Modal Logics of Ordered Trees

Ulle Endriss

Department of Computing, Imperial College London
Email: ue@doc.ic.ac.uk
Talk Overview

• Modal Logics of Ordered Trees (OTL)
  syntax and semantics; examples

• OTL as a Temporal Interval Logic
  time intervals; past and future; ontological considerations

• Technical Results
  axiomatisation and decidability

• Conclusion
  recap and discussion of future work
Modal Logics of Ordered Trees (OTL)

Models are based on ordered trees. Besides the usual propositional connectives we have a number of modal operators, for example:

\[ \Theta \varphi \quad \text{“} \varphi \text{ is true at the next righthand sibling (if any)”} \]
\[ \Diamond \varphi \quad \text{“} \varphi \text{ is true at some righthand sibling”} \]
\[ \Box \varphi \quad \text{“} \varphi \text{ is true at the parent (if any)”} \]
\[ \Diamond \varphi \quad \text{“} \varphi \text{ is true at some ancestor”} \]
\[ \Diamond \varphi \quad \text{“} \varphi \text{ is true at some child”} \]
\[ \Diamond^+ \varphi \quad \text{“} \varphi \text{ is true at some descendant”} \]

We also have modalities (\(\Theta\) and \(\Diamond\)) to refer to lefthand siblings. All corresponding box-operators are definable, for example:

\[ \Box \varphi = \neg \Diamond \neg \varphi \quad \text{“} \varphi \text{ is true at all righthand siblings”} \]
Examples

• A node $t$ is the root of the tree iff $\text{ROOT}$ is true at $t$:

$$\text{ROOT} = \Box \bot$$

• Similar formulas identify leftmost and rightmost siblings:

$$\text{LEFTMOST} = \Box \bot \quad \text{RIGHTMOST} = \Box \bot$$

• An ordered tree $T$ is a binary tree iff $\text{BINARY}$ is valid in $T$ (or, equivalently, iff $\text{BINARY}$ is globally true in a model based on $T$):

$$\text{BINARY} = (\text{LEFTMOST} \leftrightarrow \text{RIGHTMOST}) \rightarrow \text{ROOT}$$

• An ordered tree $T$ is “discretely branching” iff the formula $\text{DISCRETE}$ is valid in $T$:

$$\text{DISCRETE} = \Box (\exists A \rightarrow A) \rightarrow (\Diamond \Box A \rightarrow \Box A)$$

This is the standard axiom schema familiar from temporal logic to characterise discrete flows of time.
OTL as a Temporal Interval Logic

We can interpret OTL as a (restricted) interval logic as follows:

- *nodes* in a tree represent *time intervals*;
- *descendants* represent *subintervals*; and
- the *order declared over siblings* represents an *earlier-later ordering* over time intervals.

But this raises some questions:

- What is the meaning of, say, the $\Diamond$-operator? Is it a proper future modality?
- Are models where, say, $\varphi$ is true at some node $t$ but not at all of $t$’s children meaningful under this temporal interpretation?
If you are allowed to move up before and down after moving to the right, you can reach all the nodes to the right.

Let $\Box^* \varphi = \varphi \lor \Diamond \varphi$ and $\Box^* \varphi = \varphi \lor \Diamond^+ \varphi$. We can now define a global future modality as follows:

$\Diamond \varphi = \Box^* \Box^* \varphi$
Ontological Considerations

Consider the following two basic propositions:

(1) *The sun is shining.*

(2) *I move the pen from the table onto the OHP.*

Propositions like (1) are sometimes called *properties*; propositions like (2) are sometimes called *events* (Allen, 1984).
Properties

Properties like “The sun is shining.” are homogeneous propositions (Shoham, 1987), which we can capture in OTL as follows:

\[
\text{DOWNWARD-HEREDITARY} (\varphi) = \varphi \rightarrow \square^+ \varphi \\
\text{UPWARD-HEREDITARY} (\varphi) = \lozenge \top \rightarrow (\square^+ \varphi \rightarrow \varphi) \\
\text{HOMOGENEOUS} (\varphi) = \text{DOWNWARD-HEREDITARY} (\varphi) \land \\
\text{UPWARD-HEREDITARY} (\varphi)
\]

Then \( \varphi \) is a homogeneous proposition (with respect to a given model \( \mathcal{M} \)) iff \( \text{HOMOGENEOUS}(\varphi) \) is globally true in \( \mathcal{M} \).
Events

Events like “I move the pen from the table onto the OHP.” may be characterised as propositions that cannot be true at two intervals one of which contains the other.

Shoham (1987) calls such propositions gestalt:

\[ \text{GESTALT}(\varphi) = \varphi \rightarrow (\square \neg \varphi \land \square^+ \neg \varphi) \]
Technical Results

• A complete *axiomatisation* of the fragment of OTL excluding the transitive descendant operator ($\Diamond^+$) is available.
  
  – The most interesting axioms are:
    
    (X1) $\Diamond A \rightarrow \Diamond \Diamond A$
    
    (X2) $\Diamond \Diamond A \rightarrow (A \lor \Diamond A \lor \Diamond A)$
  
  – Problems with proving completeness for the full logic:
    transitive closure + irreflexivity + interaction

• OTL is *decidable* (proof works essentially by reduction to S2S).
  
  – Note that 2-dimensional modal logics with interacting modalities (such as products) are often undecidable.
    
    (1) $\Diamond^* \Diamond^* A \rightarrow \Diamond^* \Diamond^* A$ (right-commutativity)
    
    (2) $\Diamond^* \Box^* A \rightarrow \Box^* \Diamond^* A$ (Church-Rosser property)
  
  – Also note that modal interval logics (such as Halpern and Shoham’s logic) tend to be undecidable as well.
Conclusion

• We have introduced a simple yet expressive modal logic for talking about ordered trees.

• Original motivation: linear temporal logic + zoom

• Compromise between point- and interval-based temporal logics:
  – can model subintervals, but not overlapping intervals
  – decidable (unlike many interval logics)

• Future work:
  – prove decidability directly (rather than by reduction to S2S)
  – give a complete axiomatisation for the full logic
  – extend results to cover until-style operators
  – develop a decision procedure (possibly Tableau-based)
  – use OTL to represent (nested) communication protocols