Reduction of Economic Inequality in Combinatorial Domains

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Talk Outline

Economic inequality . . .

- is a relevant criterion for multiagent resource allocation.
- gives rise to interesting research questions.
- can be handled using integer programming.

In this talk I will show you . . .

- some of the basic definitions for economic inequality.
- a curious complexity result.

The Model

Finite sets of *agents* $\mathcal{N} = \{1, \ldots, n\}$ and of indivisible *goods* $\mathcal{G}$.
Each good needs to be allocated to exactly one agent.

Any given *allocation* A induces a *utility* of $u_i(A)$ for agent $i \in \mathcal{N}$.
So any allocation A induces a *utility vector* $(u_1(A), \ldots, u_n(A))$.

We want a *fair* allocation, i.e., one that minimises *inequality* . . .

How do you define inequality?

For instance: which is more equal, $(1, 2, 7, 7, 8)$ or $(1, 3, 5, 6, 10)$?

The Pigou-Dalton Principle

For *two* agents, it is perfectly clear what "more equal" means.
We can use this insight + a weak efficiency requirement ...

A move from allocation A to A' is called a *Pigou-Dalton transfer* if there are two agents $i, j \in \mathcal{N}$ such that:

- Only the bundles held by i and j change.
- Inequality reduces: $|u_i(A) - u_j(A)| > |u_i(A') - u_j(A')|$
- Total utility does not reduce: $u_i(A) + u_j(A) \leqslant u_i(A') + u_j(A')$

The *Pigou-Dalton Principle* postulates that any measure of fairness should value a Pigou-Dalton transfer as a (weak) improvement.

But: not yet enough to rank $(1, 2, 7, 7, 8)$ and $(1, 3, 5, 6, 10)$ ...

A.C. Pigou. *Wealth and Welfare*. Macmillan, London, 1912.

H. Dalton. The Measurement of the Inequality of Incomes. *Econ. Journal*, 1920.

The Lorenz Curve

Ideally, every single agent enjoys exactly the same utility.
The Lorenz curve is a way to visualise how far we are from this ideal.

Let $\boldsymbol{u}^*(A)$ be the *ordered utility vector* of allocation A. So this is the total utility of the k poorest agents:

$$L_k(A) \quad = \quad \sum_{i=1}^{k} u_i^*(A).$$

The vector $(L_1(A), \ldots, L_n(A))$ is called the *Lorenz curve* of A.

But: the Lorenz curves for $(1, 2, 7, 7, 8)$ and $(1, 3, 5, 6, 10)$ cross . . .

M.O. Lorenz. Methods of Measuring the Concentration of Wealth. *Publications of the American Statistical Association*, 9(70):209–219, 1905.

Inequality Indices

An *inequality index* is a function mapping allocations to $[0, 1]$, with 0 representing perfect equality and 1 representing complete inequality.

Two popular indices:

- *Gini index* $=$ *area* between line of perfect equality and Lorenz curve (divided by a suitable normalisation factor)
- *Robin Hood index* $=$ maximal *distance* between line of perfect equality and Lorenz curve (also normalised)

Now we can discern $(1, 2, 7, 7, 8)$ and $(1, 3, 5, 6, 10)$: the former is better according to Gini, the latter according to Robin Hood.

The Pigou-Dalton Problem

We are interested in the algorithmic challenges raised by these notions of inequality. Note that hardness will depend on the *language* $\mathcal{L}$ used to encode the utility functions.

PIGOU-DALTON IMPROVEMENT (PIGDAL)
Instance: Utility functions in $\mathcal{L}$, allocation A, partial allocation P.
Question: Is there an $A' \supseteq P$ s.t. (A, A') is a Pigou-Dalton transfer?

Easy results from the paper:

- PIGDAL is (at least) *NP-hard* for the *OR-language*
  But: OR is a pathological language making *everything* intractable
- PIGDAL is *polynomial* for the *XOR-language*
  But: XOR is representationally highly wasteful

What about *weighted goal languages* (compact and not pathological)?
Next: the simplest case (additive utility functions) . . .

Pigou-Dalton for Additive Utilities

A compact way of representing an *additive* utility function is to list the *weight* of each good. How hard is PigDal for this language?

Take the special case of *two agents* with *identical utility functions*. Then finding a Pigou-Dalton transfer with resulting inequality $< K$ is equivalent to the well-known NP-complete Partition problem:

Partition
Instance: $(w_1, \ldots, w_m) \in \mathbb{N}^m$, $K \in \mathbb{N}$.
Question: Is there a set $S \subseteq \{1, \ldots, m\}$ s.t. $|\sum_{i \in S} w_i - \sum_{i \notin S} w_i| < K$?

But here we are given a partition and need to find a *better partition*. Sounds just as hard, but is it?

- If the initial partition is very bad, finding a better one is easy.
- If the initial partition is pretty good, maybe this helps?

Best Known Result

Proposition 1 PigDal $\notin$ P *for additive utilities, unless* NP = coNP.

<u>Proof:</u> Recall that PigDal = Better Partition. Use the latter.

Fact: No Perfect Partition (with $\Delta = 0$) is coNP-*hard*.

For contradiction: assume *poly-time* `ALG` solves Better Partition.

Show that No Perfect Partition $\in$ NP:

- Certificate = best possible (but not perfect) partition
- Verification: use `ALG` to check no improvement possible ✓

Hence, there exists a coNP-hard problem in NP.
Thus: coNP $\subseteq$ NP, which means coNP = NP. ✓

Last Slide

- Main message: Economic inequality measures are relevant fairness criteria for work in multiagent systems. *Use them!*
- Contributions of the paper:
  - Adaptation of standard definitions form economics to the model of indivisible goods favoured in our domain
  - Complexity results for some relevant questions for certain preference representation languages
  - Modular approach to Lorentz improvements and inequality index optimisation for various representation languages in IP
- Research opportunities:
  - Complexity: several open questions
  - Algorithms: should get implemented and tested