldsymbol{A}$ satisfies EJS.*

Proof: Assume towards a contradiction that π violates EJS. Then, there must exist some $N \subseteq \mathcal{N}$ and $P \subseteq \mathcal{P}$ such that N is P-cohesive but also such that, for all agents $i \in N$, we have $share(\pi, i) < share(P, i)$. Note that, if $P \not\subseteq \pi$, this means that at the end of the algorithm either one agent $i \in N$ has been satisfied ($i \in N^\star$ when the algorithm returns) or that one project $p \in P$ has been selected ($p \in \pi$ when the algorithm returns). We distinguish these two cases.

First, consider the case where one agent has been satisfied by the end of the algorithm. Using the same notation as for the previous claim, there exists then a smallest $j \in \{1, \dots, k\}$ such that there exist $i^\star \in N \cap N_j$. Given that (N, P) has not been selecting during that run of the while loop, it means that:

$$\max_{i' \in N_j} share(P_j, i') \geq \max_{i \in N} share(P, i).$$

Since the cost of a project is split equality among its supporters, it is easy to observe that for any P-cohesive group N, and for every two agents $i, i' \in N$, we have $share(P, i) = share(P, i')$. Moreover, we also have that $share(P', i) \leq share(P, i)$ for any $P' \subseteq P \subseteq \mathcal{P}$ and $i \in \mathcal{N}$. Overall, for our specific agent $i^\star \in N \cap N_j$, we have:

$$\begin{aligned} share(\pi, i^\star) &\geq \max_{i' \in N_j} share(P_j, i') \\ &\geq \max_{i \in N} share(P, i) \\ &\geq share(P, i^\star), \end{aligned}$$

which contradicts the fact that π fails EJS.

Let us now consider the second case, i.e., when $P \cap \pi \neq \emptyset$ but $P \not\subseteq \pi$. In this case, it is important to see that if N is P-cohesive, then it is also P'-cohesive for all $P' \subseteq P$. Then, we can run the same proof considering the $(P \setminus \pi)$-cohesive group N. Iterating this argument, would either lead to the conclusion that $P \subseteq \pi$, a contradiction, or to another contradiction due to the first case we considered (when some agent of N is already satisfied). ■

Finally, it should be noted that Algorithm 1 always terminates. Indeed, after each run of the while-loop, at least one agent is added to the set $N^\star$. Moreover, if $N^\star = \mathcal{N}$, the condition of the while-loop would be violated and the algorithm would terminate. Overall at most n runs through the while-loop can occur. This concludes the proof. □

A.3 Proof of Theorem 5

Proof. Assume, that there is an algorithm $\mathbb{A}$ that always computes an allocation satisfying EJS.

We will make use of the SUBSET-SUM problem, known to be NP-hard. In this problem, we are given as input a set $S = \{s_1, \ldots, s_m\}$ of integers and a target $t \in \mathbb{N}$ and we wonder whether there exists an $X \subseteq S$ such that $\sum_{x \in X} x = t$.

Given S and t as described above, we construct $I = \langle \mathcal{P}, c, b \rangle$ and $\boldsymbol{A}$ as follows. We have m projects $\mathcal{P} = \{p_1, \ldots, p_m\}$ with the following cost function $c(p_j) = s_j$ for all $j \in \{1, \ldots, m\}$ and a budget limit $b = t$. There is moreover only one agent, who approves of all the projects.

Now, (S, t) is a positive instance of SUBSET-SUM if and only if there is a budget allocation $\pi \in \mathcal{A}(I)$ that cost is exactly b. If such an allocation π exists, then the one voter 1 is π-cohesive. Therefore, any allocation π' that satisfies EJS must give that voter $share(1, \pi') \geq share(1, \pi) = c(\pi)$. Hence, (S, t) is a positive instance of SUBSET-SUM if and only if $c(\mathbb{A}(I, \boldsymbol{A})) = b$. This way, we can use $\mathbb{A}$ to solve SUBSET-SUM in polynomial time. □

A.4 Proof of Proposition 6

Proof. Consider an instance $I = \langle \mathcal{P}, c, b \rangle$ and a profile $\boldsymbol{A}$ on which Algorithm 1 is run.

The first thing to note is that at least one agent is added to $N^\star$ during each run through the while-loop and that, if ever $N^\star = \mathcal{N}$, the condition of the while-loop is trivially satisfied. Overall, the while-loop can only be run n times.

Let us have a closer look at what is happening inside the while-loop. The main computational task here is the maximisation that goes through all subsets of agents and of projects. We will show that we can avoid going through all subsets of agents. Indeed, consider a subset of projects $P \subseteq \mathcal{P}$ and let $N \subseteq \mathcal{N}$ be the largest set of agents such that for all $i \in N$, $P \subseteq A_i$. Note such a set N can be efficiently computed (by going through all the approval ballots). Now, if some group of agents is P-cohesive, then for sure N should be P-cohesive. Moreover, note that for any P-cohesive group N', and for every two agents $i \in N$ and $i' \in N \cup N'$, we have $share(P, i) = share(P, i')$. Overall, one can, without loss of generality, only consider the group of agents N when considering the subset of projects P. This implies that the maximisation step can be computed by going through all the subsets of projects and, for each of them, only considering a single subset of agents (that is efficiently computable). □

A.5 Proof of Theorem 9

Proof. Let $\pi = \{p_1, \ldots, p_k\}$ be the budget allocation output by Rule X^e on instance I and profile $\boldsymbol{A}$, where p_1 was selected first, p_2 second etc. For any $1 \leq j \leq k$, set $\pi_j := \{p_1, \ldots, p_j\}$. Consider $N \subseteq \mathcal{N}$, a P-cohesive group, for some $P \subseteq \mathcal{P}$. We show that π satisfies Local-EJS for N. If $P \subseteq \pi$ then Local-EJS is satisfied by definition. We will thus assume that $P \not\subseteq \pi$.

Let k^* be the first round after which there exists a voter $i^* \in N$ whose load is larger than $b/n - 1/|N|$. Such a round must exist as otherwise the voters in N could afford another project from P. As we assumed $P \not\subseteq \pi$, this would mean that Rule X^e cannot have stopped. Let $\pi^* = \pi_{k^*}$ and consider an arbitrary project $p^* \in P \setminus \pi^*$. Our goal is to prove that π^* satisfies Local-EJS for N, i.e.:

$$\begin{aligned} & share(\pi^* \cup \{p^*\}, i^*) > share(P, i^*) \\ \Leftrightarrow\quad & share(\pi^*, i^*) > share(P \setminus \{p^*\}, i^*) \\ \Leftrightarrow\quad & share(\pi^* \cap P, i^*) + share(\pi^* \setminus P, i^*) > \\ & \quad share(P \cap \pi^*, i^*) + share(P \setminus (\pi^* \cup \{p^*\}), i^*) \\ \Leftrightarrow\quad & share(\pi^* \setminus P, i^*) > share(P \setminus (\pi^* \cup \{p^*\}), i^*). \quad (1) \end{aligned}$$

We will now work on each side of inequality (1) to eventually prove that it is indeed satisfied.

We start by the left-hand side of (1). Let us first introduce some notation that will allow us to eason in terms of share per unit of load. For a project $p \in \pi$, we denote by $\alpha(p)$ the smallest $\alpha \in \mathbb{R}_{>0}$ such that p was α-affordable when Rule X^e selected it. Moreover, we define $q(p)$—the share that a voter that contributes fully to p gets per unit of load—as $q(p) := 1/\alpha(p)$.

Since before round k^*, agent i^* contributed in full for all projects in π^* (as $\ell_{i^*} < b/|N|$ after each round $1, \ldots, k^*$), we know that $\alpha(p) \cdot share(\{p\}, i^*)$ equals the contribution of i^* for p, and that for any $p \in \pi^*$. We thus have:

$$\begin{aligned} & share(\pi^* \setminus P, i^*) \\ = \quad & \sum_{p \in \pi^* \setminus P} share(\{p\}, i^*) \\ = \quad & \sum_{p \in \pi^* \setminus P} \alpha(p) \cdot share(\{p\}, i^*) \cdot \frac{1}{\alpha(p)} \\ = \quad & \sum_{p \in \pi^* \setminus P} \gamma_{i^*}(p) \cdot q(p), \qquad (2) \end{aligned}$$

where $\gamma_{i^*}(p)$ denotes the contribution of i^* to any $p \in \pi$, defined such that if p has been selected at round j, i.e., $p = p_j$, then $\gamma_{i^*}(p) = \gamma_{i^*}(\pi_j, \alpha(p_j), p_j)$.

Now, let us denote by $q_{\min}$ the smallest $q(p)$ for any $p \in \pi^* \setminus P$. From (2), we get:

$$share(\pi^* \setminus P, i^*) \geq q_{\min} \sum_{p \in \pi^* \setminus P} \gamma_{i^*}(p). \quad (3)$$

We now turn to the right-hand side of (1). We introduce some additional notation for that. For every project $p \in P$, we denote by $q^*(p)$ the share per load that a voter in N receives if only voters in N contribute to p, and they all contribute in full to p, defined as:

$$q^*(p) = \frac{share(\{p\}, i)}{1/|N|} = \frac{|N|}{|\{A \in \boldsymbol{A} \mid p \in A\}|},$$

where i is any agent in N.

We have then:

$$\begin{aligned} & share(P \setminus (\pi^* \cup \{p^*\}), i^*) \\ = & \sum_{p \in P \setminus (\pi^* \cup \{p^*\})} share(\{p\}, i^*) \\ = & \sum_{p \in P \setminus (\pi^* \cup \{p^*\})} \frac{share(\{p\}, i^*)}{1/|N|} \cdot \frac{1}{|N|} \\ = & \sum_{p \in P \setminus (\pi^* \cup \{p^*\})} q^*(p) \cdot \frac{1}{|N|} \quad (4) \end{aligned}$$

Setting $q^*_{\max}$ to be the largest $q^*(p)$ for all $p \in P \setminus (\pi^* \cup \{p^*\})$, (4) gives us:

$$share(P \setminus (\pi^* \cup \{p^*\}), i^*) \leq q^*_{\max} \cdot \frac{|P \setminus (\pi^* \cup \{p^*\})|}{|N|}. \quad (5)$$

In the aim of proving inequality (1), we want to show that

$$q_{\min} \cdot \sum_{p \in \pi^* \setminus P} \gamma_{i^*}(p) > q^*_{\max} \cdot \frac{|P \setminus (\pi^* \cup \{p^*\})|}{|N|}. \quad (6)$$

Note that proving that this inequality holds, would in turn prove (1) thanks to (3) and (5). We divide the proof of (6) into two claims.

Claim 3. $q_{\min} \geq q^*_{\max}$.

Proof: Consider any project $p' \in P \setminus (\pi^* \cup \{p^*\})$. It must be the case that p' was at least $1/q^*(p)$-affordable in round $1, \ldots, k^*$, for all $p \in \pi^*$, as all voters in N could have fully contributed to it based on how we defined k^*.

As no $p' \in P \setminus (\pi^* \cup \{p^*\})$ was selected by Rule X^e, we know that all projects that have been selected must have been at least as affordable, i.e., for all $p \in \pi^*$ and $p' \in P \setminus (\pi^* \cup \{p^*\})$ we have:

$$\begin{aligned} & \alpha(p) \leq \frac{1}{q^*(p')} \\ \Leftrightarrow \quad & q(p) \geq q^*(p') \\ \Leftrightarrow \quad & q_{\min} \geq q^*_{\max}. \end{aligned}$$

This concludes the proof of our first claim. ■

Claim 4. $\sum_{p \in \pi^* \setminus P} \gamma_{i^*}(p) > \frac{|P \setminus (\pi^* \cup \{p^*\})|}{|N|}$.

Proof: From the choice of k^*, we know that the load of agent i^* at round k^* is such that:

$$\ell_{i^*}(\pi^*) + \frac{1}{|N|} > \frac{b}{n}.$$

On the other hand, since N is a P-cohesive group, we know that:

$$\frac{|P|}{|N|} = \frac{|P \setminus \{p^*\}|}{|N|} + \frac{1}{|N|} \leq \frac{b}{n}.$$

Linking these two facts together, we get:

$$\ell_{i^*}(\pi^*) > \frac{|P \setminus \{p^*\}|}{|N|}.$$

By the definition of the load, we thus have:

$$\ell_{i^*}(\pi^*) = \sum_{p_j \in \pi^*} \gamma_{i^*}(p_j) > \frac{|P \setminus \{p^*\}|}{|N|}.$$

This is equivalent to:

$$\sum_{p_j \in P \cap \pi^*} \gamma_{i^*}(p_j) + \sum_{p_j \in P \setminus \pi^*} \gamma_{i^*}(p_j) > \frac{|P \cap \pi^*|}{|N|} + \frac{|P \setminus (\pi^* \cup \{p^*\})|}{|N|} \quad (7)$$

Now, we observe that every voter in N contributed in full for every project in π^*. It follows that the contribution of every voter in N for a project $p_j \in P \cap \pi^*$ is smaller or equal the contribution needed if the voters in N would fund the project by themselves. In other words for all $p \in P \cap \pi^*$ we have:

$$\gamma_{i^*}(p) \leq \frac{1}{|N|}.$$

It follows then that:

$$\sum_{p_j \in P \cap \pi^*} \gamma_{i^*}(p_j) \leq \frac{|P \cap \pi^*|}{|N|}.$$

For (7) to be satisfied, we must have that:

$$\sum_{p_j \in \pi^* \setminus P} \gamma_{i^*}(p_j) > \frac{|P \setminus (\pi^* \cup \{p^*\})|}{|N|}$$

This concludes the proof of our second claim. ■

Putting together these two claims immediately shows that inequality (6) is satisfied, which in turn shows that (1) also is. Since P, N and p^* were chosen arbitrarily, this shows that Rule X^e satisfied Local-EJS in the unit-cost setting. □

A.6 Proof of Proposition 10

Proof. Let π be an allocation that satisfies FS. Let $i \in \mathcal{N}$ be an arbitrary agent. We distinguish two cases. First, assume $share(A_i, i) < \frac{b}{n}$. For FS to be satisfied, we must then have $share(\pi, i) \geq share(A_i, i)$. This entails that $A_i \subseteq \pi$ should be the case. Hence, the conditions for Strong-EJS are trivially satisfied for agent i.

Secondly, assume $share(A_i, i) \geq \frac{b}{n}$. Since π satisfies FS, we know that $share(\pi, i) \geq \frac{b}{n}$. Let $N \subseteq \mathcal{N}$ be a P-cohesive group, for some $P \subseteq \mathcal{P}$, such that $i \in N$. By definition of a cohesive group, we know that $c(P) \leq \frac{b}{n}|N|$. Hence, $share(P, i) \leq \frac{b}{n}$. Overall, we have:

$$share(\pi, i) \geq b/n \geq share(P, i).$$

This shows that π satisfies Strong-EJS.

Now, let π be an allocation that satisfies FS-1. First, consider an agent $i \in \mathcal{N}$ such that $share(A_i, i) < \frac{b}{n}$. For FS-1 to be satisfied, there must be a project $p \in \mathcal{P}$ such that $share(\pi \cup \{p\}, i) \geq share(A_i, i)$. This entails that $|A_i \cap \pi| \leq 1$ should be the case. Hence, the conditions for EJS-1 are trivially satisfied for agent i.

Consider now an agent $i \in \mathcal{N}$ such that $share(A_i, i) \geq \frac{b}{n}$. Since π satisfies FS-1, we know that there must be a project $p \in \mathcal{P}$ such that $share(\pi \cup \{p\}, i) \geq b/n$. Let $N \subseteq \mathcal{N}$ be a P-cohesive group, for some $P \subseteq \mathcal{P}$, such that $i \in N$. By definition of a cohesive group, we know that $c(P) \leq \frac{b}{n}|N|$. Hence, $share(P, i) \leq \frac{b}{n}$. Overall, we have:

$$share(\pi \cup \{p\}, i) \geq b/n \geq share(P, i).$$

This shows that π satisfies Strong-EJS. □

A.7 Counter Examples for the Taxonomy

We observe that Local-FS and FS-1 are independent properties, i.e., neither implies the other.

Proposition 13. *FS-1 does not imply Local-FS and Local-FS does not imply FS-1.*

Proof. Consider an instance with four projects, a budget limit of $b = 6$, and three agents, where $c(p_1) = c(p_2) = 3$, $c(p_3) = 6$ and $c(p_4) = 1$. The approvals are given by $A_1 = \{p_1, p_2\}$ and $A_2 = A_3 = \{p_3, p_4\}$. Then $\{p_1, p_2\}$ satisfies FS-1 as agent 1 already receives (more than) her fair share, while 2 and 3 receive their fair share from $\{p_1, p_2\} \cup \{p_3\}$. However, no supporter of p_4 receives their fair share from $\{p_1, p_2\} \cup \{p_4\}$. Therefore, Local-FS is violated. The other direction follows from the fact that we know of a rule satisfying Local-FS (see Theorem 3) while FS-1 is not always satisfiable. □

Following the "Strong-EJS path", by definition, we know that Strong-EJS implies EJS, which in turn implies Local-EJS and EJS-1. From Proposition 8, it is also clear that Local-EJS implies EJS-1. We can further show that Local-EJS and EJS-1 are not equivalent, even in the unit-cost setting.

Proposition 14. *EJS-1 does not imply Local-EJS, even in the unit-cost setting.*

Proof. Consider an instance with two voters, 1 and 2, and six projects $p_1, \ldots, p_6$ all of cost 1. Voter 1 approves of p_1, p_2, p_3, p_4, and p_5; and voter 2 approves of p_4, p_5 and p_6. The budget limit is $b = 4$. It can be checked that $\pi = \{p_1, p_2, p_3, p_4\}$ satisfies EJS-1 but not Local-EJS. □

Let us now turn to FS-1. While FS-1 implies the "weakest" concept of our taxonomy, we can show that it fails to imply the "second weakest" concept, namely Local-EJS.

Proposition 15. *FS-1 does not imply Local-EJS.*

Proof. Consider an instance with four projects, a budget limit of $b = 12$, and three agents where $c(p_1) = 4$, $c(p_2) = 2$, $c(p_3) = 5$, $c(p_4) = 7$. Voter 1 only approves p_4 while 2 and 3 approve p_1, p_2 and p_3. Allocation $\pi = \{p_1, p_4\}$ satisfies FS-1 but fails Local-EJS: the $\{p_2, p_3\}$-cohesive group $\{2, 3\}$ deserves a share of 3.5, but adding p_2 to π would not meet this requirement. □

We now turn to Local-FS and show that it does not imply any of the other concepts we have introduced.

Proposition 16. *Local-FS does not imply EJS-1.*

Proof. Consider an instance with three projects, a budget limit of $b = 6$, and two agents where $c(p_1) = 6$ and $c(p_2) = c(p_3)$. Moreover, 1 approves all projects, 2 only p_2 and p_3. Allocation $\pi = \{p_1\}$ satisfies Local-FS: for both p_2 and p_3, if we were to add them to π, agent 1 would have a fair share. However, it does not satisfy EJS-1: $\{2\}$ is a $\{p_2, p_3\}$-cohesive group but neither project is selected. □

This shows that Local-FS also does not imply any of Local-EJS, EJS, Strong-EJS and FS-1. We can go further and show none of these concepts imply Local-FS.

Proposition 17. *Neither Strong-EJS nor FS-1 implies Local-FS.*

Proof. Consider the following instance with five projects, a budget limit of $b = 16$, and two agents: $c(p_1) = c(p_2) = 12$, $c(p_3) = c(p_4) = 1$ and $c(p_5) = 4$ while $A_1 = \{p_1, p_2, p_3, p_4\}$ and $A_2 = \{p_1, p_5\}$. In this instance, the cohesive groups are: $\{1, 2\}$ is $\{p_1\}$-cohesive, $\{1\}$ is $\{p_3\}$-cohesive, $\{p_4\}$-cohesive and $\{p_3, p_4\}$-cohesive, and $\{2\}$ is $\{p_5\}$-cohesive. Overall, to satisfy Strong-EJS, a budget allocation should provide a share of at least 6 to agents 1 and 2. The budget allocation $\pi = \{p_1, p_5\}$ thus satisfies Strong-EJS (note that is is exhaustive). However, one can easily check that π does not satisfy the conditions of Local-FS as adding p_3 to π only provides a share of 7 to agent 1 while the lower bound for their fair share is 8.

Note that budget allocation $\pi = \{p_1, p_5\}$ does satisfy FS-1: the fair share of the voters is 8; voter 2 already has a share of 10 in π, and voter 1 would get a share of 18 in $\pi \cup \{p_2\}$. □

The above result also shows that neither EJS, Local-EJS, nor EJS-1 imply Local-FS. This completes the picture.

B ILPs for FS and EJS

We now present an ILP for finding a budget allocation $Y \subseteq \mathcal{P}$ that maximises

$$\sum_{i \in \mathcal{N}} \min\{b/n, share(Y, i)\}, \tag{13}$$

i.e., we maximise the total share of all voters but cap the share of individual voters at their fair share (b/n). Note that budget allocations satisfying fair share achieve an optimal value for (13) (which is b). Thus, this ILP finds budget allocations satisfying FS if they exist. The ILP is shown in Figure 4. Variable s_i is the share of voter i bounded by $\frac{b}{n}$. Variable y_p

$$\textbf{maximise} \quad \sum_{i=1}^{n} s_i \tag{8}$$

subject to:

$$y_p \in \{0,1\} \qquad \text{for } p \in \mathcal{P} \tag{9}$$

$$s_i \leq \sum_{p \in A_i} y_p \cdot \frac{c(p)}{|\{A \in \boldsymbol{A} \mid p \in A\}|} \qquad \text{for } i \in \mathcal{N} \tag{10}$$

$$s_i \leq {}^{b}\!/_{n} \qquad \text{for } i \in \mathcal{N} \tag{11}$$

$$\sum_{p \in \mathcal{P}} y_p \cdot c(p) \leq b \tag{12}$$

Figure 4: An ILP for maximizing the total capped share of all voters.

indicates whether project p is selected in the winning budget allocation.

Figure 5 shows an ILP for verifying whether a given budget allocation π satisfies EJS. It searches for a set $P \subseteq \mathcal{P}$ and a set $N \subseteq \mathcal{N}$ that certifies a violation of the EJS property, i.e., N is P-cohesive and all voters receive a strictly larger share from P than from π. Here, variable x_i indicates whether $i \in N$ and variable z_p indicates whether $p \in P$. Conditions (17) and (18) enforce that N is indeed P-cohesive. Condition (19) implies that $share(\pi, i) < share(P, i)$ for all $i \in N$. The inequality in Condition (19) is only strict for $\epsilon > 0$. Consequently, π fails EJS if and only if this ILP yields a solution with $\epsilon > 0$.

$$\textbf{maximise } \epsilon \tag{14}$$

subject to:

$$x_i \in \{0,1\} \qquad \text{for } i \in \mathcal{N} \tag{15}$$

$$z_p \in \{0,1\} \qquad \text{for } p \in \mathcal{P} \tag{16}$$

$$z_p + x_i - 1 \leq \mathbb{I}_{p \in A_i} \qquad \text{for } i \in \mathcal{N}, p \in \mathcal{P} \tag{17}$$

$$\frac{1}{n} \cdot \sum_{i \in \mathcal{N}} x_i \geq \frac{1}{b} \cdot \sum_{p \in \mathcal{P}} z_p \cdot c(p) \tag{18}$$

$$x_i \cdot share(\pi, i) + \epsilon \leq \sum_{p \in \mathcal{P}} z_p \cdot \frac{c(p)}{|\{A \in \boldsymbol{A} \mid p \in A\}|} \qquad \text{for } i \in \mathcal{N} \tag{19}$$

Figure 5: An ILP for verifying whether a budget allocation π satisfies EJS.