Modal Logics of Ordered Trees

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Talk Overview

- Modal Logics of Ordered Trees (OTL)
  syntax and semantics; examples
- OTL as a Temporal Interval Logic
  time intervals; past and future; ontological considerations
- Technical Results
  axiomatisation and decidability
- Conclusion
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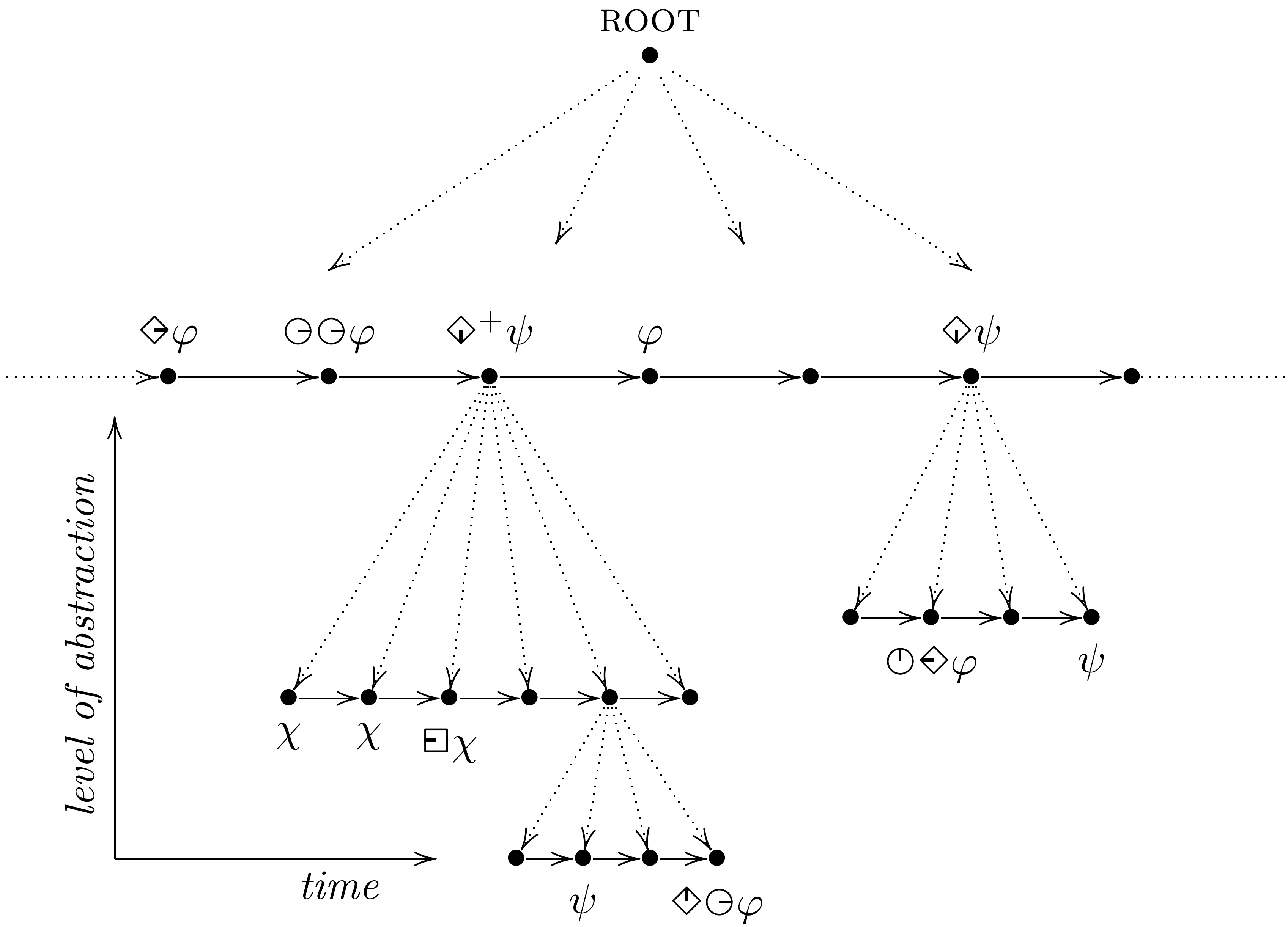
Modal Logics of Ordered Trees (OTL)

Models are based on *ordered trees*. Besides the usual propositional connectives we have a number of modal operators, for example:

$\bigcirc_{\rightarrow}\varphi$	*"φ is true at the next righthand sibling (if any)"*
$\Diamond_{\rightarrow}\varphi$	*"φ is true at some righthand sibling"*
$\bigcirc_{\uparrow}\varphi$	*"φ is true at the parent (if any)"*
$\Diamond_{\uparrow}\varphi$	*"φ is true at some ancestor"*
$\Diamond_{\downarrow}\varphi$	*"φ is true at some child"*
$\Diamond_{\downarrow}^{+}\varphi$	*"φ is true at some descendant"*

We also have modalities ($\bigcirc_{\leftarrow}$ and $\Diamond_{\leftarrow}$) to refer to *lefthand siblings*. All corresponding *box-operators* are definable, for example:

$$\Box_{\rightarrow}\varphi = \neg\Diamond_{\rightarrow}\neg\varphi \qquad \text{"}\varphi\text{ is true at all righthand siblings"}$$


ROOT
φ
ψ
χ
χ
ψ
level of abstraction
time

Examples

- A node t is the root of the tree iff ROOT is true at t:

$$\text{ROOT} \quad = \quad [\uparrow]\bot$$

- Similar formulas identify leftmost and rightmost siblings:

$$\text{LEFTMOST} \quad = \quad [\leftarrow]\bot \qquad\qquad \text{RIGHTMOST} \quad = \quad [\rightarrow]\bot$$

- An ordered tree $\mathcal{T}$ is a binary tree iff BINARY is valid in $\mathcal{T}$ (or, equivalently, iff BINARY is globally true in a model based on $\mathcal{T}$):

$$\text{BINARY} \quad = \quad (\text{LEFTMOST} \leftrightarrow \text{RIGHTMOST}) \;\rightarrow\; \text{ROOT}$$

- An ordered tree $\mathcal{T}$ is "discretely branching" iff the formula DISCRETE is valid in $\mathcal{T}$:

$$\text{DISCRETE} \quad = \quad [\rightarrow]([\rightarrow]A \rightarrow A) \rightarrow (\langle\rightarrow\rangle[\rightarrow]A \rightarrow [\rightarrow]A)$$

This is the standard axiom schema familiar from temporal logic to characterise discrete flows of time.

OTL as a Temporal Interval Logic

We can interpret OTL as a (restricted) interval logic as follows:

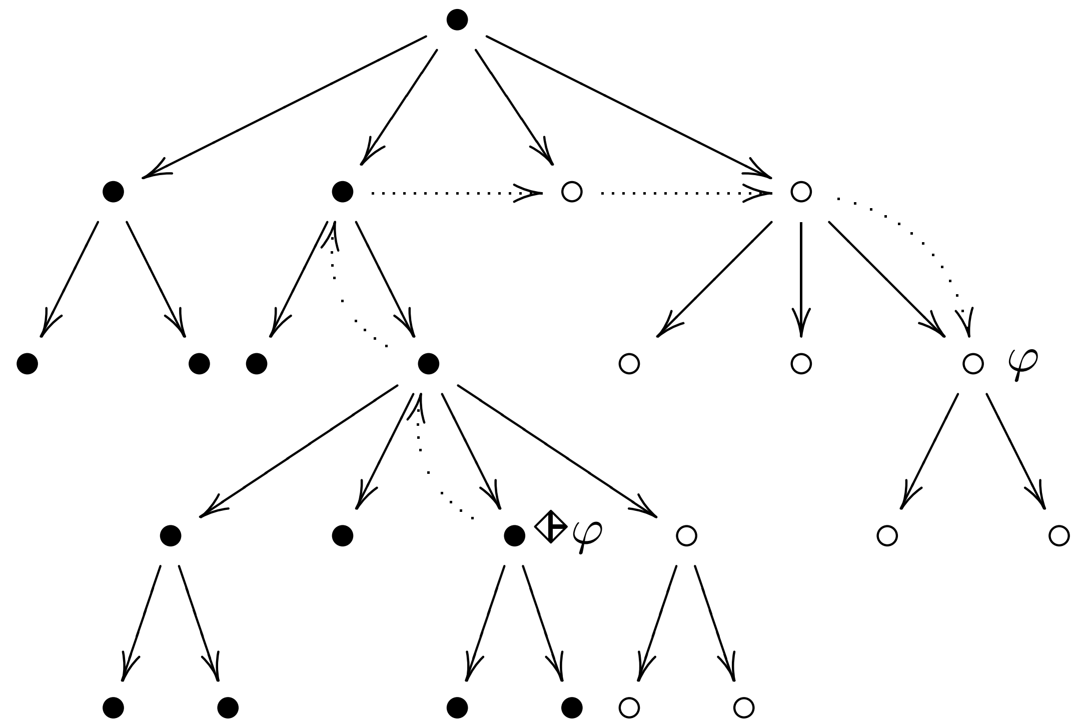
- *nodes* in a tree represent *time intervals*;
- *descendants* represent *subintervals*; and
- the *order declared over siblings* represents an *earlier-later ordering* over time intervals.

But this raises some questions:

- What is the meaning of, say, the $\Diamond\!\!\!\rightarrow$-operator? Is it a proper future modality?
- Are models where, say, φ is true at some node t but not at all of t's children meaningful under this temporal interpretation?

Past and Future

If you are allowed to move *up* before and *down* after moving to the right, you can reach *all* the nodes to the right.



Let $\Diamond_{\uparrow}^*\varphi = \varphi \vee \Diamond_{\uparrow}\varphi$ and $\Diamond_{\downarrow}^*\varphi = \varphi \vee \Diamond_{\downarrow}^{+}\varphi$.
We can now define a *global future modality* as follows:

$$\Diamond_{\updownarrow\rightarrow}\varphi \quad = \quad \Diamond_{\uparrow}^*\Diamond_{\rightarrow}\Diamond_{\downarrow}^*\varphi$$

Ontological Considerations

Consider the following two basic propositions:

(1) *The sun is shining.*

(2) *I move the pen from the table onto the OHP.*

Propositions like (1) are sometimes called *properties*; propositions like (2) are sometimes called *events* (Allen, 1984).

Properties

Properties like *"The sun is shining."* are *homogeneous* propositions (Shoham, 1987), which we can capture in OTL as follows:

$$\begin{aligned}
\textsc{downward-hereditary}(\varphi) &= \varphi \rightarrow \Box_{\downarrow}^{+}\varphi \\
\textsc{upward-hereditary}(\varphi) &= \Diamond_{\downarrow}\top \rightarrow (\Box_{\downarrow}^{+}\varphi \rightarrow \varphi) \\
\textsc{homogeneous}(\varphi) &= \textsc{downward-hereditary}(\varphi) \wedge \\
&\quad\; \textsc{upward-hereditary}(\varphi)
\end{aligned}$$

Then φ is a homogeneous proposition (with respect to a given model $\mathcal{M}$) iff $\textsc{homogeneous}(\varphi)$ is globally true in $\mathcal{M}$.

Events

Events like *"I move the pen from the table onto the OHP."* may be characterised as propositions that cannot be true at two intervals one of which contains the other.

Shoham (1987) calls such propositions *gestalt:*

$$\textsc{gestalt}(\varphi) \;=\; \varphi \;\rightarrow\; (\square\neg\varphi \wedge \square^{+}\neg\varphi)$$

Technical Results

- A complete *axiomatisation* of the fragment of OTL excluding the transitive descendant operator ($\Diamond_{\downarrow}^{+}$) is available.
  - The most interesting axioms are:

    (X1) $\Diamond_{\rightarrow} A \rightarrow \bigcirc \Diamond_{\downarrow} A$

    (X2) $\bigcirc \Diamond_{\downarrow} A \rightarrow (A \vee \Diamond_{\leftarrow} A \vee \Diamond_{\rightarrow} A)$
  - Problems with proving completeness for the full logic: transitive closure + irreflexivity + interaction

- OTL is *decidable* (proof works essentially by reduction to S2S).
  - Note that 2-dimensional modal logics with interacting modalities (such as products) are often undecidable.

    (1) $\Diamond_{\rightarrow}^{*} \Diamond_{\uparrow}^{*} A \rightarrow \Diamond_{\uparrow}^{*} \Diamond_{\rightarrow}^{*} A$ (right-commutativity)

    (2) $\Diamond_{\rightarrow}^{*} \Box_{\uparrow}^{*} A \rightarrow \Box_{\uparrow}^{*} \Diamond_{\rightarrow}^{*} A$ (Church-Rosser property)
  - Also note that modal interval logics (such as Halpern and Shoham's logic) tend to be undecidable as well.

Conclusion

- We have introduced a simple yet expressive modal logic for talking about ordered trees.
- Original motivation: linear temporal logic + zoom
- Compromise between point- and interval-based temporal logics:
  - can model subintervals, but not overlapping intervals
  - decidable (unlike many interval logics)
- Future work:
  - prove decidability directly (rather than by reduction to S2S)
  - give a complete axiomatisation for the full logic
  - extend results to cover *until*-style operators
  - develop a decision procedure (possibly Tableau-based)
  - use OTL to represent (nested) communication protocols